ALGEBRAS WITH VANISHING n-COHOMOLOGY GROUPS

MASATOSHI IKEDA, HIROSHI NAGAO
and TADASHI NAKAYAMA

Cohomology theory for (associative) algebras was first established in general higher dimensionalities by G. Hochschild [3], [4], [5]. Algebras with vanishing 1-cohomology groups are separable semisimple algebras ([3], Theorem 4.1). On extending and refining our recent results [6], [8], [12], we establish in the present paper the following:

Let $n \geqq 2$. Let A be an (associative) algebra (of finite rank) possessing a unit element 1 over a field Ω, and N be its radical. If the n-cohomology groups of A all vanish, then

α) the semisimple residue-algebra A/N is separable, and

β) for every left-ideal $\mathfrak{l}$ of A the A-left-submodule

$$1(A \times \ldots \times A \times \mathfrak{l}) \qquad (\text{with } n-2\ A\text{'s})$$

of the Kronecker product (over Ω) $A \times A \times \ldots \times A$ (with $n-1$ A's) is an (M_0)-module (see below), where the operation of A on $A \times A \times \ldots \times A$ is defined by $a(x_1 \times x_2 \times \ldots \times x_{n-1}) = ax_1 \times x_2 \times \ldots \times x_{n-1} - a \times x_1 x_2 \times \ldots \times x_{n-1} + \ldots + (-1)^{n-2}(a \times x_1 \times \ldots \times x_{n-2}x_{n-1})$, $(a, x_1, \ldots, x_{n-1} \in A)$. (On the submodule $1(A \times A \times \ldots \times A) = A(A \times A \times \ldots \times A)$ (whence on $1(A \times A \times \ldots \times A \times \mathfrak{l})$) this operation coincides, however, with the ordinary operation which simply operates, from the left, on the first component.) (In proving β) we do not need to assume A to be of finite rank.)

Conversely, if α) is the case and if

β_1) the A-left-module $1(A \times \ldots \times A \times N)$, with $n=2$ A's, is an (M_0)-module, then all the n-cohomology groups of A are 0.

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Thus either the pair α), β) or the pair α), β_1) is necessary and sufficient for A to have vanishing n-cohomology groups. Needless to say that our theorem contains [4], Theorem 8 as a very special case.

Now, we call a left-, say, module $\mathfrak{m}$ of A an (M_0)-module when the following condition is satisfied: if $\mathfrak{m}$ is A-isomorphic to a residule-module $\mathfrak{M}/\mathfrak{n}$ then $\mathfrak{M}$ is a direct sum of $\mathfrak{n}$ and a second A-submodule $\mathfrak{m}'$ (necessarily A-isomorphic with $\mathfrak{m}$), $\mathfrak{M}=\mathfrak{m}'\oplus\mathfrak{n}$. A structural characterization, for an A-left-module $\mathfrak{m}$ to be an (M_0)-module, is that 1 $\mathfrak{m}$ should be a direct sum of A-submodules A-isomorphic to principal left-ideals of A generated by idempotent elements ([9]). Hochschild [5], Theorem 1.4 asserts that all the n-cohomology groups of A vanish if and only if $A\times A\times\ldots\times A$ with nA's is an (M_0)-module when considered as an A-(double-)module with the left operation of A defined similarly as above (with n in place of $n-1$) and with the ordinary right operation of A (which simply operates, from the right, on the utmost right factor of the Kronecker product). Let us observe that in β) or β_1) the number of factors of the Kronecker product is smaller by 1 than in this statement while the last factor is an arbitrary left-ideal, in β), or the radical, in β_1). Moreover, we are dealing with one-sided modules, of A, instead of a two-sided one. We wish to note also that in case $n=2$ we are, in β), β_1), dealing simply with a left-ideal or the radical under ordinary left multiplication of A. As for the case $n=3$, $1(A\times\mathfrak{l})$ is the kernel of the (natural) homomorphism $A\times\mathfrak{l}\to\mathfrak{l}=A\times_A\mathfrak{l}$.

Now, the vanishing of the n-cohomology groups of A is expressed either by class $A\leqq n$, in the sense of Hochschild [4], or by $\dim A\leqq n-1$, in the sense of Cartan-Eilenberg [2]. Moreover, that an A-left-module $\mathfrak{m}$ is an (M_0)-module is equivalent to the fact that it is projective, i.e. of (A-left-)dimension 0, in the sense of Cartan-Eilenberg. Moreover, that the A-left-module $1(A\times\ldots\times A\times\mathfrak{m})$ with $n-2$ A's is an (M_0)-module (or projective, or of (A-left) dimension 0) turns out to be equivalent to that the A-left dimension of $\mathfrak{m}$ is not greater than $n-2$; a proof of this, together with some other results related to the present paper, will be given in a joint paper by Eilenberg and two of the present authors in Nagoya Math. Journal. So our result may be expressed also by saying that $\dim A\leqq n-1$ implies α) $\dim A/N=0$, β) $\mathrm{l.dim}_A\mathfrak{l}\leqq n-2$ ($\mathfrak{l}$ being any left-ideal in A), and is implied by α) and β_1) $\mathrm{l.dim}_A N\leqq n-2$. Further, we are informed by Eilenberg that he has an alternative treatment

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of our main theorem within the frame work of the Cartan-Eilenberg [2] theory; it will appear in Comment. Math. Helvetici.

