

## DIAGRAM ALGEBRAS, DOMINANCE TRIANGULARITY AND SKEW CELL MODULES

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### Abstract

We present an abstract framework for the axiomatic study of diagram algebras. Algebras that fit this framework possess analogues of both the Murphy and seminormal bases of the Hecke algebras of the symmetric groups. We show that the transition matrix between these bases is dominance unitriangular. We construct analogues of the skew Specht modules in this setting. This allows us to propose a natural tableaux theoretic framework in which to study the infamous Kronecker problem.

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### 1. Introduction

The purpose of this article is to develop an Okounkov–Vershik-style framework in which to study towers of diagram algebras  $(A_r)_{r \geq 0}$  over an integral domain,  $R$ , and its field of fractions,  $\mathbb{F}$ . The diagram algebras that fit into our framework include group algebras of the symmetric groups and their Hecke algebras, the Brauer and Birman–Murakami–Wenzl (BMW) algebras, walled Brauer algebras and Jones–Temperley–Lieb algebras, as well as centralizer algebras for the general linear, orthogonal and symplectic groups acting on tensor spaces.

Following [4], we observe that algebras fitting into our framework possess analogues of both the Murphy and seminormal bases of the Hecke algebras of the symmetric groups. We prove that the transition matrix between these bases is dominance unitriangular. In cases where the algebras have Jucys–Murphy elements, we prove that these elements act diagonally on the seminormal basis and triangularly with respect to dominance order on the Murphy basis. For the Hecke algebras of symmetric groups, this provides a new and very simple proof of dominance

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triangularity of the Jucys–Murphy elements (see [14, Theorem 4.6], [11, Theorem 3.32 and Proposition 3.35]). For other diagram algebras such as the Brauer algebras, walled Brauer algebras, BMW algebras or partition algebras, dominance triangularity is a new result. Dominance triangularity is an extremely useful structural result that has already found two distinct applications, which we highlight below.

There is a deeper structure of the Murphy cellular bases of diagram algebras which underlies dominance triangularity. Elements of the Murphy basis can be written using an ordered product of certain ‘branching factors’. There exist both ‘down’ and ‘up’ branching factors and a compatibility relation between them. Using this compatibility relation, one obtains a certain factorizability property of the Murphy basis elements. The compatibility and factorizability properties lead to our dominance triangularity results, but also to strong results about restrictions of cell modules, the construction of skew cell modules and, in [3], to a construction of a cellular basis for quotients of diagram algebras. The compatibility and factorizability properties were already observed in [4], but they are first exploited systematically in this paper and in [3].

The results of this paper play a crucial role in [3], where we construct new integral Murphy-type cellular bases of the Brauer algebras which decompose into bases for the kernels and images of these algebras acting on tensor space. This construction thus provides simultaneously an integral cellular basis of the centralizer algebra, and a new version of the second fundamental theorem of invariant theory. All of these results are compatible with reduction from  $\mathbb{Z}$  to an arbitrary field (characteristic two is excluded in the orthogonal case).

Given two fixed points  $\lambda$  and  $\nu$  in the  $s$ th and  $r$ th levels of the branching graph, we provide an explicit construction of an associated *skew cell module*  $\Delta_{r-s}(\nu \setminus \lambda)$ . We show that these skew cell modules possess integral bases indexed by skew tableaux (paths between the two fixed vertices in the graph) exactly as in the classical case of the symmetric group.

In the case of the partition algebra, these skew modules provide a new setting in which to study the infamous Kronecker problem. In an upcoming paper [1], the first two authors and Maud De Visscher use these skew modules to provide a uniform combinatorial interpretation for one of the largest sets of Kronecker coefficients considered to date (the Littlewood–Richardson coefficients and the Kronecker coefficients labelled by two two-line partitions are covered as important examples).

## 2. Diagram algebras

For the remainder of the paper, we shall let  $R$  be an integral domain with field of fractions  $\mathbb{F}$ . In this section, we shall define diagram algebras and recall the construction of their Murphy bases, following [4]. We first recall the definition of a cellular algebra, as in [9].

**2.1. Cellular algebras.**

**DEFINITION 2.1.** Let  $R$  be an integral domain. A *cellular algebra* is a tuple  $(A, *, \widehat{A}, \triangleright, \text{Std}(\cdot), \mathcal{A})$ , where:

- (1)  $A$  is a unital  $R$ -algebra and  $*$  :  $A \rightarrow A$  is an algebra involution, that is, an  $R$ -linear anti-automorphism of  $A$  such that  $(x^*)^* = x$  for  $x \in A$ ;
- (2)  $(\widehat{A}, \triangleright)$  is a finite partially ordered set and, for each  $\lambda \in \widehat{A}$ ,  $\text{Std}(\lambda)$  is a finite indexing set;
- (3) the set  $\mathcal{A} = \{c_{\mathbf{st}}^\lambda \mid \lambda \in \widehat{A} \text{ and } \mathbf{s}, \mathbf{t} \in \text{Std}(\lambda)\}$  is an  $R$ -basis for  $A$ .

