BULL. AUSTRAL. MATH. SOC. 20D10, 20N05
VOL. 49 (1994) [121–128]

ON CONNECTED TRANSVERSALS TO ABELIAN SUBGROUPS

MARKKU NIEMENMAA AND TOMAS KEPKA

In this paper we investigate the situation where a group G has an abelian subgroup H with connected transversals. We show that if H is finite then G is solvable. We also investigate some special cases where the structure of H is very close to the structure of a cyclic group. Finally we apply our results to loop theory and we show that if the inner mapping group of a finite loop Q is abelian then Q is centrally nilpotent.

1. INTRODUCTION

If G is a group, $H \leqslant G$ and A and B are two left transversals to H in G, then we say that A and B are H-connected if $a^{-1}b^{-1}ab \in H$ for every $a \in A$ and $b \in B$. This concept was introduced by the authors in [10] where it was used to characterise multiplication groups of loops. Naturally, connected transversals are interesting in group theory in their own right and the authors continued their investigations in [11] where they managed to prove the following two results: (1) If G is a finite group which has an abelian subgroup H such that there exist H-connected transversals A and B, then G is solvable. (2) If, in addition, $G = \langle A, B\rangle$, H is of prime power order and the core of H in G is trivial, then $Z(G) \neq 1$. In the present paper (see Theorem 4.1) we are able to prove (1) also in the case that G is infinite and H is finite (the argument of [11] based on Sylow theorems has to be replaced by other arguments and the use of Zorn's lemma) and we prove (2) without the assumption that H is of prime power order (see Proposition 6.3 and the remark after its proof). We also consider two special cases where $H \cong {C_p}^{(2)}$ and $H \cong C_p \times {C_q}^{(2)}$ (here p and q are two different prime numbers). Finally, we prove several consequences of the above results in loop theory. Perhaps the most interesting is the following result: If the inner mapping group of a finite loop Q is abelian, then Q is centrally nilpotent.

2. PRELIMINARIES

Connected transversals are defined as in the first section. The core of H in G is the largest normal subgroup of G contained in H and we denote it by $L_G(H)$. If p is a prime number then we write C_p for the cyclic group of order p and ${C_p}^{(2)} = C_p \times C_p$. In our proofs we need the following results.

Received 22nd February, 1993

LEMMA 2.1. *If A and B are H-connected transversals in G and $L_G(H) = 1$, then $1 \in A \cap B$ and $N_G(H) = H \times Z(G)$.*

LEMMA 2.2. *Let $H \leqslant G$, A and B be H-connected transversals in G, $C \subseteq A \cup B$ and $K = \langle H, C \rangle$. Then $C \subseteq L_G(K)$.*

The proofs of results can be found in [**10**, p.113–114].

LEMMA 2.3. *If $H \leqslant G$, A and B are H-connected transversals and $L_G(H) = 1$, then $Z(G) \subseteq A \cap B$. If N is a normal subgroup of G and $N \subseteq A \cap B$, then $N \leqslant Z(\langle A, B \rangle)$.*

THEOREM 2.4. *Let H be a cyclic subgroup of a group G. Then $G' \leqslant H$ if and only if there exists a pair A, B of H-connected transversals in G such that $G = \langle A, B \rangle$.*

For the proofs, see [**12**, Lemma 1.4 and Theorem 2.2].

LEMMA 2.5. *If H is a cyclic subgroup of a group G and there exist H-connected transversals A and B in G, then $G'' \leqslant L_G(H)$.*

PROOF: This follows directly from [**12**, Corollary 2.3]. ∎

In group theory our notation is standard. In Section 6 we give some applications on loop theory. For the concepts and basic results in loop theory the reader is advised to consult [**1**, **7**, **9**, **10**, **11**].

3. Connected transversals

In this section we prove several lemmas which will be used later in Section.