§ 1. Relative cohomology

Let A be an (associative) algebra, over a field Ω, and let $\mathfrak{l}$ be a left-ideal of A. Consider an A-double-module $\mathfrak{m}$ of A satisfying

$$\mathfrak{m}\mathfrak{l} = 0. \tag{1}$$

We briefly recall the notion of $\mathfrak{l}$-relative cohomology groups of A in $\mathfrak{m}$ as it was introduced in [12]; our purpose is, and was in [12], to apply it in proving our structural theorem for algebras with vanishing (ordinary) cohomology groups, but the notion is perhaps of interest by itself. Let n be a natural number, and consider the subgroup $C_{\mathfrak{l}}^n(A, \mathfrak{m})$ of the n-cochain group $C^n(A, \mathfrak{m})$ of A in $\mathfrak{m}$ consisting of all those n-cochains f—called $\mathfrak{l}$-*relative* cochains—which satisfy the condition: $f(x_1, x_2, \ldots, x_n) = 0$ when $x_n \in \mathfrak{l}$. We put $C_{\mathfrak{l}}^0(A, \mathfrak{m}) = \mathfrak{m}$. With the ordinary coboundary operator δ we have, as we see readily,

$$\delta C_{\mathfrak{l}}^{n-1}(A, \mathfrak{m}) \subseteqq C_{\mathfrak{l}}^n(A, \mathfrak{m}). \tag{2}$$

Therefore, on putting

$$Z_{\mathfrak{l}}^n(A, \mathfrak{m}) = C_{\mathfrak{l}}^n(A, \mathfrak{m}) \cap Z^n(A, \mathfrak{m}),$$
$$B_{\mathfrak{l}}^n(A, \mathfrak{m}) = \delta C_{\mathfrak{l}}^{n-1}(A, \mathfrak{m}),$$

where $Z^n(A, \mathfrak{m})$ is the ordinary n-cocycle group, we obtain the group (in fact, Ω-module)

$$H_{\mathfrak{l}}^n(A, \mathfrak{m}) = Z_{\mathfrak{l}}^n(A, \mathfrak{m}) / B_{\mathfrak{l}}^n(A, \mathfrak{m})$$

which we want to call the $\mathfrak{l}$-*relative n-cohomology group* of A in $\mathfrak{m}$; if we speak simply of a cohomology group, we shall always mean an ordinary, non-relative (i.e. 0-relative) cohomology group.

We consider $C_{\mathfrak{l}}^n(A, \mathfrak{m})$ as an A-double-module, defining, for $f \in C_{\mathfrak{l}}^n(A, \mathfrak{m})$,

$$\begin{aligned} (xf)(x_1, \ldots, x_n) &= xf(x_1, \ldots, x_n), \\ (fx)(x_1, \ldots, x_n) &= xf(x_1, \ldots, x_n) - \delta f(x, x_1, \ldots, x_n) \end{aligned} \tag{3}$$

$(x, x_1, \ldots, x_n \in A)$. Then we obtain the following generalization of Hochschild's reduction theorem:

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(4) $$H_{\mathfrak{l}}^{n+r}(A, \mathfrak{m}) \simeq H^{r}(A, C_{\mathfrak{l}}^{n}(A, \mathfrak{m})),$$

where the right-hand side is the ordinary r-cohomology group of A in $C_{\mathfrak{l}}^{n}(A, \mathfrak{m})$. The proof is exactly the same as in the ordinary case.

§ 2. The modules P^{n-1}, $Q_{\mathfrak{l}}^{n-1}$.

Let $n \geqq 2$ and define, after Hochschild, the Kronecker product, over Ω,

(5) $$P^{n-1} = A \times A \times \ldots \times A \times A \qquad (\text{with } n-1 \ A\text{'s})$$

the following left operation of A:

(6) $$\begin{aligned} & x(x_1 \times x_2 \times \ldots \times x_{n-2} \times x_{n-1}) \\ & \quad = x x_1 \times x_2 \times \ldots \times x_{n-1} - x \times x_1 x_2 \times \ldots \times x_{n-1} \\ & \qquad + \ldots + (-1)^n (x \times x_1 \times \ldots \times x_{n-2} x_{n-1}). \end{aligned}$$

Let $\mathfrak{m}$ be an A-left-module. We consider it as an A-double-module with

(7) $$\mathfrak{m} A = 0.$$

The (ordinary) (n-1)-cochain group $C^{n-1}(A, \mathfrak{m}) = C_0^{n-1}(A, \mathfrak{m})$ may be identified with the Ω-module $L(P^{n-1}, \mathfrak{m})$ of all Ω-linear mappings of P^{n-1} into $\mathfrak{m}$. The A-double-module structure of $C^{n-1}(A, \mathfrak{m})$, as defined in (3), reads as follows in $L(P^{n-1}, \mathfrak{m})$: for $\varphi \in L(P^{n-1}, \mathfrak{m})$, $x \in A$, $u \in P^{n-1}$

(8) $$\begin{aligned} (x\varphi)(u) &= x\varphi(u), \\ (\varphi x)(u) &= \varphi(xu) \end{aligned}$$

(Observe (6) and (7)). The (ordinary) reduction theorem ((4) with $\mathfrak{l} = 0$, $r = 1$ and n replaced by $n-1$) gives

(9) $$H^{n}(A, \mathfrak{m}) \simeq H^{1}(A, L(P^{n-1}, \mathfrak{m})).$$

If here $H^{n}(A, \mathfrak{m})$ vanishes, so does $H^{1}(A, L(P^{n-1}, \mathfrak{m}))$ naturally. This means, however, that every (A-left-module) extension of the A-left-module $\mathfrak{m}$ by the A-left-module P^{n-1} splits; the proof is exactly the same as in Hochschild [5], § 1. As $\mathfrak{m}$ is any A-left-module, this proves

LEMMA 1. *If the* (*ordinary*) *n-cohomology groups of A in A-double-modules annihilated by A on the right all vanish, then the A-left-module* P^{n-1} (*as defined in* (5) *with* (6)) *is an* (M_0)*-module; for the notion of an* (M_0)*-module cf. the introduction.*

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Remark 1. In case A possesses a unit element 1, every P^i $(i \geqq 1)$ is an (M_0)-module as an A-left-module under the operation (6). This holds indeed without any assumption on cohomology groups of A, and follows in fact readily either from (9) and [4], Theorem 1 combined or from an inductive argument which makes use of the homomorphisms of P^i similar to those of $Q^i_{\mathfrak{l}}$ (see below) used in our proof of Theorem 4 below (cf. also [11], Lemma 4.1 and [6], Proposition 2).

Next, let $\mathfrak{l}$ be a left-module of A, and put

$$Q^{n-1}_{\mathfrak{l}} = A \times \ldots \times A \times \mathfrak{l} \qquad (\text{with } n-2\ A\text{'s}). \tag{10}$$

This is a left-module of A under the operation (6), where we let x_{n-1} now stand for an element of $\mathfrak{l}$. If in particular $\mathfrak{l}$ is a left-ideal of A, as we shall assume in the rest of this section, then $Q^{n-1}_{\mathfrak{l}}$ is an A-submodule of the A-left-module P^{n-1} (with the operation (6)).

