EXTREME POINTS IN $H^1(R)$

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1. Let R be an open Riemann surface. f belongs to $H^1(R)$ if f is holomorphic on R and if the subharmonic function $|f|$ has a harmonic majorant on R. Let p be in R and define $||f||$ to be the value at p of the least harmonic majorant of $|f|$. $||f||$ is a norm on the linear space $H^1(R)$, and with this norm $H^1(R)$ is a Banach space **(7)**. The unit ball of $H^1(R)$ is the closed convex set of all f in $H^1(R)$ with $||f|| \leqslant 1$. Problem: *What are the extreme points of the unit ball of* $H^1(R)$? de Leeuw and Rudin have given a complete solution to this problem where R is the open unit disk **(1)**. The purpose of this paper is to give a necessary condition for an extreme point of the unit ball of $H^1(R)$ when R is conformally equivalent to the interior of a compact bordered Riemann surface.

2. We let L^1 be the Banach space of Lebesgue measurable functions on the unit circle that are Lebesgue summable. The norm of f in L^1 is

$$\int |f|\, d\sigma = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|\, d\theta.$$

We let H^1 be the closed subspace of functions f in L^1 whose Fourier coefficients vanish for negative indices:

$$\int e^{-in\theta} f\, d\sigma = 0$$

for $n < 0$. Because of the uniformization theorem, $H^1(R)$ has a canonical representation as a closed subspace of H^1 **(7)**. I give a sketch of a proof of this.

I shall assume that $H^1(R)$ contains non-constant functions. Then R carries a non-constant positive harmonic function, and therefore the universal covering surface of R is the open unit disk D. Let t be the analytic function from D onto R with the property that (D, t) is a regular covering surface of R and $t(0) = p$, and let G be the group of cover transformations of (D, t). G is the group of fractional linear transformations T that take D onto D with $t(T) = t$.

Let T be a fractional linear transformation that takes the open unit disk onto itself. Then T is also a homeomorphism of the unit circle Γ, and T^{-1} takes Lebesgue measurable subsets of Γ into sets of the same kind and does it in such a way that Lebesgue null sets are taken into Lebesgue null sets. Thus $Tf = f(T)$ is Lebesgue measurable when f is, and $Tf = Tg$ a.e. when $f = g$ a.e.

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Consequently Tf is defined when f is in L^1. In addition, Tf belongs to L^1 when f belongs to L^1 and

$$\int Tf\, d\sigma = \int fP_T\, d\sigma,$$

where P_T is the Poisson kernel

$$P_T(e^{i\theta}) = \mathrm{Re}[(e^{i\theta} + T(0))/(e^{i\theta} - T(0))]. \tag{1}$$

Return to the group G of cover transformations of (D, t). H^1/G is the closed subspace of functions in H^1 that are invariant under G. Thus f belongs to H^1/G if and only if f belongs to H^1 and $Tf = f$ a.e. when T is in G. Let f be in $H^1(R)$ and let $g = f(t)$ be the function we get by lifting f to D. g belongs to $H^1(D)$ and is invariant under G. Let h be the radial limit of g. Then h belongs to H^1/G, and the linear transformation $f \to g \to h$ is an isometry from $H^1(R)$ onto H^1/G.

I shall work in the following with H^1/G and not with $H^1(R)$.

3. I have defined H^1/G, and the spaces L^p/G and H^p/G are defined in the same way. Thus L^∞/G is the closed subspace of functions in L^∞ that are invariant under G. There is another way to describe these spaces **(3)**. Let Σ be the sigma-field of Lebesgue measurable subsets of Γ, and let Σ/G be the sigma-field of X in Σ with $\sigma(X\Delta T^{-1}X) = 0$ for all T in G: then L^p/G is the closed subspace of functions in L^p that are measurable with respect to the smaller sigma-field Σ/G.

The conditional expectation given Σ/G will be denoted by E **(3, 6)**. E is the projection of L^1 onto the linear set of Σ/G measurable functions in L^1 given by

$$\int_X Ef\, d\sigma = \int_X f\, d\sigma \tag{2}$$

when X is in Σ/G. (The right side of (2) is a bounded complex measure on Σ/G that is absolutely continuous with respect to σ, and therefore by the Radon–Nikodym theorem there is a Σ/G measurable function Ef in L^1 such that (2) holds.) When $1 \leqslant p \leqslant \infty$, Ef is in L^p when f is, and thus, because L^p/G is the linear set of functions in L^p that are Σ/G measurable, E is a projection of L^p onto L^p/G. In particular, E carries L^∞ onto L^∞/G. The norm of E as a linear transformation of L^p is 1; E commutes with complex conjugation, i.e.

$$\overline{Ef} = E\bar{f}; \tag{3}$$

E is formally self-adjoint, i.e.

$$\int Ef\bar{g}\, d\sigma = \int f\overline{Eg}\, d\sigma \tag{4}$$

when f is in L^p and g is in L^q; and E has the property

$$E(fg) = fEg,$$

when f is in L^p/G and g is in L^q. Here $1 \leqslant p \leqslant \infty$ and q is the dual exponent $p/(p-1)$.

