

## PARTITIONS OF NATURAL NUMBERS AND THEIR WEIGHTED REPRESENTATION FUNCTIONS

SHUANG-SHUANG LI , YU-QING SHAN  and XIAO-HUI YAN  

(Received 11 August 2023; accepted 7 September 2023; first published online 27 October 2023)

### Abstract

For any positive integers  $k_1, k_2$  and any set  $A \subseteq \mathbb{N}$ , let  $R_{k_1, k_2}(A, n)$  be the number of solutions of the equation  $n = k_1 a_1 + k_2 a_2$  with  $a_1, a_2 \in A$ . Let  $g$  be a fixed integer. We prove that if  $k_1$  and  $k_2$  are two integers with  $2 \leq k_1 < k_2$  and  $(k_1, k_2) = 1$ , then there does not exist any set  $A \subseteq \mathbb{N}$  such that  $R_{k_1, k_2}(A, n) - R_{k_1, k_2}(\mathbb{N} \setminus A, n) = g$  for all sufficiently large integers  $n$ , and if  $1 = k_1 < k_2$ , then there exists a set  $A$  such that  $R_{k_1, k_2}(A, n) - R_{k_1, k_2}(\mathbb{N} \setminus A, n) = 1$  for all positive integers  $n$ .

2020 *Mathematics subject classification*: primary 11B34.

*Keywords and phrases*: representation function, partition, Sárközy problem.

### 1. Introduction

Let  $\mathbb{N}$  be the set of all nonnegative integers. For a set  $A \subseteq \mathbb{N}$ , let  $R_1(A, n)$ ,  $R_2(A, n)$  and  $R_3(A, n)$  denote the number of solutions of  $a_1 + a_2 = n, a_1, a_2 \in A$ ;  $a_1 + a_2 = n, a_1, a_2 \in A, a_1 < a_2$  and  $a_1 + a_2 = n, a_1, a_2 \in A, a_1 \leq a_2$ , respectively. For  $i = 1, 2, 3$ , Sárközy asked whether there exist two sets  $A$  and  $B$  with  $|(A \cup B) \setminus (A \cap B)| = +\infty$  such that  $R_i(A, n) = R_i(B, n)$  for all sufficiently large integers  $n$ . We call this problem the Sárközy problem. In 2002, Dombi [2] proved that the answer is negative for  $i = 1$  and positive for  $i = 2$ . For  $i = 3$ , Chen and Wang [1] proved that the answer is also positive. In 2004, Lev [3] provided a new proof by using generating functions. Later, Sándor [5] determined the partitions of  $\mathbb{N}$  into two sets with the same representation functions by using generating functions. In 2008, Tang [6] provided a simple proof by using the characteristic function.

In 2012, Yang and Chen [7] first considered the Sárközy problem with weighted representation functions. For any positive integers  $k_1, \dots, k_t$  and any set  $A \subseteq \mathbb{N}$ , let  $R_{k_1, \dots, k_t}(A, n)$  be the number of solutions of the equation  $n = k_1 a_1 + \dots + k_t a_t$  with  $a_1, \dots, a_t \in A$ . They posed the following question.

---

This work is supported by the National Natural Science Foundation of China (Grant Nos. 12101009 and 12371005), Anhui Provincial Natural Science Foundation (Grant No. 2108085QA02) and University Natural Science Research Project of Anhui Province (Grant No. 2022AH050171).

© The Author(s), 2023. Published by Cambridge University Press on behalf of Australian Mathematical Publishing Association Inc.



**PROBLEM 1.1** [7, Problem 1]. Does there exist a set  $A \subseteq \mathbb{N}$  such that  $R_{k_1, \dots, k_t}(A, n) = R_{k_1, \dots, k_t}(\mathbb{N} \setminus A, n)$  for all  $n \geq n_0$ ?

They answered this question for  $t = 2$  and proved the following results.

**THEOREM 1.2** [7, Theorem 1]. *If  $k_1$  and  $k_2$  are two integers with  $k_2 > k_1 \geq 2$  and  $(k_1, k_2) = 1$ , then there does not exist any set  $A \subseteq \mathbb{N}$  such that  $R_{k_1, k_2}(A, n) = R_{k_1, k_2}(\mathbb{N} \setminus A, n)$  for all sufficiently large integers  $n$ .*

**THEOREM 1.3** [7, Theorem 2]. *If  $k$  is an integer with  $k > 1$ , then there exists a set  $A \subseteq \mathbb{N}$  such that*

$$R_{1, k}(A, n) = R_{1, k}(\mathbb{N} \setminus A, n) \quad (1.1)$$

for all integers  $n \geq 1$ .