Further, let

$$A = A^0 \oplus \mathfrak{l} \tag{11}$$

be a decomposition of A into a direct sum of $\mathfrak{l}$ and an Ω-submodule A^0. With $x \in A$ we denote by x^0 the A^0-component of x in this decomposition. For $x, y \in A$, put

$$xy = (xy)^0 + \lambda(x, y) \qquad (\lambda(x, y) \in \mathfrak{l}). \tag{12}$$

Then we have, for $x, y, z \in A$,

$$\lambda(xy, z) = \lambda(x, (yz)^0) + x\lambda(y, z), \tag{13}$$

$$\lambda(x, l) = xl \qquad (l \in \mathfrak{l}). \tag{14}$$

Now let $\mathfrak{M}$ be an A-left-module and $\mathfrak{m}$ an A-submodule of $\mathfrak{M}$. Let

$$\mathfrak{M} = \mathfrak{M}^1 \oplus \mathfrak{m} \tag{15}$$

be a direct decomposition of $\mathfrak{M}$ into $\mathfrak{m}$ and an Ω-submodule $\mathfrak{M}^1$. With $u \in \mathfrak{M}$, let u^1 be the $\mathfrak{M}^1$-component of u with respect to (15). Putting, for $x \in A$, $u \in \mathfrak{M}$,

$$xu = (xu)^1 + \mu(x, u) \qquad (\mu(x, u) \in \mathfrak{m}), \tag{16}$$

we have, similarly as above,

$$\mu(xy, u) = \mu(x, (yu)^1) + x\mu(y, u), \tag{17}$$

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(18) $$\mu(x, m) = xm \qquad (m \in \mathfrak{m}).$$

Now assume that $\mathfrak{M}/\mathfrak{m}$ is A-isomorphic to our A-left-module $Q_{\mathfrak{l}}^{n-1}$. With $q \in Q_{\mathfrak{l}}^{n-1}$, let q^* be the element of $\mathfrak{M}^1$ whose class modulo $\mathfrak{m}$ corresponds to q by the (once fixed) A-isomorphism of $\mathfrak{M}/\mathfrak{m}$ and $Q_{\mathfrak{l}}^{n-1}$. For the sake of brevity, we write $\mu(x, q)$ in the place of $\mu(x, q^*)$.

Making $\mathfrak{m}$ an A-double-module with

(19) $$\mathfrak{m}A = 0,$$

we consider the $\mathfrak{l}$-relative $(n+1)$-cocycle f in $\mathfrak{m}$ defined by

(20) $$f(x_0, x_1, \ldots, x_{n-2}, y, z) = \mu(x_0, x_1 \times \ldots \times x_{n-2} \times \lambda(y, z^0)).$$

Indeed, we have

$$\begin{aligned}
&\delta f(w, x_0, x_1, \ldots, x_{n-2}, y, z)\\
&\quad = w\mu(x_0, x_1 \times \ldots \times x_{n-2} \times \lambda(y, z^0)) - \mu(wx_0, x_1 \times \ldots \times x_{n-2} \times \lambda(y, z^0))\\
&\qquad + \sum_{i=0}^{n-3} (-1)^i \mu(w, x_0 \times \ldots \times x_i x_{i+1} \times \ldots \times x_{n-2} \times \lambda(y, z^0))\\
&\qquad + (-1)^n (\mu(w, x_0 \times \ldots \times x_{n-3} \times \lambda(x_{n-2}y, z^0))\\
&\qquad - \mu(w, x_0 \times \ldots \times x_{n-3} \times \lambda(x_{n-2}, (yz)^0)))\\
&\quad = w\mu(x_0, x_1 \times \ldots \times x_{n-2} \times \lambda(y, z^0)) - (\mu(w, x_0(x_1 \times \ldots \times x_{n-2} \times \lambda(y, z^0)))\\
&\qquad + w\mu(x_0, x_1 \times \ldots \times x_{n-2} \times \lambda(y, z^0))) + (\mu(w, x_0(x_1 \times \ldots \times x_{n-2} \times \lambda(y, z_0))\\
&\qquad - (-1)^n \mu(w, x_0 \times \ldots \times x_{n-3} \times x_{n-2}\lambda(y, z^0)))\\
&\qquad + (-1)^n \mu(w, x_0 \times \ldots \times x_{n-3} \times x_{n-2}\lambda(y, z^0)) = 0,
\end{aligned}$$

by (17), (6) and (13). Thus

(21) $$f \in Z_{\mathfrak{l}}^{n+1}(A, \mathfrak{m}).$$

Suppose that there exists $h \in C_{\mathfrak{l}}^n(A, \mathfrak{m})$ such that

(22) $$\delta h = f,$$

i.e.

(23) $$\mu(x_0, x_1 \times \ldots \times x_{n-2} \times \lambda(y, z^0)) = \delta h(x_0, x_1, \ldots, x_{n-2}, y, z).$$

Put, in view of (11),

(24) $$\begin{aligned}&\nu(x_0, x_1 \times x_2 \times \ldots \times x_{n-1})\\ &\quad = (-1)^{n-1} h(x_0, x_1, \ldots, x_{n-1}) + \mu(x_0, x_1 \times \ldots \times x_{n-2} \times (x_{n-1} - x_{n-1}^0)).\end{aligned}$$

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Then clearly

$$\nu(x_0,\ x_1\times\ldots\times x_{n-2}\times l)=\mu(x_0,\ x_1\times\ldots\times x_{n-2}\times l)\qquad(l\in\mathfrak{l}). \tag{25}$$