4. The basic lemma about extreme points of the unit ball of H^1/G is due to de Leeuw and Rudin **(1, 5)**, and is as follows.

LEMMA 1. *Let f be in H^1/G with*

$$\int |f|\, d\sigma = 1.$$

Then f is not an extreme point of the unit ball of H^1/G if and only if there is a non-constant real-valued function ϕ in L^∞/G with ϕf in H^1/G.

I shall give the proof because it is not long.

Suppose that there is such a ϕ. Subtracting from ϕ the constant

$$\int \phi |f|\, d\sigma,$$

we can assume that

$$\int \phi |f|\, d\sigma = 0.$$

Multiplying ϕ by a positive constant, we can also assume that $-1 \leqslant \phi \leqslant 1$. But now $f \pm \phi f$ belongs to the unit ball of H^1/G, and f is not an extreme point of the unit ball of H^1/G for $f \neq f \pm \phi f$ and $f = \frac{1}{2}(f + \phi f) + \frac{1}{2}(f - \phi f)$.

To see that the condition is also necessary, suppose that there is a non-zero function g in H^1/G with $f \pm g$ in the unit ball of H^1/G. Let $\phi = g/f$. Then

$$\int (|1 + \phi| + |1 - \phi| - 2)|f|\, d\sigma = 0,$$

which gives $|1 + \phi| + |1 - \phi| = 2$, which in turn gives $-1 \leqslant \phi \leqslant 1$. ϕ is non-constant for

$$\int |f|\, d\sigma = \int (1 + \phi)|f|\, d\sigma,$$

and hence

$$\int \phi |f|\, d\sigma = 0.$$

I in H^∞ is said to be an inner function if $|I| = 1$, and F in H^1 is said to be an outer function if FH^∞ is dense in H^1. Let f be in H^1 with

$$\int |f|\, d\sigma = 1,$$

and let $f = IF$ be an inner outer factoring of f **(1, 5)**. (There is an argument free of function theory for this factoring in **(2)**.) When I is non-constant, $\phi = I + \bar{I}$ satisfies the conditions of Lemma 1. On the other hand, if f is outer and ϕf belongs to H^1, where ϕ is in L^∞, then ϕ must belong to H^∞ since fH^∞ is dense in H^1. Thus if ϕ is also real, ϕ must be a constant. This is the de Leeuw–Rudin theorem: *Let f be in H^1 with*

$$\int |f|\, d\sigma = 1.$$

Then f is an extreme point of the unit ball of H^1 if and only if f is an outer function.

Now let f be in H^1/G with

$$\int |f|\, d\sigma = 1.$$

When f is outer, f is an extreme point of the unit ball of H^1/G. On the other hand, if f is not outer, we cannot infer as before that f is not extreme in the unit ball of H^1/G, because $I + \bar{I}$ may not belong to L^∞/G. Lemma 2 is a dual version of Lemma 1, and says that being an extreme point of the unit ball of H^1/G is the same as behaving in a manner resembling that of an outer function.

LEMMA 2. *Let f be in H^1/G and let*

$$\int |f|\, d\sigma = 1.$$

Then f is an extreme point of the unit ball of H^1/G if and only if the linear space

$$\bar{f}\,\overline{E(H_0^\infty)} + C + f\,E(H_0^\infty) \tag{5}$$

is dense in L^1/G.

H_0^∞ is the subspace of functions g in H^∞ with

$$\int g\, d\sigma = 0,$$

and C is the one-dimensional space of constant functions.

Proof. Suppose (5) is not dense in L^1/G. Then there is an h in L^∞/G with $h \neq 0$ that annihilates (5). Because

$$\int h\, d\sigma = 0,$$

h is non-constant. $\bar{h}$ also annihilates (5), and therefore there is a non-constant real-valued function ϕ in L^∞/G that annihilates (5). We have, by (3) and (4) and $E(\phi f) = \phi f$, that

$$\int \phi f g\, d\sigma = \int \phi f E g\, d\sigma = 0$$

for all g in H_0^∞, and thus ϕf belongs to H^1/G (since H^1/G is the subspace of functions in L^1/G whose Fourier coefficients vanish for negative indices). Hence (Lemma 1) f is not an extreme point of the unit ball of H^1/G.

