Glasgow Math. J. **48** (2006) 29–35. © 2006 Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089505002831. Printed in the United Kingdom

ON THE WEAKLY PRECOMPACT AND UNCONDITIONALLY CONVERGING OPERATORS

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(Received 5 February, 2005; accepted 3 August, 2005)

Abstract. In this paper we present some results about wV (weak property V of Pełczyński) or property wV^* (weak property V^* of Pełczyński) in Banach spaces. We show that E has property wV if for any reflexive subspace F of E^*, ${}^{\perp}F$ has property wV. It is shown that G has property wV if under some condition $K_{w^*}(E^*, F^*)$ contains the dual of G. Moreover, it is proved that E^* contains a copy of c_0 if and only if E contains a copy of ℓ_1 where E has property wV^*. Finally, the identity between $L(C(\Omega, E), F)$ and $WP(C(\Omega, E), F)$ is investigated.

2000 *Mathematics Subject Classification.* 46B25, 46B28.

1. Introduction. In order to prevent any doubt, we shall fix some terminology. Throughout in this article $E, F, G, \ldots$ will denote Banach spaces and E^* the dual of E. The term operator means a bounded linear map. A series $\sum x_n$ in E is said to be *weakly unconditionally Cauchy* (w.u.C) if for each $x^* \in E^*$, $\sum | x^*(x_n) | < \infty$. An operator $T : E \to F$ is said to be an *unconditionally converging operator* if T maps w.u.C series in E into unconditionally convergent series in F. T is said to be a *weakly precompact operator* if $T(B_E)$ is a weakly precompact set in F (i.e., for any bounded sequence (x_n) in E, $(T(x_n))$ has a weakly Cauchy subsequence). Let $L(E, F)$, $K(E, F)$, $W(E, F)$, $WP(E, F)$ and $K_{w^*}(E^*, F)$ denote the Banach space of operators, compact operators, weakly compact operators, weakly precompact operators and weak* to weak continuous compact operators between two Banach spaces respectively. For a compact Hausdorff space Ω, $C(\Omega, E)$ is the Banach space of all continuous E-valued functions on Ω with the supremum norm. A subset H of E^* is said to be a *V-set* if

$$\lim_n \sup_{x^* \in H} |x^*(x_n)| = 0,$$

where $\sum x_n$ is any w.u.C series in E. A Banach space E has the *Property V* if any V-set in E^* is relatively weakly compact. A Banach space E has the *wV-property* if any V-set in E^* is a weakly precompact set [**19**]. For the notions and terminology used and not defined in this paper see [**4**] and [**5**].

One of the most important problems of Banach space theory is to recognize the classical properties in Banach spaces. The study of V and V^* properties go back to Pełczyński [**13**]; also wV and wV^* properties go back to E. Saab and P. Saab [**19**].

A complete characterization of their equivalences and properties has been obtained through the efforts of [**2**], [**3**], [**6**], [**7**], [**8**], [**14**], [**18**] and [**19**]. In [**13**] it was shown that the Banach space E has property V if and only if every unconditionally converging operator from E to any Banach space F is a weakly compact operator. Also E has Property V^* if and only if any conjugate unconditionally converging operator from E^* into F^* is a weakly compact operator ([**6**] and [**7**]). In the present paper we show that E has property wV if some special subspace of it has property wV. Weakly precompact operators from $C(\Omega, E)$ to F will be characterized; in fact we show that the condition $L(C(\Omega, E), F) = WP(C(\Omega, E), F)$ is equivalent to E containing a complemented copy of ℓ_1. It is well known that E contains a complemented copy of ℓ_1 if and only if E^* contains a copy of c_0. We shall show that the complemented condition can be replaced by the wV^* property of E.

2. Property wV**.** It is well known that E has Property V if and only if E has property wV and E^* is weakly sequentially complete [**19**].