Further

$$\begin{aligned}
&w\nu(x_0,\ x_1\times\ldots\times x_{n-1})\\
&\quad=(-1)^{n-1}wh(x_0,\ x_1,\ \ldots,\ x_{n-1})+w\mu(x_0,\ x_1\times\ldots\times x_{n-2}\times(x_{n-1}-x^0_{n-1})),\\
&\nu(wx_0,\ x_1\times\ldots\times x_{n-1})\\
&\quad=(-1)^{n-1}h(wx_0,\ x_1,\ \ldots,\ x_{n-1})+\mu(wx_0,\ x_1\times\ldots\times x_{n-2}\times(x_{n-1}-x^0_{n-1}))\\
&\quad=(-1)^{n-1}h(wx_0,\ x_1,\ \ldots,\ x_{n-1})+\mu(w,\ x_0(x_1\times\ldots\times x_{n-2}\times(x_{n-1}-x^0_{n-1})))\\
&\qquad+w\mu(x_0,\ x_1\times\ldots\times x_{n-2}\times(x_{n-1}-x^0_{n-1})),\\
&\nu(w,\ x_0(x_1\times\ldots\times x_{n-1}))\\
&\quad=(-1)^{n-1}(\delta h(w,\ x_0,\ x_1,\ \ldots,\ x_{n-1})\\
&\qquad-wh(x_0,\ x_1,\ \ldots,\ x_{n-1})+h(wx_0,\ x_1,\ \ldots,\ x_{n-1})\\
&\qquad+(-1)^n\mu(w,\ x_0\times x_1\times\ldots\times x_{n-3}\times\lambda(x_{n-2},\ x^0_{n-1}))\\
&\qquad+\lambda(w,\ x_0(x_1\times\ldots\times x_{n-2}\times(x_{n-1}-x^0_{n-1})),
\end{aligned}$$

by (17), (6) and (12), whence

$$\begin{aligned}
&w\nu(x_0,\ x_1\times\ldots\times x_{n-1})-\nu(wx_0,\ x_1\times\ldots\times x_{n-1})\\
&\quad+\nu(w,\ x_0(x_1\times\ldots\times x_{n-1}))=(-1)^{n-1}\delta h(w,\ x_0,\ x_1,\ \ldots,\ x_{n-1})\\
&\quad+(-1)^n\mu(w,\ x_0\times x_1\times\ldots\times x_{n-3}\times\lambda(x_{n-2},\ x^0_{n-1}))=0,
\end{aligned} \tag{26}$$

by (23).

Introduce now an Ω-module $\mathfrak{N}$ isomorphic to P^{n-1}, and let, for $u\in P^{n-1}$, $u^{\S}$ be the element of $\mathfrak{N}$ which corresponds to u. Put

$$\mathfrak{S}=\mathfrak{N}\oplus\mathfrak{m} \tag{27}$$

and define, for $u\in P^{n-1}$, $m\in\mathfrak{m}$,

$$x(u^{\S}+m)=(xu)^{\S}+\nu(x,\ u)+xm. \tag{28}$$

(26) secures that this makes $\mathfrak{S}$ an A-left-module. (25) shows that its sub-module $(Q^{n-1}_{\mathfrak{l}})^{\S}\oplus\mathfrak{m}$ is A-isomorphic to $\mathfrak{M}$.

Thus we have the first half of the following theorem whose second half may be seen simply by tracing the above calculation in reverse order.

THEOREM 1. *Let $\mathfrak{M}$ be an A-left-module and $\mathfrak{m}$ be its A-submodule. Let $\mathfrak{l}$ be a left-ideal of A and assume that $\mathfrak{M}/\mathfrak{m}$ is A-isomorphic to the A-left-*

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module $Q_{\mathfrak{l}}^{n-1}$ (*as defined in* (10), *with* (6)). *Define* $f \in Z_{\mathfrak{l}}^{n+1}(A, \mathfrak{m})$ *by* ((12), (16) *and*) (20), *where we consider* $\mathfrak{m}$ *as an A-double-module by putting* $\mathfrak{m}A = 0$. *If, and only if, the class of* f *in* $H_{\mathfrak{l}}^{n+1}(A, \mathfrak{m})$ *is* 0, *the A-left-module* $\mathfrak{M}$ *may be imbedded, as A-left-module, into an A-left-module L such that there is an A-left-isomorphism of* $\mathfrak{L}/\mathfrak{m}$ *with* P^{n-1} *which coincides on* $\mathfrak{M}/\mathfrak{m}$ *with the given isomorphism of* $\mathfrak{M}/\mathfrak{m}$ *and* $Q_{\mathfrak{l}}^{n-1}(\subseteqq P^{n-1})$.

Now, assume that our algebra A is such that its (ordinary) n-cohomology groups are all 0. Then its $\mathfrak{l}$-relative $(n+1)$-cohomology groups are all 0, by virtue of the reduction theorem (4). So the condition ($f \sim 0$ in $H_{\mathfrak{l}}^{n+1}(A, \mathfrak{m})$) is always satisfied in this case. We see thus that every extension, of $\mathfrak{m}$, by $Q_{\mathfrak{l}}^{n+1}$ can be imbedded in an extension by P^{n-1}. But our same assumption implies, as we have seen in Lemma 1, that the A-left-module P^{n-1} is an (M_0)-module; cf. also Remark there. The splitting of an extension by P^{n-1} entails naturally the splitting of the imbedded extension by $Q_{\mathfrak{l}}^{n-1}$. Thus we have

THEOREM 2. *Suppose that the (ordinary) n-cohomology groups of A all vanish. Then, with any left-ideal* $\mathfrak{l}$ *of A, the A-left-module* $Q_{\mathfrak{l}}^{n-1}$ (*as defined in* (10) *with* (6)) *is an* (M_0)-*module* (*or what is the same, the unitary A-left-module* $1Q_{\mathfrak{l}}^{n-1}$ *is an* (M_0)-*module, provided that A possesses unit element 1*).

For the last reduction to a unitary module, see e.g. [9], Lemmas 1, 2. We also remark that for the submodule $1P^{n-1}$, or AP^{n-1} generally, of P^{n-1} (whence for $1Q_{\mathfrak{l}}^{n-1}$, or $AQ_{\mathfrak{l}}^{n-1}$) the operation (6) of A coincides with the ordinary operation which simply operates, from the left, on the utmost left component.