Furthermore, if  $0 \in A$ , then (1.1) holds for all integers  $n \geq 1$  if and only if

$$A = \{0\} \cup \left( \bigcup_{i=0}^{\infty} [(k+1)k^{2i}, (k+1)k^{2i+1} - 1] \right),$$

where  $[x, y] = \{n : n \in \mathbb{Z}, x \leq n \leq y\}$ .

Later, Li and Ma [4] proved the same results by using generating functions.

Let  $g$  be a fixed integer. In this paper, we consider whether there exists a set  $A \subseteq \mathbb{N}$  such that  $R_{k_1, k_2}(A, n) - R_{k_1, k_2}(\mathbb{N} \setminus A, n) = g$  for all  $n \geq n_0$ . First, we answer this problem in the negative if  $k_1$  and  $k_2$  are two integers with  $2 \leq k_1 < k_2$  and  $(k_1, k_2) = 1$ .

**THEOREM 1.4.** *Let  $g$  be a fixed integer. If  $k_1$  and  $k_2$  are two integers with  $2 \leq k_1 < k_2$  and  $(k_1, k_2) = 1$ , then there does not exist any set  $A \subseteq \mathbb{N}$  such that*

$$R_{k_1, k_2}(A, n) - R_{k_1, k_2}(\mathbb{N} \setminus A, n) = g$$

for all sufficiently large integers  $n$ .

Similar to Theorem 1.3, we seek a set  $A \subseteq \mathbb{N}$  such that  $R_{1, k}(A, n) - R_{1, k}(\mathbb{N} \setminus A, n) = g$  for all integers  $n \geq 1$ . In fact, if  $|g| > 1$ , then such a set  $A$  does not exist by the simple observation that  $0 \leq R_{1, k}(A, n) \leq 1$  and  $0 \leq R_{1, k}(\mathbb{N} \setminus A, n) \leq 1$  for all positive integers  $n < k$ . So we only need to consider the case  $g = 1$ .

**THEOREM 1.5.** *If  $k$  is an integer with  $k > 1$ , then there exists a set  $A \subseteq \mathbb{N}$  such that*

$$R_{1, k}(A, n) - R_{1, k}(\mathbb{N} \setminus A, n) = 1 \quad (1.2)$$

for all integers  $n \geq 1$ .

Furthermore, (1.2) holds for all integers  $n \geq 1$  if and only if

$$A = \{0\} \cup \left( \bigcup_{i=0}^{\infty} [k^{2i}, k^{2i+1} - 1] \right).$$

### 2. Proofs

**LEMMA 2.1.** *Let  $k_1 < k_2$  be two positive integers,  $\{a(n)\}_{n=-\infty}^{+\infty}$  be a sequence of integers with  $a(n) = 0$  for  $n < 0$  and  $A \subseteq \mathbb{N}$ . Then the equality*

$$R_{k_1, k_2}(A, n) - R_{k_1, k_2}(\mathbb{N} \setminus A, n) = a(n) \tag{2.1}$$

holds for all nonnegative integers  $n$  if and only if

$$\chi_A\left(\left\lfloor \frac{n}{k_1} \right\rfloor\right) + \chi_A\left(\left\lfloor \frac{n}{k_2} \right\rfloor\right) = 1 + \sum_{j=0}^{k_1-1} (a(n-j) - a(n-k_2-j))$$

holds for all nonnegative integers  $n$ , where  $\chi_A(i)$  is the characteristic function of  $A$ , that is,  $\chi_A(i) = 1$  if  $i \in A$  and  $\chi_A(i) = 0$  if  $i \notin A$ .