On the other hand, if f is not an extreme point of the unit ball of H^1/G, let ϕ be the function given by Lemma 1, and let $h = \phi - c$, where

$$c = \int \phi\, d\sigma.$$

Then $h \neq 0$ and h annihilates (5).

5. The things set down in this section are given with more details in **(3)**. Let f be in L^2 and let f^* be the function conjugate to f. f^* is the function in L^2 with Fourier series

$$\textstyle\sum_{n<0} ic_n\, e^{in\theta} + \sum_{n>0} -ic_n\, e^{in\theta},$$

where

$$\textstyle\sum c_n\, e^{in\theta}$$

is the Fourier series of f. When T is a fractional linear transformation that takes D onto D,

$$T(f^*) = (Tf)^* - \int f P_T{}^*\, d\sigma,$$

where P_T is the Poisson kernel (1). Therefore, when f is in L^2/G and T is in G,

$$T(f^*) = f^* - \int f v_T \, d\sigma, \tag{6}$$

where

$$v_T = E(P_T{}^*). \tag{7}$$

v_T is a bounded real function since $P_T{}^*$ is. Let N be the complex linear span of the functions v_T (T in G). N is contained in L^∞/G, and the orthogonal complement in L^2/G of N consists of the functions in L^2/G whose conjugates also belong to L^2/G. Let N^2 be the L^2 closure of N. Then we have the orthogonal decomposition

$$L^2/G = N^2 \oplus \overline{H_0{}^2/G} \oplus C \oplus H_0{}^2/G. \tag{8}$$

Let $d(N)$ be the dimension of N, let G' be the commutator subgroup of G, and let $r(G/G')$ be the smallest number of elements that will generate the homology group G/G'. When R is not simply connected, $r(G/G')$ is the first Betti number of R. (6) implies that

$$v_{ST} = v_S + v_T \tag{9}$$

for all S and T in G, and this in turn implies that $d(N) \leqslant r(G/G')$. Let $d(R)$ be the largest number of bounded harmonic functions on R with cohomologously independent conjugate differentials. Then $d(N) = d(R)$ always. When R is the interior of a compact bordered Riemann surface that is not simply connected, $d(R)$ is finite and is equal to $r(G/G')$, and, conversely, a Riemann surface with this property is conformally equivalent to the interior of a compact bordered Riemann surface.

Let χ be a homomorphism of G into the multiplicative group of unimodular complex numbers. A non-zero vector f in H^1 is called an eigenvector of G with eigenvalue χ if $Tf = \chi(T) f$ for all T in G. The following lemma is a simple consequence of (6) and (9), and is a paraphrase of a well-known fact about compact bordered Riemann surfaces.

LEMMA 3. *Suppose $d(N)$ is finite and is equal to $r(G/G')$. Let χ be any homomorphism of G into the multiplicative group of unimodular complex numbers. Then there is a unit u in the algebra H^∞ with* $\log |u|$ *in N that is an eigenvector of G with eigenvalue χ.*

With the hypothesis of Lemma 3 it is also true that

$$E(H_0{}^\infty) = N + H_0{}^\infty/G, \tag{10}$$

$$E(H^\infty) = N + H^\infty/G, \tag{11}$$

and that there is a function k in H^∞/G with

$$kE(H^\infty) = H^\infty/G. \tag{12}$$

(10) and (11) are simple consequences of (7) and (8). To prove that there is

a function k such that (12) is true is a little complicated; however (12) will be used only for exposition.

Let f be in H^1/G and assume the hypothesis of Lemma 3. *When f is an outer function, fH^∞/G is dense in H^1/G and fH_0^∞/G is dense in H_0^1/G.* There are several proofs of this, and among them is one using (12) that I find attractive. Let ϕ in kH^∞ be such that $E(\phi) = 1$. Now let g be in H^1/G. Because f is outer, there is b in H^∞ (and in H_0^∞ if g is in H_0^1/G) with

$$\int |fb - g|\, d\sigma \leqslant \epsilon.$$

$E(\phi b)$ is in H^∞/G (and in H_0^∞/G if b is in H_0^∞ by (2)) and

$$\begin{aligned}\int |fE(\phi b) - g|\, d\sigma &= \int |E(f\phi b - g\phi)|\, d\sigma \\ &\leqslant \int |f\phi b - g\phi|\, d\sigma \leqslant \epsilon ||\phi||_\infty,\end{aligned}$$

showing that fH^∞/G is dense in H^1/G.