LEMMA 3.1. *Let K be a subgroup of G and let A and B be subsets of G such that $1 \in A \cap B$, $AB \subseteq BK$, $A^{-1}B \subseteq BK$, $BA \subseteq AK$ and $B^{-1}A \subseteq AK$. Then $\langle A, B \rangle \subseteq AK = BK$.*

PROOF: Now $A \subseteq BK$ and thus $AK \subseteq BK^2 = BK$. Likewise, $BK \subseteq AK^2 = AK$, hence $AK = BK = E$. Now denote $F = A \cup B \cup A^{-1} \cup B^{-1}$. Clearly, $A^{-1} \subseteq E$ and $B^{-1} \subseteq E$. Now $AA \subseteq AE = ABK \subseteq BK^2 = BK = E$ and in a similar way $A^{-1}A \subseteq E$. Thus $FA \subseteq E$ and now $F^2 = FF \subseteq FE = FAK \subseteq EK = E$. By induction, it is clear that $F^n \subseteq E$ and thus $\langle A, B \rangle \subseteq E$. ∎

From now on in this section we assume that H is a subgroup of G and there exist H-connected transversals A and B in G. We write $E = \langle A, B \rangle$ and if $H \leqslant K$ we denote $C = A \cap K$, $D = B \cap K$, $F = \langle C, D \rangle$, $V = H \cap F$ and $W = H \cap L_G(K)$. Naturally, C and D are H-connected in K and V -connected transversals in F. Moreover, $F \leqslant L_G(K)$ by Lemma 2.2 and $K = HL_G(K)$. Finally, $K/L_G(H) \cong H/W$. In the following lemma we prove some technical results.

LEMMA 3.2. *(1) If $V = 1$, then $F = C = D \subseteq A \cap B$. (2) If $W = 1$, then $F = C = D = L_G(K) \leqslant Z(E)$. (3) If $W = 1$ and $H < K$, then $Z(E) > 1$.*

PROOF: (1) If $a, b \in C$, then there exist elements $c, d \in C$ and $e \in D$ such that $c^{-1}ab$, $d^{-1}a^{-1}$ and $e^{-1}a$ are elements of H. Clearly, these three elements belong to $V = 1$ and thus $ab = c$, $a^{-1} = d$ and $a = e$. We have shown that $C = D$ is a subgroup , hence $F = C = D \subseteq A \cap B$. (2) Now $W = 1$ implies $V = 1$ and by (1) $F = C = D$. Clearly, $D = L_G(K)$ and $[D, A] \leqslant H \cap D = 1$ and $[D, B] \leqslant H \cap D = 1$, hence $D \leqslant Z(E)$. (3) This follows directly from (2). ∎

LEMMA 3.3. *Let $G = \langle A, B\rangle$, $L_G(H) = 1$ and $K = N_G(H)$. If $H \cap L_G(K) = 1$ then $Z(G/Z(G)) = 1$.*

PROOF: By Lemma 2.1, $K = N_G(H) = HZ(G)$ and since $H \cap L_G(K) = 1$, it follows that $L_G(K) = Z(G)$. By Lemma 3.2, $A \cap K = B \cap K = Z(G)$. Now we write $\overline{G} = G/Z(G)$. Since $L_{\overline{G}}(\overline{H})$ is trivial, it follows from Lemma 2.3 that $\overline{A} \cap \overline{B}$ contains $Z(\overline{G})$. Then denote $E = N_G(K)$. Since $\overline{E} = \overline{H}Z(\overline{G})$, by Lemma 2.1, we conclude that $\overline{A} \cap \overline{E} = \overline{B} \cap \overline{E} = Z(\overline{G})$. But now $\overline{A} \cap \overline{E} = \overline{A \cap E}$, hence $A \cap E = B \cap E$ is a normal subgroup of G and by Lemma 2.3, $A \cap E \leqslant Z(\langle A, B\rangle) = Z(G)$. Thus $Z(\overline{G})$ is trivial and the proof is complete. ∎

LEMMA 3.4. *Let H be an abelian maximal subgroup of G and assume that H is not normal in G and $1 \in A \cap B$. Then $AZ(G) = BZ(G)$ is a subgroup of G and $G \neq \langle A, B\rangle$.*

PROOF: It is easy to see that $N_G(H) = H$ and $Z(G) \leqslant H$. If $a \in A$, then $b^{-1}a \in H$ for some $b \in B$. Since $a^{-1}b^{-1}ab \in H$ it follows that $b^{-1}a \in H \cap aHb^{-1} = H \cap bHb^{-1} = T$. If $N_G(T) = H$, then $b \in H$ and $a = b = 1$. If $N_G(T) = G$, then $C_G(T) = G$, hence $T \leqslant Z(G)$. Thus $a \in BZ(G)$ and we have shown that $A \subseteq BZ(G)$. In a similar way, $B \subseteq AZ(G)$.