**PROOF.** Let  $f(x)$  be the generating function associated with  $A$ , that is,

$$f(x) = \sum_{a \in A} x^a = \sum_{i=0}^{\infty} \chi_A(i)x^i.$$

Then,

$$\begin{aligned} & \sum_{n=0}^{\infty} (R_{k_1, k_2}(A, n) - R_{k_1, k_2}(\mathbb{N} \setminus A, n))x^n \\ &= f(x^{k_1})f(x^{k_2}) - \left(\frac{1}{1-x^{k_1}} - f(x^{k_1})\right)\left(\frac{1}{1-x^{k_2}} - f(x^{k_2})\right) \\ &= \frac{f(x^{k_1})}{1-x^{k_2}} + \frac{f(x^{k_2})}{1-x^{k_1}} - \frac{1}{(1-x^{k_1})(1-x^{k_2})}. \end{aligned}$$

Let

$$p(x) = \sum_{n=0}^{\infty} a(n)x^n.$$

It follows that (2.1) holds for all nonnegative integers  $n$  if and only if

$$\frac{f(x^{k_1})}{1-x^{k_2}} + \frac{f(x^{k_2})}{1-x^{k_1}} - \frac{1}{(1-x^{k_1})(1-x^{k_2})} = p(x),$$

that is,

$$f(x^{k_1})\frac{1-x^{k_1}}{1-x} + f(x^{k_2})\frac{1-x^{k_2}}{1-x} = \frac{1}{1-x} + (1-x^{k_2})\frac{1-x^{k_1}}{1-x}p(x). \tag{2.2}$$

Note that

$$f(x^{k_1})\frac{1-x^{k_1}}{1-x} = (1+x+\dots+x^{k_1-1})\sum_{n=0}^{\infty} \chi_A(n)x^{k_1n} = \sum_{n=0}^{\infty} \chi_A\left(\left\lfloor \frac{n}{k_1} \right\rfloor\right)x^n,$$

$$f(x^{k_2}) \frac{1-x^{k_2}}{1-x} = (1+x+\dots+x^{k_2-1}) \sum_{n=0}^{\infty} \chi_A(n)x^{k_2n} = \sum_{n=0}^{\infty} \chi_A\left(\left\lfloor \frac{n}{k_2} \right\rfloor\right)x^n,$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

and

$$(1-x^{k_2}) \frac{1-x^{k_1}}{1-x} p(x) = (1-x^{k_2})(1+x+\dots+x^{k_1-1}) \sum_{n=0}^{\infty} a(n)x^n$$

$$= \sum_{n=0}^{\infty} \left( \sum_{j=0}^{k_1-1} (a(n-j) - a(n-k_2-j)) \right) x^n.$$

It follows from (2.2) that for all nonnegative integers  $n$ ,

$$\chi_A\left(\left\lfloor \frac{n}{k_1} \right\rfloor\right) + \chi_A\left(\left\lfloor \frac{n}{k_2} \right\rfloor\right) = 1 + \sum_{j=0}^{k_1-1} (a(n-j) - a(n-k_2-j)).$$

This completes the proof of Lemma 2.1. □

**LEMMA 2.2.** *Let  $n_0$  be a positive integer and  $k_1 < k_2$  be two positive integers with  $(k_1, k_2) = 1$  and  $A \subseteq \mathbb{N}$  be a set with*

$$\chi_A\left(\left\lfloor \frac{i}{k_1} \right\rfloor\right) + \chi_A\left(\left\lfloor \frac{i}{k_2} \right\rfloor\right) = 1 \quad \text{for all } i \geq k_1 + k_2 + n_0. \tag{2.3}$$

*If  $n \geq k_1 + k_2 + n_0$  and  $\chi_A(n) + \chi_A(n+1) = 1$ , then  $k_2 \mid n+1$ .*

**PROOF.** Since  $\chi_A(n) + \chi_A(n+1) = 1$ , it follows that

$$\chi_A\left(\left\lfloor \frac{(n+1)k_1-1}{k_1} \right\rfloor\right) + \chi_A\left(\left\lfloor \frac{(n+1)k_1}{k_1} \right\rfloor\right) = \chi_A(n) + \chi_A(n+1) = 1. \tag{2.4}$$

By (2.3),

$$\chi_A\left(\left\lfloor \frac{(n+1)k_1-1}{k_1} \right\rfloor\right) + \chi_A\left(\left\lfloor \frac{(n+1)k_1-1}{k_2} \right\rfloor\right) = 1$$

and

$$\chi_A\left(\left\lfloor \frac{(n+1)k_1}{k_1} \right\rfloor\right) + \chi_A\left(\left\lfloor \frac{(n+1)k_1}{k_2} \right\rfloor\right) = 1.$$