I now give an argument that does not use (12). The finite-dimensional space fN has only the vector 0 in common with H^1. For if fv is in H^1, where v is in N, then v must be in H^∞ since fH^∞ is dense in H^1, and hence by (8)

$$\int |v|^2\, d\sigma = 0.$$

Thus there is a constant K such that

$$\int |h|\, d\sigma \leqslant K \int |fv + h|\, d\sigma$$

for all v in N and h in H^1. Now let g and b be as before. Then by (11)

$$Eb = v + h,$$

where v is in N and h is in H^∞/G (and in H_0^∞/G if b is in H_0^∞), and

$$\begin{aligned}\int |fh - g|\, d\sigma &\leqslant K \int |fv + fh - g|\, d\sigma \\ &= K \int |E(fb - g)|\, d\sigma \leqslant K \int |fb - g|\, d\sigma \leqslant K\epsilon.\end{aligned}$$

6. THEOREM 1. *Suppose $d(N)$ is finite and equal to $r(G/G')$. Let f be in H^1/G with*

$$\int |f|\, d\sigma = 1$$

and let n be the codimension in H^1/G of the L^1 closure of fH^∞/G. Then for f to be an extreme point of the unit ball of H^1/G it is necessary that $2n \leqslant d(N)$.

Proof. Let $f = IF$ be an inner outer factoring of f. Though f is invariant under G, I and F need not be. What is true is that I and F are eigenvectors of G with eigenvalues $\bar{\chi}$ and χ. (An inner outer factoring is determined up to constant unimodular factors, and the collections of inner and outer functions are carried onto themselves when composed with a fractional linear transformation that takes the open unit disk onto itself.) Let u be the unit given by Lemma 3. Then $f = Iuu^{-1}F$, Iu belongs to H^∞/G, $u^{-1}F$ belongs to H^1/G, and $u^{-1}F$ is an outer function. Because $u^{-1}FH^\infty/G$ is dense in H^1/G, the L^1

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closure of fH^∞/G is IuH^1/G. We are not going to work with the L^1 closure of fH^∞/G, but with the L^1 closure of fH_0^∞/G. The L^1 closure of fH_0^∞/G is IuH_0^1/G because $u^{-1}FH_0^\infty/G$ is dense in H_0^1/G, and the co-dimension of this space in H_0^1/G is again n. Since the co-dimension of H_0^1/G in H^1/G is 1, the co-dimension of IuH_0^1/G in IuH^1/G is 1 too, the co-dimension of IuH_0^1/G in H^1/G is $n + 1$, and the co-dimension of IuH_0^1/G in H_0^1/G is n.

Let S be the L^1 closure of $\overline{H_0^1/G} + H_0^1/G$. When f is in H_0^1/G and T is in G,

$$\int f v_T \, d\sigma = \int f E(P_T^*) \, d\sigma = \int f P_T^* \, d\sigma = -\int f^* P_T \, d\sigma = i\int f P_T \, d\sigma = i \int f \, d\sigma = 0.$$

Therefore

$$\int f v \, d\sigma = 0$$

for f in S and v in N, and N and S have only 0 in common. (8) shows that $N + C + S$ is dense in L^1/G, and as N is finite dimensional, $N + C + S$ is also closed in L^1/G. Thus we have the direct sum decompositions

$$L^1/G = N + C + S,$$
$$L_0^1/G = N + S,$$

and the co-dimension of S in L_0^1/G is $d(N)$.

Let M be the L^1 closure of $\bar{f}\,\overline{H_0^\infty/G} + fH_0^\infty/G$. M is contained in S, and the co-dimension of M in S is $2n$. Here is one proof. Let $0 < r < 1$. There is a constant $K(r)$ such that

$$\left(\int |Pg|^r \, d\sigma\right)^{1/r} \leqslant K(r) \int |g| \, d\sigma$$

for all g in L^1, where P, when restricted to trigonometric polynomials, is given by