If $a \in A$ and $b \in B$, then there exists $c \in B$ such that $c^{-1}ab \in H$. Since $a^{-1}b^{-1}ab \in H$ and $a^{-1}c^{-1}ac \in H$, it follows that $c^{-1}abaH = c^{-1}aabH = c^{-1}acH = aH$. Thus $a^{-1}c^{-1}aba \in H$, hence $c^{-1}ab \in H \cap aHa^{-1}$. As in the first part of the proof we conclude that $c^{-1}ab \in Z(G)$. This means that $AB \subseteq BZ(G)$ and in the same way $BA \subseteq AZ(G)$.

If again $a \in A$ and $b \in B$, then there exists $c \in B$ such that $c^{-1}a^{-1}b \in H$. Now $c^{-1}a^{-1}baH = c^{-1}a^{-1}abH = c^{-1}acH = aH$, hence $a^{-1}c^{-1}a^{-1}ba \in H$. It follows that $c^{-1}a^{-1}b \in H \cap aHa^{-1}$. As before, $c^{-1}a^{-1}b \in Z(G)$ and thus $A^{-1}B \subseteq BZ(G)$. Of course, we also have $B^{-1}A \subseteq AZ(G)$. By Lemma 3.1, $\langle A, B\rangle \subseteq AZ(G) = BZ(G)$ and now it is easy to see that $AZ(G)$ is a subgroup of G. If $G = \langle A, B\rangle$, then $G = AZ(G)$ which means that $H = Z(G)$. Since H is not normal in G, we conclude that $\langle A, B\rangle$ is a proper subgroup of G. ∎

https://doi.org/10.1017/S0004972700016166 Published online by Cambridge University Press

Lemma 3.5. *Assume that $G = \langle A, B\rangle$, H is an abelian subgroup of G and $1 \in A \cap B$. If $R = \bigcap\{K : H < K \leqslant G\}$ and $H < R$, then H is normal in R.*

Proof: If H is not normal in R, then $N_G(H) = H$ and $Z(R) \leqslant H$. If $T < H$ and $N_G(T) > H$, then $T \leqslant Z(R)$. Now we can proceed as in the proof of Lemma 3.4 and we can show that $G = \langle A, B\rangle = AZ(R) = BZ(R)$. Since $Z(R) \leqslant H$ and A, B are transversals to H in G we conclude that $Z(R) = H$. Thus H is normal in R, a contradiction. ∎

4. Main results

Theorem 4.1. *Let H be a finite abelian subgroup of a group G such that there exist H-connected transversals A and B in G. Then G is solvable.*

Proof: We show that there exists a mapping $t : N \to N$ such that $G^{(t(n))} = 1$, where $n = |H|$. From Lemma 2.5, it follows that we can put $t(1) = 1$ and $t(2) = t(3) = 3$. For $n \geqslant 4$ our proof is by induction. We first write $m = \max\{t(k) : 1 \leqslant k < n\}$. If $L_G(H) \neq 1$, then $H/L_G(H)$ and $G/L_G(H)$ satisfy the assumptions of our theorem and thus $G^{(m+1)} = 1$. Thus we may assume that $L_G(H) = 1$. This means that $1 \in A \cap B$. Now we divide the proof into three parts.