It follows from (2.4) that

$$\chi_A\left(\left\lfloor \frac{(n+1)k_1-1}{k_2} \right\rfloor\right) + \chi_A\left(\left\lfloor \frac{(n+1)k_1}{k_2} \right\rfloor\right) = 1.$$

Let  $t$  and  $r$  be integers with

$$(n+1)k_1 = tk_2 + r, \quad 0 \leq r \leq k_2 - 1.$$

If  $r \geq 1$ , then

$$1 = \chi_A\left(\left\lceil \frac{(n+1)k_1 - 1}{k_2} \right\rceil\right) + \chi_A\left(\left\lceil \frac{(n+1)k_1}{k_2} \right\rceil\right) = 2\chi_A(t),$$

which is a contradiction. Hence,  $r = 0$  and  $(n+1)k_1 = tk_2$ . Noting that  $(k_1, k_2) = 1$ , we have  $k_2 \mid n+1$ . This completes the proof of Lemma 2.2.  $\square$

**PROOF OF THEOREM 1.4.** Let  $g$  be an integer and let  $k_1, k_2$  be integers with  $2 \leq k_1 < k_2$  and  $(k_1, k_2) = 1$ . Suppose that

$$R_{k_1, k_2}(A, n) - R_{k_1, k_2}(\mathbb{N} \setminus A, n) = g \tag{2.5}$$

for all integers  $n \geq n_0$ . Let  $\{a(n)\}_{n=-\infty}^{+\infty}$  be a sequence of integers with  $a(n) = 0$  for  $n < 0$  and  $a(n) = g$  for all integers  $n \geq n_0$ . It follows from Lemma 2.1 that for all integers  $i \geq k_1 + k_2 + n_0$ ,

$$\chi_A\left(\left\lceil \frac{i}{k_1} \right\rceil\right) + \chi_A\left(\left\lceil \frac{i}{k_2} \right\rceil\right) = 1. \tag{2.6}$$

If  $A$  is a finite set, then  $R_{k_1, k_2}(A, n) = 0$  for all sufficiently large integers  $n$ , and  $R_{k_1, k_2}(\mathbb{N} \setminus A, n)$  cannot be a fixed constant as  $n \rightarrow +\infty$ , which implies that (2.5) cannot hold. So  $A$  is an infinite set. Similarly,  $\mathbb{N} \setminus A$  is also an infinite set.

Since  $2 \leq k_1 < k_2$ , it follows that there exists an integer  $t > 1$  such that  $k_2 < k_1^t$ . Note that both  $A$  and  $\mathbb{N} \setminus A$  are infinite sets. So there exists an integer  $n = k_1^\alpha k_2^\beta h - 1 > (k_1 + k_2 + n_0)^{t+1}$  such that  $n \in A$  and  $n + 1 \notin A$ , where  $\alpha$  and  $\beta$  are nonnegative integers and  $h$  is a positive integer with  $(h, k_1 k_2) = 1$ . It follows from (2.6) and Lemma 2.2 that  $k_2 \mid n + 1$  and  $\beta \geq 1$ . Since

$$(k_1 + k_2 + n_0)^{t+1} < n < k_1^\alpha k_2^\beta h < k_1^{t(\alpha+\beta)} h,$$

it follows that  $k_1^{\alpha+\beta} > k_1 + k_2 + n_0$  or  $h > k_1 + k_2 + n_0$ . Hence, for any  $0 \leq i \leq \beta$ ,

$$k_1^{\alpha+i} k_2^{\beta-i} h \geq k_1^{\alpha+\beta} h > k_1 + k_2 + n_0. \tag{2.7}$$

By (2.6),

$$\chi_A\left(\left\lceil \frac{k_1^{\alpha+1} k_2^\beta h}{k_1} \right\rceil\right) + \chi_A\left(\left\lceil \frac{k_1^{\alpha+1} k_2^\beta h}{k_2} \right\rceil\right) = 1 \tag{2.8}$$

and

$$\chi_A\left(\left\lceil \frac{k_1^{\alpha+1} k_2^\beta h - k_1}{k_1} \right\rceil\right) + \chi_A\left(\left\lceil \frac{k_1^{\alpha+1} k_2^\beta h - k_1}{k_2} \right\rceil\right) = 1. \tag{2.9}$$