A subset H of E is said to be a V^*-*set* if

$$\lim_n \sup_{x \in H} |x_n^*(x)| = 0,$$

where $\sum x_n^*$ is any w.u.C series in E^*. A Banach space E has the *Property* V^* if any V^*-subset of E is relatively weakly compact. A Banach space E has the wV^*-*property* if any V^*-subset of E is weakly precompact set [**19**]. E. Saab and P. Saab proved that E has Property V^* if and only if E has property wV^* and E is weakly sequentially complete [**19**].

The following result characterizes the V and V^* properties.

PROPOSITION 2.1.

(a) [**13**] *E^* has Property V^* if E has Property V.*

(b) [**13**] *E has Property V^* if E^* has Property V.*

(c) [**19**] *E has property wV if and only if for any Banach space F any unconditionally converging operator $T : E \to F$ has a weakly precompact adjoint.*

(d) [**19**] *E has property wV^* if and only if for any Banach space F any operator $T : F \to E$ is a weakly precompact operator if its adjoint is an unconditionally converging operator.*

In the following proposition we show that E has property wV if some subspace of it has property wV. The next lemma provides the basic criterion for weak precompactness of bounded sequences.

LEMMA 2.2. [**9**] *Let E be a Banach space, F a reflexive subspace of E and $Q : E \to E/F$ the canonical quotient map. Let $(x_n) \subseteq E$ be a bounded sequence such that $(Q(x_n))$ is a weakly Cauchy sequence. Then (x_n) is a weakly precompact set.*

PROPOSITION 2.3. *Let F be a reflexive subspace of E^*. Assume that ${}^{\perp}F = \{x \in E : y^*(x) = 0, \forall y^* \in F\}$ has property wV. Then E has property wV.*

Proof. Let $Q : E^* \to E^*/F$ be the canonical quotient map and $i : E^*/F \to ({}^{\perp}F)^*$ the natural surjective isomorphism. It is well known that $iQ : E^* \to ({}^{\perp}F)^*$ is weak* to weak* continuous [**12**], and so there is an operator $S : {}^{\perp}F \to E$ such that $iQ = S^*$. Suppose $T : E \to G$ is any unconditionally converging operator from E to any Banach space G. Then TS is an unconditionally converging operator too. From the assumption

https://doi.org/10.1017/S0017089505002831 Published online by Cambridge University Press

$(TS)^* = S^*T^* = iQT^*$ is weakly precompact. i is a surjective isomorphism and so QT^* is weakly precompact. It follows that for any bounded sequence $(z_n^*) \subseteq G^*$ there is a subsequence (y_n^*) of (z_n^*) such that $(QT^*(y_n^*))$ is a weak Cauchy sequence. According to Lemma 2.2 $(T^*(y_n^*))$ has a weak Cauchy subsequence. Consequently, T^* is a weakly precompact operator. □

The Banach space of operators $S(E, F)$ with the $\mathcal{K}$-*property* is a subspace of $L(E, F)$ in which weak convergence and pointwise weak convergence on sequences coincide. See [**1**] and [**11**].

DEFINITION 2.4. $A \subseteq S(E, F)$ is said to be a *quasi-V- set* if the following conditions satisfy:

(a) $\lim_n \sup_{T\in A} \mid T(x_n \otimes y^*) \mid = \lim_n \sup_{T\in A} \mid y^*(T(x_n)) \mid = 0$,

(b) $\lim_n \sup_{T\in A} \mid T(x \otimes y_n^*) \mid = \lim_n \sup_{T\in A} \mid y_n^*(T(x)) \mid = 0$, where $\sum x_n$ and $\sum y_n^*$ are w.u.C series in E and F^* respectively.