(1) Assume that $Z(G) = 1$ and $H \cap L_G(K) > 1$ whenever $H < K$. By induction, $G^{(m)} \leqslant L_G(K)$. If we write $R = \bigcap\{K : H < K \leqslant G\}$, then $G^{(m)} \leqslant L_G(R)$. If $R = H$, then $G^{(m)} \leqslant L_G(H) = 1$. Thus we assume that $H < R$. Since $Z(G) = 1$, we have $N_G(H) = H$ by Lemma 2.1. Thus H is not normal in R and $Z(R) < H$. We write $C = A \cap R$ and $D = B \cap R$. Then C and D are H-connected transversals in R. By Lemma 3.4, $CZ(R) = DZ(R)$ is a subgroup of R. It follows that $[C, D] \leqslant CZ(R) \cap H = Z(R)$. Clearly, $Z(R) \leqslant L_R(H)$ and if we write $\overline{R} = R/L_R(H)$, then $\overline{R} = \overline{CH}$, where $\overline{C} = \overline{D}$ is an abelian subgroup of $\overline{R}$. By the theorem of Ito [6, p.674–675], $\overline{R}'' = 1$, hence $R^{(3)} = 1$. Since $G^{(m)} \leqslant R$, we have $G^{(m+3)} = 1$.

(2) Now assume that $Z(G) > 1$ and $H \cap L_G(K) > 1$, where $K = N_G(H) = HZ(G)$. If we write $\overline{G} = G/L_G(K)$, then $\overline{G}^{(m)} = 1$, hence $G^{(m)} \leqslant L_G(K) \leqslant K$ and $G^{(m+1)} = 1$.

(3) Now we write $E = \langle A, B\rangle$ and $\mathcal{A} = \{P \leqslant G : H \leqslant P,\ P \cap A = P \cap B$ is a subgroup of $Z(E)\}$. Now $H \in \mathcal{A}$, $\mathcal{A}$ is ordered by inclusion and clearly we can apply Zorn's lemma. Let F be a maximal element of $\mathcal{A}$. Then $C = F \cap A = F \cap B$ is a subgroup of $Z(E)$, C is naturally an abelian group and $F = CH$, hence $F'' = 1$ by Ito's theorem. If $L_G(F) \cap H > 1$, then $G^{(m)} \leqslant L_G(F) \leqslant F$ and $G^{(m+2)} = 1$. Thus we can assume that $L_G(F) \cap H = 1$. Now we write $\overline{G} = G/L_G(F)$ and let K be a subgroup of G such that $F \leqslant K$ and $L_{\overline{G}}(\overline{K}) \cap \overline{H}$ is trivial. Now $\overline{H} = \overline{F} \cong H$ and since $L_G(F) \leqslant L_G(K)$, it follows that $L_{\overline{G}}(\overline{K}) = \overline{L_G(K)}$ and $L_G(F) \leqslant L_G(K) \cap F$. On the

other hand, from the fact that $L_{\overline{G}}(\overline{K}) \cap \overline{H}$ is trivial it follows that $L_G(F) = L_G(K) \cap F$ and thus $L_G(K) \cap H = L_G(F) \cap H = 1$. By Lemma 3.2, $K \cap A = K \cap B = L_G(K) \leqslant Z(E)$ and we have shown that $K \in \mathcal{A}$, hence $K = F$. Thus it is clear that $\overline{H} \cap L_{\overline{G}}(\overline{V})$ is not trivial whenever $\overline{H} < \overline{V} \leqslant \overline{G}$. If $Z(\overline{G})$ is trivial, then $\overline{G}^{(m+3)}$ is trivial by (1) of this proof, hence $G^{(m+5)} = 1$. If $Z(\overline{G})$ is not trivial, then $\overline{G}^{(m+1)}$ is trivial by (2) and thus $G^{(m+3)} = 1$. From the three parts of our proof it now follows that we can put $t(n) = m + 5$. The proof is complete. ∎

We put an end to this section by considering the case where H is elementary abelian of order p^2.