$$P\left(\sum c_n e^{in\theta}\right) = \sum_{n>0} c_n e^{in\theta}$$

(**8**, p. 254). P takes L^1 into H_0^r, where H_0^r is the closure in L^r of trigonometric polynomials of the form $\sum_{n>0} c_n e^{in\theta}$. Let m be finite and $\leqslant n$ and let g_1 through g_m be in H_0^1/G and be linearly independent modulo IuH_0^1/G. The $2m$ functions $\bar{g}_1$ through $\bar{g}_m$ and g_1 through g_m belong to S, and we wish to show that they are linearly independent modulo M. Suppose that $\sum c_k \bar{g}_k + \sum d_k g_k$ belongs to M. Approximating this function by functions taken from $\bar{f}\,\overline{H_0^\infty/G} + f H_0^\infty/G$ and then applying the linear transformation P shows that $g = \sum d_k g_k$ belongs to the L^r closure of fH_0^∞. The L^r closure of fH_0^∞ is IH_0^r, and thus $g = Ih$, where h is in H_0^r. Because h is in L^1, h is in fact in H_0^1 (**8**, p. 278). Hence $g = Iuu^{-1}h$ is in IuH_0^1/G, and thus the d_k are all 0 because of the assumed linear independence. We now have $\sum \bar{c}_k g_k$ in M, and the argument just given shows that the c_k are also all 0. Thus the co-dimension of M in S is $\geqslant 2n$, and clearly it cannot be larger than this.

I shall use Lemma 2 to complete the proof of Theorem 1. The co-dimension of M in L_0^1/G is $2n + d(N)$. Moreover (by (10)),

$$\bar{f}\,\overline{E(H_0^\infty)} + f\,E(H_0^\infty) = \bar{f}N + fN + \bar{f}\,\overline{H_0^\infty/G} + f\,H_0^\infty/G \tag{13}$$

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and (13) is contained in

$$\bar{f}N + fN + M. \tag{14}$$

The dimension of $\bar{f}N + fN$ is $\leqslant 2d(N)$, and therefore the co-dimension of M in (14) is $\leqslant 2d(N)$, and thus, to have (13) dense in L_0^1/G, it is necessary that $2n + d(N) \leqslant 2d(N)$.

Here is a corollary to Theorem 1.

THEOREM 2. *Let R be conformally equivalent to an annulus and let f be in H^1/G with*

$$\int |f|\, d\sigma = 1.$$

Then f is an extreme point of the unit ball of H^1/G if and only if f is an outer function.

Proof. $d(N) = r(G/G') = 1$.

Gamelin in **(4)** shows that there are extreme points of the unit ball of H^1/G that are not outer functions when R is the interior of a compact bordered Riemann surface with first Betti number $\geqslant 2$.

So far nothing has been said about the anatomy of an inner function, and in particular no use has been made of this anatomy (because I wanted to give a proof of Theorem 1 that did not use it). When the anatomy of an inner function is taken into account, however, there is another version of Theorem 1.

Assume the conditions of Theorem 1. The inner factor I of f has a factoring $I = BS$ where B is a Blaschke product and S is a singular inner function **(5)**. *When S appears, n is infinite.* For let k be a positive integer. Then there is a factoring,

$$S = W^k, \tag{15}$$

where W is a singular inner function too. Now S and B are eigenvectors of G because I is (for the factoring of an inner function into a Blaschke product and a singular inner function is determined up to constant unimodular factors, and the collections of Blaschke products and singular inner functions are carried onto themselves when composed with a fractional linear transformation that takes the open unit disk onto itself). In turn (15) implies that W is an eigenvector of G also. (For let T be in G. Then

$$TS = S(T) = \lambda^k S,$$

where λ is a unimodular complex number, and hence

$$\Pi_1^k(TW - \lambda_j \lambda W) = (TW)^k - \lambda^k W^k = TS - \lambda^k S = 0,$$

where λ_1 through λ_k are the k roots of 1. This implies that one of the factors in the product is the zero vector since H^∞ does not have zero divisors.) Let u

and v be units in H^∞ such that Bu and Wv belong to H^∞/G (Lemma 3). Then the L^1 closure of fH^∞/G is $(Wv)^k BuH^1/G$, and

$$(Wv^k)BuH^1/G < (Wv)^{k-1}BuH^1/G < \ldots < WvBuH^1/G < BuH^1/G$$

is a strictly ascending chain of $k + 1$ (closed) subspaces from $(Wv)^k BuH^1/G$ to BuH^1/G, showing that $k \leqslant n$. *When S does not appear, n is equal to the number of zeros on R of f* (by a like argument). *Thus for f to be an extreme point of the unit ball of H^1/G it is necessary that f have no singular part and that twice the number of zeros of f on R does not exceed the first Betti number of R.* This condition on the zeros of f is also given by Gamelin in **(4)**. Gamelin's methods, however, do not show that an extreme point has no singular part. Moreover, an argument is needed to get rid of the singular part, since a singular inner function can be an eigenvector of G and have the property that there is no non-constant inner function that is invariant under G and divides it.

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