Remark 2. As will be shown by one of us in a paper to appear in Journ. Polytech. Osaka City Univ., our above argument may best be seen from the exact sequence

$$H^n(A, \mathfrak{m}) = H^1(A, L(P^{n-1}, \mathfrak{m})) \xrightarrow{\varphi} H^1(A, L(Q_{\mathfrak{l}}^{n-1}, \mathfrak{m})) \xrightarrow{\delta} H_{\mathfrak{l}}^{n+1}(A, \mathfrak{m})$$

for an A-double-module $\mathfrak{m}$ with $\mathfrak{m}A = 0$. The mapping φ is merely a restriction. The mapping δ is obtained by extending, first, any cocycle in $Z^1(A, L(Q_{\mathfrak{l}}^{n-1}, \mathfrak{m}))$ to a cochain in $C^1(A, L(P^{n-1}, \mathfrak{m}))$ and then taking the coboundary of the extension. Our argument in the first half of the present section was to make this last construction explicit (and to interprete it module-theoretically). We see without much difficulty that our construction of f in (20) corresponds

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to what we obtain in the alluded construction by taking the extended cochain so as to vanish on $L(A\times\ldots\times A\times A^0, \mathfrak{m})$ (where we consider $L(P^{n-1}, \mathfrak{m})$ as the direct sum of $L(A\times\ldots\times A^0, \mathfrak{m})$ and $L(Q_1^{n-1}, \mathfrak{m})$).

§ 3. Lemmas

Let again A be an algebra over a field Ω.

LEMMA 2. *Suppose that A possesses a unit element. An A-left-module $\mathfrak{m}$ is an (M_0)-module if and only if $1\mathfrak{m}$ is a direct summand of a free A-left-module. In case A is of finite rank over the ground field Ω, $\mathfrak{m}$ is an (M_0)-module if and only if $1\mathfrak{m}$ is a direct sum of (a finite or infinite number of) A-submodules A-isomorphic to principal left-ideals of A generated by primitive idempotent elements.*

Proof. A-left-module annihilated by A is an (M_0)-module, and the direct sum of two A-left-module is an (M_0)-module if and only if each summand is an (M_0)-module; see [9]. It follows then easily that we have only to consider, in order to prove our lemma, an A-left-module satisfying $1\mathfrak{m}=\mathfrak{m}$. Clearly every such A-left-module $\mathfrak{m}$ is A-isomorphic to a residue-module of a free A-left-module, say $\mathfrak{m}_0$. If $\mathfrak{m}$ is an (M_0)-module, $\mathfrak{m}$ is a direct summand of $\mathfrak{m}_0$. On the other hand, a free A-left-module is evidently an (M_0)-module. Hence its direct summand is an (M_0)-module.

The second half of the lemma follows immediatly from the first half in case $\mathfrak{m}$ is of finite rank over Ω. For the case $(\mathfrak{m}:\Omega)=\infty$ we merely refer to [9], since in our applications of the second half of the lemma, which we shall make below, $\mathfrak{m}$ will always be of finite rank over Ω.

LEMMA 3. *Let Λ be an extension field of Ω and n be a natural number. All n-cohomology groups of the algebra A_Λ over Λ vanish, if and only if all n-cohomology groups of A vanish.*

Proof. The lemma could be derived from Hochschild [5], Theorem 1.4 and the fact that, with an A-module $\mathfrak{m}$, the A_Λ-module m_Λ is an (M_0)-module (of A_Λ) if, and only if, the A-left-module $\mathfrak{m}$ is an (M_0)-module (of A); this last fact follows readily from (the first half of) Lemma 2. But our lemma can readily be seen directly as follows, as has been communicated by Hochschild. If $\mathfrak{m}$ is an A-double-module, the natural mapping of $H^n(A, \mathfrak{m})$ into $H^n(A_\Lambda, \mathfrak{m}_\Lambda)$

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is evidently an (into-)isomorphism; in fact we have $H^n(A_\Lambda, \mathfrak{m}_\Lambda) = H^n(A, \mathfrak{m})_\Lambda$. Hence $H^n(A_\Lambda, \mathfrak{m}_\Lambda) = 0$ implies $H^n(A, \mathfrak{m}) = 0$. Conversely, let f be an n-cocycle on A_Λ in an A_Λ-double module $\mathfrak{M}$, and suppose that $H^n(A, \mathfrak{M}) = 0$. Let $\bar{f}$ be the restriction of f to A. Then $\bar{f} = \delta g$, where g is an $(n-1)$-cochain for A in $\mathfrak{M}$. Let g_Λ be the natural extension of g to an $(n-1)$-cochain for A_Λ in $\mathfrak{M}$. Then, clearly, $f = \delta g_\Lambda$. Thus $H^n(A, \mathfrak{M}) = 0$ implies $H^n(A_\Lambda, \mathfrak{M}) = 0$.

§ 4. Separability of A/N

So far we did not assume that A is finite over Ω (except in the latter half of Lemma 2). But we assume now that our algebra A, over Ω, is of finite rank, and possesses a unit element 1. Let N be the radical of A.

Let

$$1 = \sum_{\kappa=1}^{k} \sum_{i=1}^{m_\kappa} e_{\kappa,i} \tag{29}$$

be a decomposition of 1 into mutually orthogonal idempotent elements in A such that the left-ideals $Ae_{\kappa,i}$ and $Ae_{\lambda,j}$ are A-isomorphic (or, equivalently, the right-ideals $e_{\kappa,i}A$ and $e_{\lambda,j}A$ are A-isomorphic) when, and only when, $\kappa = \lambda$. Put

$$e_\kappa = e_{\kappa,1}$$

for the sake of simplicity; for general structural theory of algebras see [7] e.g.