Lemma 4.2. *Let $G = \langle A, B\rangle$ and $H \leqslant G$ such that $H \cong C_p{}^{(2)}$, then $G' \leqslant N_G(H)$.*

Proof: We divide the proof into three parts. (1) If $L_G(H) > 1$, then $H/L_G(H)$ is cyclic and $G' \leqslant H$ by Theorem 2.4. Thus we may assume that $L_G(H) = 1$ (then $1 \in A \cap B$). (2) Assume now that $Z(G) = 1$. If $H < K$, then $H \cap L_G(K) > 1$ by Lemma 3.2 and $HL_G(K)/L_G(K) = K/L_G(K)$ is cyclic , hence $G' \leqslant K$. Thus $G' \leqslant R = \bigcap\{K : K > H\}$ and naturally R is normal in G. Now $N_G(H) = H$ and thus $H < R$ and H is not normal in R. This is a contradiction to Lemma 3.5 , hence we may assume that $Z(G) > 1$. (3) Now consider $T = N_G(H) = HZ(G)$. If $H \cap L_G(T) = 1$, then $L_G(T) = Z(G)$ by Lemma 3.2. Thus the core of $T/Z(G)$ in $G/Z(G)$ is trivial and by (2) of this proof $Z(G/Z(G)) > 1$. On the other hand, this is not possible because of Lemma 3.3. Thus $H \cap L_G(T) > 1$ and then $HL_G(T)/L_G(T)$ is cyclic and again by Theorem 2.4, $G' \leqslant HL_G(T) = T = N_G(H)$. ∎

5. A special case

In this section we consider the situation where G is a finite group, H is a subgroup of G such that $H \cong C_p \times C_q{}^{(2)}$ and there exist H-connected transversals A and B in G (here p and q are two different prime numbers).

Theorem 5.1. *If G is a finite group, $G = \langle A, B\rangle$ and $H \cong C_p \times C_q{}^{(2)}$, then $L_G(H) \neq 1$.*

Proof: Assume by induction that G is a counterexample of smallest possible order. Thus $L_G(H) = 1$. We first show that $Z(G) \neq 1$. If $Z(G) = 1$, then $N_G(H) = H$ by Lemma 2.1. If $H < K$, then $H \cap L_G(K) = R \neq 1$ by Lemma 3.2. If H/R is cyclic, then $G' \leqslant HL_G(K) = K$ by Theorem 2.4. If H/R is not cyclic, then $H/R \cong C_q{}^{(2)}$ and $G' \leqslant N_G(K)$ by Lemma 4.2. Thus $G' \leqslant \bigcap N_G(K)$, where K ranges over all subgroups of G which properly contain H. If $T = \bigcap K$, then $G' \leqslant N_G(T)$. Thus $N_G(T)$ is normal in G. If $T = H$, then $N_G(T) = N_G(H) = H$ and H is normal in G, a contradiction. Thus $H < T$ and we now have a contradiction to Lemma 3.5.