Since  $k_1^\alpha k_2^\beta h = n + 1 \notin A$  and  $k_1^\alpha k_2^\beta h - 1 = n \in A$ , it follows from (2.8) and (2.9) that

$$\chi_A(k_1^{\alpha+1} k_2^{\beta-1} h - 1) + \chi_A(k_1^{\alpha+1} k_2^{\beta-1} h) = 1.$$

By Lemma 2.2,  $k_2 \mid k_1^{\alpha+1}k_2^{\beta-1}h$  and so  $\beta \geq 2$ . Continuing this procedure yields

$$\chi_A(k_1^{\alpha+\beta}h - 1) + \chi_A(k_1^{\alpha+\beta}h) = 1.$$

By (2.7) and Lemma 2.2, we also have  $k_2 \mid k_1^{\alpha+\beta}h$ , which is impossible. Hence, there does not exist any set  $A \subseteq \mathbb{N}$  such that (2.5) holds for all sufficiently large integers  $n$ . This completes the proof of Theorem 1.4.  $\square$

**PROOF OF THEOREM 1.5.** Suppose that there is a set  $A$  such that

$$R_{1,k}(A, n) - R_{1,k}(\mathbb{N} \setminus A, n) = 1 \tag{2.10}$$

for all integers  $n \geq 1$ . Then  $0 \in A$  and (2.10) holds for all integers  $n \geq 0$ . Let  $\{a(n)\}_{n=-\infty}^{+\infty}$  be a sequence of integers with  $a(n) = 0$  for  $n < 0$  and  $a(n) = 1$  for  $n \geq 0$ . By Lemma 2.1,

$$R_{1,k}(A, n) - R_{1,k}(\mathbb{N} \setminus A, n) = a(n)$$

for all nonnegative integers  $n$  if and only if

$$\chi_A(n) + \chi_A\left(\left\lfloor \frac{n}{k} \right\rfloor\right) = 1 + a(n) - a(n - k)$$

for all nonnegative integers  $n$ , that is,

$$\begin{aligned} \chi_A(n) + \chi_A(0) &= 2 \quad \text{for } 0 \leq n \leq k - 1, \\ \chi_A(n) + \chi_A\left(\left\lfloor \frac{n}{k} \right\rfloor\right) &= 1 \quad \text{for } n \geq k. \end{aligned}$$

Thus,

$$A = \{0\} \cup \left( \bigcup_{i=0}^{\infty} [k^{2i}, k^{2i+1} - 1] \right). \quad \square$$

### References

- [1] Y.-G. Chen and B. Wang, ‘On additive properties of two special sequences’, *Acta Arith.* **110** (2003), 299–303.
- [2] G. Dombi, ‘Additive properties of certain sets’, *Acta Arith.* **103** (2002), 137–146.
- [3] V. F. Lev, ‘Reconstructing integer sets from their representation functions’, *Electron. J. Combin.* **11** (2004), Article no. R78.
- [4] Y.-L. Li and W.-X. Ma, ‘Partitions of natural numbers with the same weighted representation functions’, *Colloq. Math.* **159** (2020), 1–5.
- [5] C. Sándor, ‘Partitions of natural numbers and their representation functions’, *Integers* **4** (2004), Article no. A18.
- [6] M. Tang, ‘Partitions of the set of natural numbers and their representation functions’, *Discrete Math.* **308** (2008), 2614–2616.
- [7] Q.-H. Yang and Y.-G. Chen, ‘Partitions of natural numbers with the same weighted representation functions’, *J. Number Theory* **132** (2012), 3047–3055.

SHUANG-SHUANG LI, Office of Scientific Research,  
Anhui Normal University, Wuhu 241002, PR China  
e-mail: [ddlshuang@163.com](mailto:ddlshuang@163.com)

YU-QING SHAN, School of Mathematics and Statistics,  
Anhui Normal University, Wuhu 241002, PR China  
e-mail: [syq357660@163.com](mailto:syq357660@163.com)

XIAO-HUI YAN, School of Mathematics and Statistics,  
Anhui Normal University, Wuhu 241002, PR China  
e-mail: [yanxiaohui\\_1992@163.com](mailto:yanxiaohui_1992@163.com)