We first assume that A/N is separable (over Ω). By Wedderburn's theorem, there exists a (semisimple separable) subalgebra $\bar{A}$ of A such that

$$A = \bar{A} \oplus N; \tag{31}$$

this is in fact a consequence of the fact the 2-cohomology groups of A/N all vanish. The idempotent elements $e_{\kappa,i}$ may, and shall be taken from $\bar{A}$. Further, taking $\mathfrak{l} = N$ in (10), we consider the A-left-module

$$Q_N^{n-1} = A \times A \times \ldots \times A \times N \qquad (\text{with } n-2 \ A\text{'s}). \tag{32}$$

(with the operation of A as defined in (6)). As N is a two-sided ideal, Q_N^{n-1} may be regarded as an A-right-module with the ordinary operation of A from the right (which simply operates, from the right, on the last component). Indeed Q_N^{n-1} becomes then an A-double-module. It is in particular an A-$\bar{A}$-

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(double-) module. We assert generally

LEMMA 4. *An A-$\bar{A}$-module $\mathfrak{m}$ with $\mathfrak{m}\bar{A}=\mathfrak{m}$ is an (M_0)-module, as A-$\bar{A}$-module, if and only if it is an (M_0)-module as A-left-module.*

The proof depends, naturally, on the separability of $\bar{A}$ and is rather simple. See [8].

Now we consider the case where the irreducible representations of A in Ω are all absolutely irreducible. This is equivalent to that $(e_\kappa Ae_\kappa/e_\kappa Ne_\kappa : \Omega)=1$ for every κ, and further to that the semisimple residue-algebra A/N is a direct sum of matric algebras (of degrees m_κ) over Ω. Moreover, A/N is certainly separable and we have (31).

On assuming, with a certain $n\geqq 2$, that all the n-cohomology groups of A vanish, we have that the A-left-module Q_N^{n-1} is, by Theorem 1, an (M_0)-module, and hence, by Lemma 4, the A-$\bar{A}$-module Q_N^{n-1} is an (M_0)-module. The same is the case with the unitary A-$\bar{A}$-module $1Q_N^{n-1}$. Then, by virtue of Lemma 2, applied to the Kronecker product algebra of A and an inverse-isomorphic image of $\bar{A}$, $1Q_N^{n-1}$ is a direct sum of A-$\bar{A}$-submodules isomorphic to A-$\bar{A}$-modules of the form $Ae_\kappa\times e_\lambda\bar{A}$. Denoting by $t_{\kappa,\lambda}$ the number of components isomorphic to $Ae_\kappa\times e_\lambda\bar{A}$, we write, symbolically,

$$1Q_N^{n-1}\simeq\sum_{\kappa,\lambda}t_{\kappa,\lambda}(Ae_\kappa\times e_\lambda\bar{A}). \tag{33}$$

Then we have, for each pair μ, ν, the $e_\mu Ae_\mu-e_\nu\Omega(=e_\nu\bar{A}e_\nu)$-isomorphism

$$e_\mu Q_N^{n-1}e_\nu\simeq\sum_\kappa t_{\kappa,\nu}(e_\mu Ae_\kappa\times e_\nu\Omega). \tag{34}$$

Hence

$$(e_\mu Q_N^{n-1}e_\nu : \Omega)=\sum_\kappa t_{\kappa\nu}c_{\mu\kappa}, \tag{35}$$

where

$$c_{\mu\kappa}=(e_\mu Ae_\kappa : \Omega) \tag{36}$$

are the Cartan invariants of A (See [1] or [10] for instance).

On the other hand, we consider Q_N^{n-1} as an A-left-module under the ordinary left-operation. Associating $x_1\times x_2\times\ldots\times x_{n-1}\in Q_N^{n-1}$ with the element $x_1(x_2\times\ldots\times x_{n-1})$ of $AQ_N^{n-2}=1Q_N^{n-2}$ with our left-operation (6) ($n-1$ being replaced by $n-2$), we have an A-homomorphic mapping of Q_N^{n-1}, under the

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ordinary left-operation, upon $1Q_N^{n-2}$ (either under the operation (6) or the ordinary operation; they coincide on $1Q_N^{n-2}$). The mapping is also A-right-homomorphic, under the ordinary right-operation, and its kernel is exactly $1Q_N^{n-1}$; cf. [11], Lemma 4.1, or [6], Proposition 2. It induces thus a homomorphism from $e_\mu A\times A\times\ldots\times A\times Ne_\nu$ ($\subseteqq Q_N^{n-1}$) onto $e_\mu Q_N^{n-2}e_\nu$, and the kernel is $e_\mu Q_N^{n-1}e_\nu$. Hence we have

(37) $$(e_\mu Q_N^{n-1}e_\nu : \Omega)=r_\mu(A:\Omega)^{n-3}(s_\nu-m_\nu)-(e_\mu Q_N^{n-2}e_\nu : \Omega),$$

where

(38) $$r_\mu=(e_\mu A:\Omega)=\sum_\lambda c_{\mu\lambda}m_\lambda,$$

(39) $$s_\nu=(Ae_\nu:\Omega)=\sum_\kappa c_{\kappa\nu}m_\kappa;$$

observe that $(Ne_\nu:\Omega)=s_\nu-m_\nu$. Similarly we have

(40) $$(e_\mu Q_N^{n-2}e_\nu : \Omega)=r_\mu(A:\Omega)^{n-4}(s_\nu-m_\nu)-(e_\mu Q_N^{n-3}e_\nu : \Omega),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(e_\mu Q_N^2e_\nu : \Omega)=r_\mu(s_\nu-m_\nu)-(e_\mu Q_N^1e_\nu : \Omega),$$

and

$$(e_\mu Q_N^1e_\nu : \Omega)=(e_\mu Ne_\nu : \Omega)=c_{\mu\nu}-\delta_{\mu\nu}.$$

Hence

(41) $$(e_\mu Q_N^{n-1}e_\nu : \Omega)=r_\mu(s_\nu-m_\nu)\Big(\sum_{i=0}^{n-3}\pm(A:\Omega)^i\Big)+(-1)^n(c_{\mu\nu}-\delta_{\mu\nu}).$$

(In case $n=2$ the vacuous sum on the right-hand side is to mean 0.) These are independent of our cohomology assumption. Combining this (41) with the relation (35), which we derived from the assumption that the n-cohomology groups vanish, we have that their right-hand sides are equal. Putting

(42) $$q_\nu^{(n)}=(s_\nu-m_\nu)\Big(\sum_{i=0}^{n-3}\pm(A:\Omega)^i\Big)$$

and observing (38), we thus have

(43) $$\sum_\lambda c_{\mu\lambda}m_\lambda q_\nu^{(n)}+(-1)^n c_{\mu\lambda}-\sum_\lambda c_{\mu\lambda}t_{\lambda\nu}=(-1)^n\delta_{\mu\nu}.$$