Thus we may assume that $Z(G) > 1$. Now let $x \in Z(G)$ such that $x \neq 1$ and $|x| = r$, where r is a prime number and consider the group $K = H\langle x\rangle$. If $H \cap L_G(K) = 1$, then $L_G(K) \leqslant Z(G)$ by Lemma 3.2 and, in fact, $L_G(K) = \langle x\rangle$. By induction, the core of $H\langle x\rangle/\langle x\rangle$ in $G/\langle x\rangle$ is nontrivial. Hence we have a normal subgroup D of G such that $\langle x\rangle < D \leqslant H\langle x\rangle$. But now $D \leqslant L_G(K)$, a contradiction. Thus $H \cap L_G(K) > 1$ and the order of $L_G(K)$ is one of the following: pq^2r, pqr, q^2r, pr or qr. If $r \neq p$ and $r \neq q$ then we immediately have the characteristic Sylow p-subgroup (or the characteristic Sylow q-subgroup) in $L_G(K)$ and since $L_G(K)$ is normal in G, we conclude that $L_G(H) > 1$, which is not possible. Thus $r = p$ or $r = q$. If $r = q$, then the order of $L_G(K)$ is one of the following: pq^3, pq^2, pq, q^2 or q^3. In the three first cases we can proceed by using the characteristic Sylow p-subgroup of $L_G(K)$. If $|L_G(K)| = q^2$ or $= q^3$, then $HL_G(K)/L_G(K)$ is cyclic and by Theorem 2.4, $G' \leqslant HL_G(K) = K$. Thus K is normal in G and the Sylow p-subgroup of K is normal in G, hence $L_G(H) > 1$, a contradiction. Thus we may finally assume that $r = p$ and $Z(G)$ is a p-group. Now the order of $L_G(K)$ is one of the following: p^2q^2, p^2q, pq^2, pq or p^2. In the four first cases we have the characteristic Sylow q-subgroup in $L_G(K)$ and this leads to a contradiction as before. If $|L_G(K)| = p^2$, then we write $\overline{G} = G/L_G(K)$. Now $\overline{H} = HL_G(K)/L_G(K) = K/L_G(K) \cong {C_q}^{(2)}$. If $L_{\overline{G}}(\overline{H})$ is not trivial, then we can proceed as in the first part of the proof of Lemma 4.2 and we conclude that $G' \leqslant HL_G(K) = K$. Now K is normal in G and K has the characteristic Sylow q-subgroup, hence $L_G(H) > 1$. Thus we can assume that $L_{\overline{G}}(\overline{H})$ is trivial. Then by Lemma 2.1, $N_{\overline{G}}(\overline{H}) = \overline{H}Z(\overline{G})$ and $\overline{H} \cap Z(\overline{G})$ is trivial. By Lemma 4.2, $\overline{G}' \leqslant N_{\overline{G}}(\overline{H})$, hence $G' \leqslant T = N_G(HL_G(K)) = N_G(K)$ and thus T is normal in G. Clearly, $N_{\overline{G}}(\overline{H}) = \overline{T}$, hence $\overline{H}Z(\overline{G}) = \overline{T}$. Now we write $N/L_G(K) = Z(\overline{G})$ and thus $KN = T$. Now by Lemma 2.3, $Z(\overline{G}) \subseteq \overline{A} \cap \overline{B}$ and then $\overline{A} \cap \overline{T} = \overline{B} \cap \overline{T} = Z(\overline{G})$. It follows that $(A \cap T)L_G(K) = N$. Let Q be the subgroup of order q^2 in H. Now Q is characteristic in K, hence Q is normal in T. Thus $T = Q \times N$ and $Q \leqslant Z(T)$. On the other hand, $Z(T) = C_T(T) \leqslant C_T(H) \leqslant N_T(H) \leqslant N_G(H) = HZ(G)$. Since T is normal in G, we know that $Z(T)$ is normal in G and thus $Q \leqslant Z(T) \leqslant L_G(HZ(G))$. Since $Z(G)$ is a p-group, it follows that $L_G(HZ(G))$ has a characteristic Sylow q-subgroup Q and then $Q \leqslant L_G(H)$. This is our final contradiction and the proof is complete. ∎

6. Application to loop theory

We say that a groupoid Q is a loop if Q has unique division and a neutral element (thus loops are nonassociative versions of groups). The mappings $L_a(x) = ax$ and $R_a(x) = xa$ define two permutations on Q for every $a \in Q$ and the permutation group $M(Q) = \langle L_a, R_a : a \in Q\rangle$ is called the multiplication group of Q. By $I(Q)$ we denote the stabilizer of the neutral element and $I(Q)$ is called the inner mapping group of Q. It is rather easy to see that the core of $I(Q)$ in $M(Q)$ is trivial and $A = \{L_a : a \in Q\}$ and $B = \{R_a : a \in Q\}$ are $I(Q)$ - connected transversals in $M(Q)$. The concept of the multiplication group of a loop was introduced by Bruck [1] and he used this concept to investigate the structure of loops. Glauberman [3] and [4] studied very thoroughly loops of odd order and Conway [2], Griess [5] and Liebeck [8] have investigated the connection between certain finite simple groups and finite Moufang loops. In [**10**, Theorem 4.1] Kepka and Niemenmaa gave the following characterisation of multiplication groups of loops.

THEOREM 6.1. *A group G is isomorphic to the multiplication group of a loop if and only if there exist a subgroup H satisfying $L_G(H) = 1$ and H-connected transversals A and B satisfying $G = \langle A, B \rangle$.*

By combining theorems 5.1 and 6.1 we get the following

COROLLARY 6.2. *If Q is a finite loop then it is not possible that $I(Q) \cong C_p \times {C_q}^{(2)}$, where p and q are two different prime numbers.*

REMARK. In [10] and [12] it was shown that $I(Q)$ can not be a nontrivial cyclic group. On the other hand, $I(Q) \cong C_2 \times C_2$ is possible as was shown in [**10**, p.120].

