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THE NEVANLINNA-PICK THEOREM AND A NON-POSITIVE DEFINITE MATRIX

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Let $\{z_j\}$ be an interpolation sequence in the open unit disc and $\{w_j\}$ a bounded sequence. In this note, it is shown that there is a function F in $H^\infty + C$ satisfying $\|F\|_\infty \le 1$ and $\tilde{F}(z_j) - w_j \longrightarrow 0$ as $j \longrightarrow \infty$ if and only if there exists a compact matrix $[t_{ij}]$ such that $[1 - w_i\bar{w}_j / 1 - z_i\bar{z}_j] \ge [a_{ij}]$ on $\mathbb{N} \times \mathbb{N}$ where $[a_{ij}] = [w_j\bar{t}_{ji} + \bar{w}_i t_{ij}] + [t_{ij}][(1 - |z_i|^2)^{1/2}(1 - |z_j|^2)^{1/2} / 1 - \bar{z}_i z_j]^{-1}[\bar{t}_{ji}]$.

Let U be the open unit disc and ∂U the unit circle. For $0 < p \le \infty$, the spaces $L^p(d\theta/2\pi)$ will be denoted simply by L^p , and the corresponding Hardy classes by H^p . Let C denote the space of continuous complex valued functions on ∂U . It is well-known that $H^\infty + C$ is a closed subalgebra of L^∞ . We shall identify a function in H^∞ or $H^\infty + C$ with its holomorphic or harmonic extension to U . The space

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$M(H^\infty + C)$ consists of $M(H^\infty)$ with U deleted.

For a function F in $H^\infty + C$, $Z(F)$ denotes the zero set of F in $M(H^\infty + C)$. If b is an interpolating Blaschke product, then $Z(b)$ is an interpolation subset for H^∞. Let ℓ^∞ be the space of all bounded sequences of complex numbers and $\ell_0^\infty = \{w \in \ell^\infty ; \lim_j w_j = 0\}$. We shall prove the following theorem.

THEOREM. *Let b be an interpolating Blaschke product having zeros at $z_1, z_2, z_3, \ldots$ and u a continuous function on $Z(b)$. Then there exists a unique sequence $\{w_j\}$ in ℓ^∞ modulo ℓ_0^∞ such that $f(z_j) = w_j$ $j = 1, 2, \ldots$, $f = u$ on $Z(b)$ and f in H^∞ and the following are equivalent:*

(1) there exists F in $H^\infty + C$ such that $\|F\|_\infty \le \varepsilon$ and $F \mid Z(b) = u$;

(2) there exists F in $H^\infty + C$ such that $\|F\|_\infty \le \varepsilon$ and $\{F(z_j)\} - \{w_j\}$ in ℓ_0^∞;

(3) there exists a compact matrix $[t_{ij}]$ such that $[\varepsilon - w_i\bar{w}_j / 1 - z_i\bar{z}_j] \ge [a_{ij}]$ on $\mathbb{N} \times \mathbb{N}$ where $[a_{ij}] = [w_j\bar{t}_{ji} + \bar{w}_i t_{ij}] + [t_{ij}][(1 - |z_i|^2)^{1/2}(1 - |z_j|^2)^{1/2}/1 - \bar{z}_i z_j]^{-1}[\overline{t_{ji}}]$ and $\varepsilon \ge 0$.

To prove the theorem we require four lemmas.

LEMMA 1. *If b is an interpolating Blaschke product with zeros $\{z_j\}$ then for any continuous function u on $Z(b)$ there exists a unique sequence $\{w_j\}$ in ℓ^∞ modulo ℓ_0^∞ such that $f(z_j) = w_j$ $j = 1, 2, \ldots$, $f = u$ on $Z(b)$ and f in H^∞.*

Proof. Since $Z(b)$ is an interpolation subset for H^∞, there exists an $f \in H^\infty$ such that $f = u$ on $Z(b)$. Put $w_j = f(z_j)$ $j = 1, 2, \ldots$ If $g \in H^\infty$ satisfies $g = u$ on $Z(b)$ then $\{f(z_j) - g(z_j)\} \in \ell_0^\infty$ because $Z(b) = \overline{\{z_j\}} \setminus \{z_j\}$ where $\overline{\{z_j\}}$ is the closure of $\{z_j\}$ in $M(H^\infty)$.

For an inner function b, put $K = H^2 \ominus bH^2$. The orthogonal projection in L^2 with range K will be denoted by P. For f a function in $H^\infty + C$ let S_f denote the operator $PM_f|K$ where M_f is the multiplication on L^2 that it determines.

LEMMA 2. *For f a function in H^∞ $\|S_f\| = \|f + bH^\infty\|$ and $\|S_f\|_e = \|f + b(H^\infty + C)\|$, where the essential norm $\|S_f\|_e$ of S_f is the distance to the compact operators.*

Proof. Theorem 1 in [4] shows $\|S_f\| = \|f + bH^\infty\|$. We shall show $\|S_f\|_e = \|f + b(H^\infty + C)\|$. The proof is similar to the calculation of the essential norm of a Hankel operator (see [1, p. 608]). We can show that $\|S_f\|_e \geq \|f + b(H^\infty + C)\|$ because $S_z^{*n} \longrightarrow 0$ strongly. For the converse inequality use Theorem 2 in [4].

If b is the Blaschke product with zeros $\{z_j\}$, the functions

$$k_j(z) = (1 - |z_j|^2)^{1/2}/(1 - \bar{z}_j z)$$

form a normalized (although not orthonormal) basis for K. We require an important property of interpolating sequences which was proved by Clark [2, Lemma 3.2].