This means that the matrix $(c_{\mu\nu})_{\mu\nu}$ has an inverse with integral coefficients, and we have

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THEOREM 3. *Let A be an algebra, of finite rank, over a field Ω possessing a unit element. Assume that the irreducible representations of A in Ω are all absolutely irreducible, and that all the n-cohomology groups of A vanish, for a certain natural number n. Then the matrix $C=(c_{\kappa\lambda})$ of Cartan invariants of A, with integral coefficients $c_{\kappa\lambda}$, has a determinant $|C| = \pm 1$.*

Next we prove

LEMMA 5. *Let A be an algebra, of finite rank, over a field Ω possessing a unit element* 1. *If the semisimple residue-algebra A/N modulo the radical N is inseparable, over Ω, then the determinant of the matrix of the Cartan invariants of A_Λ is divisible by the characteristic p of Ω, where Λ is the algebraic closure of Ω.*

To prove this lemma, we may, and shall, restrict ourselves to the case where Ω has no separable (algebraic) proper extension, since any separable extension of the ground field preserves the inseparability of the residue-algebra modulo the radical. Thus we assume that the algebraic closure Λ is purely inseparable over Ω. Then, by a theorem of Noether-Köthe, every division algebra (of finite rank) over Ω is commutative. Thus every $e_\kappa Ae_\kappa/e_\kappa Ne_\kappa$ ($\kappa=1, 2, \ldots, k$) is commutative, and is in fact a purely inseparable (proper or improper) extension of Ω. Hence each $(e_\kappa Ae_\kappa/e_\kappa Ne_\kappa)_\Lambda$ is completely primary and each e_κ remains primitive in A_Λ. The composition residue-modules of the left-module $A_\Lambda e_\kappa/N_\Lambda e_\kappa$ of A_Λ are all isomorphic to $A_\Lambda e_\kappa/Me_\kappa$, where M is the radical of A_Λ, and are $(e_\kappa Ae_\kappa/e_\kappa Ne_\kappa : \Omega)$ in number. Here $(e_\kappa Ae_\kappa/e_\kappa Ne_\kappa : \Omega)$ is a power of p, say p^{a_κ}. Hence, for each λ, the number $c_{\kappa\lambda}$ of composition residue-modules of $A_\Lambda e_\lambda = (Ae_\lambda)_\Lambda$ isomorphic to $A_\Lambda e_\kappa/Me_\kappa$ is equal to p^{a_κ} times the number of composition residue-modules of Ae_λ isomorphic to Ae_λ/Ne_λ. So each $c_{\kappa\lambda}$ is a multiple of p^{a_κ}. Now, as A/N is assumed to be inseparable, there exists a suffix κ such that $e_\kappa Ae_\kappa/e_\kappa Ne_\kappa$ is a (purely inseparable) proper extension of Ω. Each $c_{\kappa\lambda}$ with this κ is thus a multiple of p^{a_κ} with $a_\kappa \geqq 1$, and the determinant of the matrix $(c_{\mu\lambda})_{\mu\lambda}$ is certainly divisible by p, which proves our lemma.

Combining Theorem 3 and Lemma 5 we obtain the case with unit element of the following theorem, to which the case without unit element can easily be reduced by Hochschild [4], Theorem 3.

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THEOREM 4. *Let A be an algebra, of finite rank, over a field Ω. If the n-cohomology groups of A all vanish, with a certain natural number n, then the residue-algebra A/N modulo the radical N is separable.*

§ 5. Sufficiency proof

The sufficiency part of our main theorem was proved already in [8]. However, we repeat its proof briefly, for the sake of completeness. Let A be an algebra finite over its ground Ω. Assume that A/N is separable, where N is, as before, the radical of A. Then we have (31), where $\bar{A}$ is a (separable semisimple) subalgebra of A (isomorphic with A/N). Then we have (cf. a somewhat similar relation (9) for $\mathfrak{m}$ with $\mathfrak{m}A=0$).

LEMMA 6. *Let $\mathfrak{m}$ be a double-module of A satisfying*

$$\mathfrak{m}N=0. \tag{44}$$

Let $\bar{L}(Q_N^{n-1}, \mathfrak{m})$ be the module of all $\bar{A}$-right-homomorphic mappings of Q_N^{n-1} into $\mathfrak{m}$ (where we consider Q_N^{n-1} under the ordinary right operation of $\bar{A}$), and consider it as a double-module of A by the operations given in (8) *(where we consider Q_N^{n-1} under the left operation* (6) *of A). Then*

$$H^n(A, \mathfrak{m}) \simeq H^1(A, \bar{L}(Q_N^{n-1}, \mathfrak{m})) \qquad (n \geqq 2). \tag{45}$$

We repeat the proof in [8] briefly. We first prove, that if f is an n-cochain of A in an arbitrary double-module of A, with $n \geqq 1$, such that $\delta f(x_1, x_2, \ldots, x_{n+1})=0$ whenever $x_{n+1} \in \bar{A}$, then there exists an $n-1$ cochain g of A in the same module such that $(f-\delta g)(x_1, x_2, \ldots, x_n)=0$ whenever $x_n \in \bar{A}$. This we may see by a well known argument on considering the right operation of A on Q_N^{n-1} symmetric to (6), using naturally the separability of $\bar{A}$; cf. for a similar, but a little different, situation Hochschild [5], Lemma 10.1 and, more generally, Rose [13] or Shih [14], pp. 6–7.