LEMMA 2.5. *Suppose that H^* is a subspace of $S(E, F)$. Then every V-set in H^* is a quasi-V-set.*

Proof. Suppose that $A \subseteq H^*$ is a V-set. $\sum x_n \otimes y^*$ is a w.u.C series in H because $H^* \subseteq L(E, F) \subseteq (E \otimes F^*)^*$, and so $\sum \mid T(x_n \otimes y^*) \mid = \sum \mid T(x_n)(y^*) \mid < \infty$. Hence, $\sum x_n \otimes y^*$ is a w.u.C series. Therefore, $(x_n \otimes y^*)$ converges uniformly to zero on A. On the other hand $\sum \mid T(x \otimes y_n^*) \mid = \sum \mid T(x)(y_n^*) \mid < \infty$. Similarly $(x \otimes y_n^*)$ converges uniformly to zero on A. □

THEOREM 2.6. *Suppose that F^* is a separable Banach space and $S(E, F^*)$ is a space of operators with the $\mathcal{K}$-property. Suppose $A \subseteq S(E, F^*)$ is a quasi-V-set. Then A is weakly precompact if E and F have property V.*

Proof. Let $(h_n) \subseteq A$ be an arbitrary sequence. Since F^* is separable, one can consider $Y \subseteq F$ as a countable separating set for F^*. For each w.u.C series $\sum x_n$ and $\sum y_n^*$ in E and F^* respectively, we have

$$(h_n^*(y))(x_n) = h_n(x_n \otimes y) \to 0, \tag{1}$$
$$(h_n(x))(y_n) = h_n(x \otimes y_n) \to 0. \tag{2}$$

Therefore, $(h_n^*(y))$ is a V-set of E^*. Hence, by the countability of Y there is a subsequence of (h_n) which we denote again by (h_n) such that $(h_n^*(y))$ is weakly convergent for each $y \in Y$. We claim that $(h_n(x))$ is weakly Cauchy for each $x \in E$. From (2) $(h_n(x))$ is a V-set in F^* and so it has a weakly convergent subsequence. We claim that $(h_n(x))$ has only one weak cluster point. To see this, suppose that z_1^* and z_2^* are two weak cluster points for $(h_n(x))$. There are two subsequences $(h_{k(n)}(x))$ and $(h_{p(n)}(x))$ such that

$$z_1^* = weak - lim\, h_{k(n)}(x), \quad z_2^* = weak - lim\, h_{p(n)}(x).$$

Now for any $y \in Y$ we have

$$\begin{aligned} z_1^*(y) &= \lim h_{k(n)}(x)(y) \\ &= \lim h_{k(n)}^*(y)(x) \\ &= \lim h_n^*(y)(x) \\ &= \lim h_{p(n)}^*(y)(x) \\ &= \lim h_{p(n)}(x)(y) \\ &= z_2^*(y). \end{aligned}$$

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Then $z_1^* = z_2^*$ since Y is a separating set. According to the definition of $S(E, F^*)$, the sequence (h_n) is a weakly Cauchy sequence which proves that A is a weakly precompact set. □

COROLLARY 2.7. *Suppose that E^* and F have property V. Then any quasi-V-set of $K_{w^*}(E^*, F^*)$ is weakly precompact.*

Proof. We recall that $K_{w^*}(E^*, F^*)$ has the $\mathcal{K}$-property [**11**]. Consider a quasi-V-set H of $K_{w^*}(E^*, F^*)$. According to the Eberlein-Smulian theorem [**4**] we consider the case $H = (h_n)$, and so we can assume that F^* is separable. Then Theorem 2.6 completes the proof. □

COROLLARY 2.8. *Let E^* and F have property V and let $K_{w^*}(E^*, F^*)$ contain the dual of a Banach space G. Then G has property wV.*

Proof. Suppose that $(h_n) \subseteq G^*$ is a V-set. Similar to what is done in the proof of Theorem 2.6 one can assume that $(h_n^*(y))$ is a weakly convergent sequence for each $y \in Y$ where Y is a countably separating set for H and H is a separable subspace of F^* containing all the ranges of the h_n's. The rest of the proof is similar to the proof of Theorem 2.6. □

3. Property wV^*. The concept of V^*-set was introduced, as a dual concept to that of V-set which was first studied by A. Pełczyński in his fundamental paper [**13**]. F. Bombal in [**3**] proved that every closed subspace of an order continuous Banach lattice has property wV^*. We should like to extend this result to Banach spaces from the case of an order continuous Banach lattice.