LEMMA 3. *Let b be an interpolating Blaschke product. Then the map $G : k \longrightarrow (a_1, a_2, \ldots)$ with $\{a_n\}$ given by*

$$k = \sum_{j=1}^{\infty} a_j k_j$$

is a bounded invertible operator of K onto ℓ^2.

LEMMA 4. *Let f be in H^∞ and $\varepsilon \geq 0$. $\|S_f\|_e \leq \varepsilon$ if and only if there exists a compact matrix $[t_{ij}]$ such that $[\varepsilon - w_i\bar{w}_j/1 - z_i\bar{z}_j] \geq [a_{ij}]$ on $\mathbb{N} \times \mathbb{N}$ where $[a_{ij}] = [w_j\bar{t}_{ji} + \bar{w}_i t_{ij}] + [t_{ij}][(1 - |z_i|^2)^{1/2}(1 - |z_j|^2)^{1/2}/1 - \bar{z}_i z_j]^{-1}[\bar{t}_{ji}]$.*

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Proof. If $\|S_f\|_e \le \varepsilon$ then $\|f + b(H^\infty + C)\| \le \varepsilon$ by Lemma 2. By Theorem 4 in [1] and Theorem 2 in [4], there exists a compact operator T on K such that $\|S_f + T\| \le \varepsilon$. Hence

$$\varepsilon^2 I - S_f S_f^* - A \ge 0$$

and

$$A = S_f T^* + T S_f^* + T T^*$$

where I denotes the identity operator on K. Let $\{e_j\}$ denote the orthonormal basis of ℓ^2 given by $e_j = (\delta_{1j}, \delta_{2j}, \ldots)$. Then

$$\begin{aligned}[(Ak_i, k_j)] &= [(G^* A G e_i, e_j)]\\ &= [(G^* S_f T^* G e_i, e_j) + (G^* T S_f^* G e_i, e_j) + (G^* T G (G^* G)^{-1} G^* T^* G e_i, e_j)]\\ &= [f(z_j)(G^* T^* G e_i, e_j) + \overline{f(z_i)}(G^* T G e_i, e_j)]\\ &\quad + [(G^* T G e_i, e_j)]\,[(G^* G e_i, e_j)]^{-1}\,[(G^* T^* G e_i, e_j)]\,.\end{aligned}$$

Put $[t_{ij}] = [(G^* T G e_i, e_j)]$ and $[a_{ij}] = [(Ak_i, k_j)]$, then the lemma follows. The converse follows by reversing the above steps.

Proof of Theorem. By Lemma 1 there exists a unique sequence $\{w_j\}$ in ℓ^∞ modulo ℓ_0^∞ such that $f(z_j) = w_j$ $j = 1, 2, \ldots$, $f = u$ on $Z(b)$ and f in H^∞.

(1) $\Longrightarrow$ (2). Let $F \in H^\infty + C$ such that $\|F\|_\infty \le \varepsilon$ and $F|Z(b) = u$, then $F - f = 0$ on $Z(b)$. We will prove that $F(z_j) - f(z_j) \longrightarrow 0$ as $j \longrightarrow \infty$. Suppose $\{F(z_j) - f(z_j)\} \notin \ell_0^\infty$. Then there exists a subsequence $\{s_j\}$ in $\{z_j\}$ and a nonzero complex number a such that $F(s_j) - f(s_j) \longrightarrow a$ as $j \longrightarrow \infty$. Moreover there exists a subnet $\{t_j\}_\Lambda$ in $\{s_j\}$ such that $t_j \xrightarrow[\Lambda]{} \phi$ in $M(H^\infty)$ and $t_j \xrightarrow[\Lambda]{} \phi(z) = \alpha$. Since $F \in H^\infty + C$, we can write $F = g + v$ for some $g \in H^\infty$ and $v \in C$. Then

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$$F(t_j) - f(t_j) = g(t_j) + v(t_j) - f(t_j)$$

$$\underset{\Lambda}{\longrightarrow} \phi(g) + v(\alpha) - \phi(f) = \phi(F - f) = 0 .$$

This contradicts $a \neq 0$ and it follows that $F(z_j) - f(z_j) \longrightarrow 0$ $j \longrightarrow \infty$.

(2) $\Longrightarrow$ (3). Since $(F - f)(z_j) \longrightarrow 0$, if $\phi \in Z(b)$ then $\phi(F - f) = 0$ and hence $F - f \in b(H^\infty + C)$ by [3, Theorem 1]. This and Lemma 2 imply $\|S_f\|_e \leq \varepsilon$. Now Lemma 4 implies (3).

(3) $\Longrightarrow$(1). Use Lemmas 2 and 4.

References

[1] S. Axler, I.D. Berg, N. Jewell, and A. Shields, "Approximation by compact operators and the space $H^\infty + C$ ", *Ann. of Math.*, 109 (1979), 601-612.

[2] D.N. Clark, "On interpolating sequences and the theory of Hankel and Toeplitz matrices", *J. Funct. Anal.* 5 (1970), 247-258.

[3] C. Guillory, K. Izuchi and D. Sarason, "Interpolating Blaschke products and division in Douglas algebras", *Proc. Roy. Irish Acad. Sect. A*, 84 (1984), 1-7.

[4] D. Sarason, "Generalized interpolation in H^∞ ", *Trans. Amer. Math. Soc.*, 127 (1967), 179-203.

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