Let now $n \geqq 2$ and $\bar{C}^n(A, \mathfrak{m})$ be the module of all n-cochains f of A in $\mathfrak{m}$ such that $f(x_1, x_2, \ldots, x_n)=0$ when $x_n \in \bar{A}$, and put $\bar{Z}^n(A, \mathfrak{m}) = Z^n(A, \mathfrak{m}) \cap \bar{C}^n(A, \mathfrak{m})$. The above observation shown that $H^n(A, \mathfrak{m})$ is isomorphic to $\bar{Z}^n(A, \mathfrak{m})/(\delta C^{n-1}(A, \mathfrak{m}) \cap \bar{C}^n(A, \mathfrak{m}))$. For each $f \in \bar{Z}^n(A, \mathfrak{m})$ we put

$$f^{\blacktriangle}(x_1)(x_2 \times \ldots \times x_{n-1} \times y) = f(x_1, \ldots, x_{n-1}, y) \qquad (y \in N). \tag{46}$$

Then we see, by the assumed property of $\mathfrak{m}$, that $f^{\blacktriangle} \in Z^1(A, \bar{L}(Q_N^{n-1}, \mathfrak{m}))$. If

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$f \in \delta C^{n-1}(A, \mathfrak{m}) \cap \bar{C}^n(A, \mathfrak{m})$ and $f = \delta g'$ ($g' \in C^{n-1}(A, \mathfrak{m})$), then our above consideration, with n replaced by $n-1$, shows the existence of $h \in C^{n-2}(A, \mathfrak{m})$ such that $g' - \delta h \in \bar{C}^{n-1}(A, \mathfrak{m})$. Setting $g = g' - \delta h$ and $g^{\blacktriangle}(x_1 \times \ldots \times x_{n-2} \times y) = g(x_1, \ldots, x_{n-2}, y)$ $(y \in N)$, we have $g^{\blacktriangle} \in \bar{L}(Q_N^{n-1}, \mathfrak{m})$. We have further $f^{\blacktriangle} = \delta g^{\blacktriangle}$.

Conversely, if $F \in Z^1(A, \bar{L}(Q_N^{n-1}, \mathfrak{m}))$, we have $F = f^{\blacktriangle}$ where f is an element of $\bar{C}^n(A, \mathfrak{m})$, in fact of $\bar{Z}^n(A, \mathfrak{m})$, with $f(x_1, \ldots, x_{n-1}, y) = F(x_1)(x_2 \times \ldots \times x_{n-1} \times y)$ $(y \in N)$. If here $F = \delta G$ with $G \in \bar{L}(Q_N^{n-1}, \mathfrak{m})$, then $f = \delta g$ where g is an element of $\bar{C}^{n-1}(A, \mathfrak{m})$ with $g(x_1, \ldots, x_{n-2}, y) = G(x_1 \times \ldots \times x_{n-2} \times y)$ $(y \in N)$. These assure the asserted isomorphism (45).

Now, on the other hand, the right-hand side of (45) is 0, when and only when every enlargement of the A-$\bar{A}$-module $\mathfrak{m}$ by the A-$\bar{A}$-module Q_N^{n-1} splits; see e.g. [11], Lemma 3.1, and observe that, every enlargement is $\bar{A}$-inessential.

Further, if the n-cohomology group of A in every double-module satisfying (44) vanishes, then the n-cohomology group of A in any double-module vanishes; this may easily be seen by considering a normal series of a given double-module in which every residue-module satisfies (44) and applying a well-known argument of considering residue-modules. Similarly, if the enlargements by Q_N^{n-1} of every module satisfying (44) split, so do those of any (A-$\bar{A}$-)module.

Combining these facts we see that (under the assumption of the separability of A/N, among others) the n-cohomology groups of A are all 0 if (and only if) the A-$\bar{A}$-module Q_N^{n-1} is an (M_0)-module. Here the (M_0)-module property of Q_N^{n-1} as A-left-module (under (6)) suffices, as we have seen in Lemma 6. So we have the latter half of our following main theorem whose former half has been proved in §§3, 4.

Main Theorem. *Let $n \geqq 2$. Let A be an algebra, finite over a field Ω, possessing a unit element 1, and let N be its radical. If all the n-cohomology groups of A vanish, then*

α) A/N is separable, and

β) for every left-ideal $\mathfrak{l}$ of A the A-left-module $1Q_{\mathfrak{l}}^{n-1}$ is an (M_0)-module, where $Q_{\mathfrak{l}}^{n-1}$ is given in (10) *and the left-operation of A on $Q_{\mathfrak{l}}^{n-1}$ is that given in* (6).

Conversely, if α) is the case, and if

β_1) the A-left-module $1Q_N^{n-1}$ is an (M_0)-module, then all the n-cohomology

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groups of A vanish.

(The problem of algebras without unit element, whose n-cohomology groups all vanish, may easily be reduced to that of algebras with unit element; see [4], Theorem 3.)

As an immediate consequence of (the first half of) our theorem, we mention the following corollary which generalizes, as well as clarifies the back ground of, Hochschild [5], Theorem 11.2.

COROLLARY. *Let A be a non-semisimple quasi-Frobenius algebra over a field Ω. For every natural number n there exists an A-double module $\mathfrak{m}$ with $H^n(A, \mathfrak{m}) \neq 0$.*

Proof. Suppose, contrary to the assertion, that for some n all the n-cohomology groups of A are 0; we may assume naturally that $n \geqq 2$. Then the A-left-module $1Q_N^{n-1}$ is an (M_0)-module. Hence it is an (M_u)-module; the definition of (M_u)-modules is dual to that of (M_0)-modules, and quasi-Frobenius algebras are characterized as algebras (with unit element) whose (M_0)-left-modules are always (M_u)-left-modules and conversely ([9]). Consider the module Q_N^{n-1} as an A-left-module under the ordinary left-operation of A. It has our (M_u)-module $1Q_N^{n-1}$ as an A-submodule. Hence we have $Q_N^{n-1} = \mathfrak{Q} \oplus 1Q_N^{n-1}$ with an A-submodule $\mathfrak{Q}$ of Q_N^{n-1}. $\mathfrak{Q}$ is A-isomorphic to $Q_N^{n-1}/1Q_N^{n-1}$, whence to the module $1Q_N^{n-2}$. On the other hand, as a direct summand in an (M_0)-module $Q_N^{n-1}(\simeq A^{(Q_N^{n-2}:\Omega)})$, $\mathfrak{Q}$ is an (M_0)-module. Thus the A-left-module $1Q_N^{n-2}$ is an (M_0)-module. Repeating this argument $n-2$ times, we see that the A-left-module N is an (M_0)-module, whence an (M_u)-module, which certainly could not be the case, unless N were 0.

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Osaka University

Osaka City University

Nagoya University

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