Our next application deals with the central nilpotency of a finite loop Q. For this concept and related results, we advise the reader to consult [**1**, **9**, **11**]. We first introduce the following purely group theoretical result.

PROPOSITION 6.3. *Let H be an abelian subgroup of a finite group G such that there exist H-connected transversals A and B in G and assume that $G = \langle A, B \rangle$. Then H is subnormal in G.*

PROOF: Assume that G is a counterexample of smallest order. If $L_G(H) > 1$, then $H/L_G(H)$ is subnormal in $G/L_G(H)$, hence H is subnormal in G. Thus we may assume that $L_G(H) = 1$ (this means that $1 \in A \cap B$.) Then assume that $Z(G) = 1$ and let $H < K \leqslant G$. By Lemma 3.2, $H \cap L_G(K) > 1$. Thus $HL_G(K)/L_G(K)$ is subnormal in $G/L_G(K)$ and since $K = HL_G(K)$, we conclude that K is subnormal in G. Clearly, $R = \bigcap\{K : H < K \leqslant G\}$ is subnormal in G. If $R = H$, then H is subnormal in G. If $H < R$, then by Lemma 3.5, H is normal in R. Thus we may assume that $Z(G) > 1$. Now $HZ(G)/Z(G)$ is subnormal in $G/Z(G)$, hence $HZ(G)$ is subnormal in G. But then H is subnormal in G and we are ready. ∎

From the preceding proposition it follows that if G is a finite group, $H < G$ is abelian, $L_G(H) = 1$ and there exist H-connected transversals A and B such that $G = \langle A, B \rangle$ then $Z(G) \neq 1$. This improves the result of Theorem 3.4 in [11] and now proceeding as in Section 4 in [11] we can prove the following interesting result in loop theory.

https://doi.org/10.1017/S0004972700016166 Published online by Cambridge University Press

COROLLARY 6.4. *Let Q be a finite loop such that the inner mapping group $I(Q)$ is abelian. Then Q is centrally nilpotent.*

REFERENCES

[1] R.H. Bruck, 'Contributions to the theory of loops', *Trans. Amer. Math. Soc.* **60** (1946), 245–354.

[2] J.H. Conway, 'A simple construction for the Fischer-Griess monster group', *Invent. Math.* **79** (1985), 513–540.

[3] G. Glauberman, 'On loops of odd order', *J. Algebra* **1** (1964), 374–396.

[4] G. Glauberman, 'On loops of odd order II', *J. Algebra* **8** (1968), 393–414.

[5] R.L. Griess, 'A Moufang loop, the exceptional Jordan algebra and a cubic form in 27 variables', *J. Algebra* **131** (1990), 281–293.

[6] B. Huppert, *Endliche Gruppen I* (Springer-Verlag, Berlin, Heidelberg, New York, 1967).

[7] T. Kepka and M. Niemenmaa, 'On conjugacy classes in finite loops', *Bull. Austral. Math. Soc.* **38** (1988), 171–176.

[8] M.W. Liebeck, 'The classification of finite simple Moufang loops', *Math. Proc. Camb. Phil. Soc.* **102** (1987), 33–47.

[9] M. Niemenmaa, 'Problems in loop theory for group theorists', in *Groups St.Andrews 1989 Vol.2*, London Math. Soc. Lecture Notes Series **160** (Cambridge University Press, Cambridge, 1991), pp. 396–399.

[10] M. Niemenmaa and T. Kepka, 'On multiplication groups of loops', *J. Algebra* **135** (1990), 112–122.

[11] M. Niemenmaa and T. Kepka, 'On connected transversals to abelian subgroups in finite groups', *Bull. London Math. Soc.* **24** (1992), 343–346.

[12] M. Niemenmaa and T. Kepka, 'On loops with cyclic inner mapping groups', *Arch. Math.* (to appear).

Department of Mathematics
University of Oulu
SF-90570 Oulu
Finland

MFF - UK
Prague
Czechoslovakia