PROPOSITION 3.1. *Suppose that E has property wV^*. Then any closed subspace F of E has property wV^*.*

Proof. Let $H \subseteq F$ be a V^*-set. Consider a w.u.C series $\sum x_n^*$ in E^*. It is easy to see that $\sum \tilde{x_n^*}$ is a w.u.C series, where $\tilde{x_n^*}$ is the restriction of x_n^* to F. Therefore $(\tilde{x_n^*})$ converges uniformly to zero on H. Then (x_n^*) converges uniformly to zero on H. That H is a weakly precompact set in F follows directly from this and the fact that H is a V^*-set in E. □

A key ingredient in the proof of the next proposition will be the isometric embedding occurring in the following lemma.

LEMMA 3.2 [**10**]. *Let E_0 be a separable subspace of E. Then there is a separable subspace Z of E that contains E_0 and an isometric embedding $J : Z^* \to E^*$ such that $J(z^*)(z) = z^*(z)$ for each z in Z and z^* in Z^*. In particular $J(Z^*)$ is 1-complemented in E^*.*

In [**3**] it is shown that E has property wV^* if and only if every closed separable subspace of E has property wV^*, where E has the separable complementation property. Here, we prove this result without the separable complementation property.

PROPOSITION 3.3. *E has property wV^* if and only if any closed separable subspace of E has property wV^*.*

Proof. $(\Rightarrow)$. This follows from Proposition 3.1.

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($\Leftarrow$). Suppose $H \subseteq F$ is a bounded subset that is not weakly precompact. We shall show that H is not a V^*-set in E. From Rosenthal's ℓ_1 Theorem [**15**], H contains a subsequence (x_n) as a copy of ℓ_1. Let $F = [x_n]$ be the closed linear span of (x_n) which is certainly a separable subspace of E. There is a separable subspace Z of E and an isometric embedding J which satisfies the conditions of Lemma 3.2. That (x_n) is not a V^*-set in E follows from this and the assumption that Z has property wV^*. Consequently, there is a (w.u.C.) series $\sum x_n^*$ in Z^* such that

$$\lim_n \sup_k |z_n^*(x_k)| \neq 0.$$

Choose $x_k^* = Jz_k^*$. Then it is easy to see that $\sum x_k^*$ is a w.u.C series in E^* and

$$\begin{aligned} \lim_k \sup_n |x_k^*(x_n)| &= \lim_k \sup_n |Jz_k^*(x_n)| \\ &= \lim_k \sup_n |z_k^*(x_n)| \\ &\neq 0 \end{aligned}$$

and the proof is complete. □

The following result provides us with a criterion for non-weakly precompactness of an operator. To get started, we first provide a way of characterizing a V^*-set.

LEMMA 3.4 [**6**]. *A subset H of E is a V^*-set if and only if the image of any operator $T : E \to \ell_1$ on H is relatively compact.*

THEOREM 3.5. *$T : E \to F$ is not a weakly precompact operator if and only if T fixes a copy of ℓ_1.*

Proof. ($\Leftarrow$). Without loss of generality we can assume that $T : \ell_1 \to \ell_1$ is an isomorphism. We claim that $\{e_n : n \in \mathbf{N}\}$ is a bounded sequence in ℓ_1 but $(T(e_n))$ has no weak Cauchy subsequence. On the contrary, from the Schur property of ℓ_1 [**4**, p. 85], it has a norm Cauchy subsequence and thus a norm convergent subsequence. Now T is an isomorphism and thus $(e_n)_n$ has a norm convergence subsequence in ℓ_1, which is a contradiction.

($\Rightarrow$). Since T is not weakly precompact there is a bounded sequence (x_n) in E such that $(T(x_n))$ has no weak Cauchy subsequence. From Rosenthal's ℓ_1 Theorem, there is a subsequence $(T(x_{n_k}))$ equivalent to the unit vector basis of ℓ_1. (x_{n_k}) cannot have a weak Cauchy subsequence. Again using Rosenthal's ℓ_1 Theorem there is a subsequence $(x_{n'})$ such that $(x_{n'})$ and $(T(x_{n'}))$ are equivalent to the unit vector basis of ℓ_1. Then $\tilde{T} : [x_{n'}] \to [Tx_{n'}]$ is an isomorphism, where $\tilde{T}$ is the restriction of T to $[x_{n'}]$. Therefore, T fixes a copy of ℓ_1. □

THEOREM 3.6. *Suppose that E has property wV^* and (x_n) is a bounded but not weakly precompact sequence in E. Then there is a subsequence (x_{n_k}) equivalent to the unit vector basis of ℓ_1 such that $[x_{n_k}]$ is complemented in E.*

Proof. From the assumption (x_n) is not a V^*-set in E. According to Lemma 3.4, there is an operator $T : E \to \ell_1$ such that (Tx_n) is not relatively compact in ℓ_1. Hence, T is not a weakly precompact operator. Now according to Theorem 3.5, T fixes a copy of ℓ_1 and so $[Tx_{n_k}]$ is complemented in ℓ_1 by a projection $Q : \ell_1 \to [Tx_{n_k}]$ [**4**, p. 55].

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Now consider the following compositions

$$E \xrightarrow{T} \ell_1 \xrightarrow{Q} [T(x_{n_k})] \xrightarrow{\tilde{T}^{-1}} [x_{n_k}],$$

where $\tilde{T}$ is the restriction of T on $[x_{n_k}]$. Now $\tilde{T}^{-1}QT$ is the required projection. □

We recall that E^* has a copy of c_0 if and only if E has a complemented copy of ℓ_1 [**4**, 48]. This leads to our next result.

Corollary 3.7. *Suppose E has property wV^*. Then E^* contains a copy of c_0 if and only if E contains a copy of ℓ_1.*

Proof. ($\Rightarrow$). This follows from the previous paragraph.

($\Leftarrow$). Since E contains a copy of ℓ_1 there is a sequence $(x_n) \subseteq E$ equivalent to the unit vector basis of ℓ_1. Therefore, it is not a weakly precompact set in E. The assertion we are after follows quickly from Theorem 3.6. □

Our final result now follows.

Theorem 3.8. *Suppose F has property wV^*. Then one of the two following statements holds.*

(a) *E contains a complemented copy of ℓ_1.*

(b) *$L(C(\Omega, E), F) = WP(C(\Omega, E), F)$.*

Proof. Suppose that (b) does not hold. Then there is a bounded operator $T : C(\Omega, E) \to F$ which is not weakly precompact. According to Theorem 3.5, T fixes a copy of ℓ_1; i.e., there is a sequence (f_n) in $C(\Omega, E)$ such that (f_n) and $(T(f_n))$ are equivalent to the unit vector basis of ℓ_1. $(T(f_n))$ is not weakly precompact in F, where F has property wV^*. Hence, according to Theorem 3.6, $[Tf_n]$ is complemented in F by a projection P. Then the following composition.

$$C(\Omega, E) \xrightarrow{T} F \xrightarrow{P} [T(f_n)] \xrightarrow{\tilde{T}^{-1}} [f_n],$$

is a projection, where $\tilde{T}$ is the restriction of T to $[f_n]$. Also $\tilde{T}^{-1}PT$ is a projection from $C(\Omega, E)$ to $[f_n]$. This means that $C(\Omega, E)$ has a complemented copy of ℓ_1. Consequently, E has a complemented copy of ℓ_1 [**17**]. □

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