Let  $A^{\triangleright\lambda}$  denote the  $R$ -module with basis  $\{c_{\mathbf{st}}^\mu \mid \mu \triangleright \lambda \text{ and } \mathbf{s}, \mathbf{t} \in \text{Std}(\mu)\}$ .

- (4) The following two conditions hold for the basis  $\mathcal{A}$ .
  - (a) Given  $\lambda \in \widehat{A}$ ,  $\mathbf{t} \in \text{Std}(\lambda)$  and  $a \in A$ , there exist coefficients  $r(a; \mathbf{t}, \mathbf{v}) \in R$ , for  $\mathbf{v} \in \text{Std}(\lambda)$ , such that, for all  $\mathbf{s} \in \text{Std}(\lambda)$ ,

$$c_{\mathbf{st}}^\lambda a \equiv \sum_{\mathbf{v} \in \text{Std}(\lambda)} r(a; \mathbf{t}, \mathbf{v}) c_{\mathbf{sv}}^\lambda \pmod{A^{\triangleright\lambda}}. \tag{2.1}$$

- (b) If  $\lambda \in \widehat{A}$  and  $\mathbf{s}, \mathbf{t} \in \text{Std}(\lambda)$ , then  $(c_{\mathbf{st}}^\lambda)^* \equiv (c_{\mathbf{ts}}^\lambda) \pmod{A^{\triangleright\lambda}}$ .

The tuple  $(A, *, \widehat{A}, \triangleright, \text{Std}(\cdot), \mathcal{A})$  is a *cell datum* for  $A$ . The basis  $\mathcal{A}$  is called a *cellular basis* of  $A$ .

If  $A$  is a cellular algebra over  $R$ , and  $R \rightarrow S$  is a homomorphism of integral domains, then the specialization  $A^S = A \otimes_R S$  is a cellular algebra over  $S$ , with cellular basis

$$\mathcal{A}^S = \{c_{\mathbf{st}}^\lambda \otimes 1_S \mid \lambda \in \widehat{A}, \text{ and } \mathbf{s}, \mathbf{t} \in \text{Std}(\lambda)\}.$$

In particular,  $A^\mathbb{F}$  is a cellular algebra. Since the map  $a \mapsto a \otimes 1_\mathbb{F}$  is injective, we regard  $A$  as contained in  $A^\mathbb{F}$  and we identify  $a \in A$  with  $a \otimes 1_\mathbb{F} \in A^\mathbb{F}$ .

An order ideal  $\Gamma \subset \widehat{A}$  is a subset with the property that if  $\lambda \in \Gamma$  and  $\mu \triangleright \lambda$ , then  $\mu \in \Gamma$ . It follows from the axioms of a cellular algebra that, for any order ideal  $\Gamma$  in  $\widehat{A}$ ,

$$A^\Gamma = \text{Span}\{c_{\mathbf{st}}^\lambda \mid \lambda \in \Gamma, \mathbf{s}, \mathbf{t} \in \text{Std}(\lambda)\}$$

is an involution-invariant two-sided ideal of  $A$ . In particular  $A^{\triangleright\lambda}$  and

$$A^{\triangleright\lambda} = \text{Span}\{c_{\mathbf{st}}^\mu \mid \mu \in \widehat{A}, \mathbf{s}, \mathbf{t} \in \text{Std}(\mu) \text{ and } \mu \triangleright \lambda\}$$

are involution-invariant two-sided ideals.

**DEFINITION 2.2.** Let  $A$  be a cellular algebra over  $R$  and let  $\lambda \in \widehat{A}$ . The *cell module*  $\Delta(\lambda)$  is the right  $A$ -module defined as follows. As an  $R$ -module,  $\Delta(\lambda)$  is free with basis indexed by  $\text{Std}(\lambda)$ , say,  $\{c_{\mathbf{t}}^\lambda \mid \mathbf{t} \in \text{Std}(\lambda)\}$ . The right  $A$ -action is given by

$$c_{\mathbf{t}}^\lambda a = \sum_{\mathbf{v} \in \widehat{A}^\lambda} r(a; \mathbf{t}, \mathbf{v}) c_{\mathbf{v}}^\lambda,$$

where the coefficients  $r(a; \mathbf{t}, \mathbf{v})$  are those of Equation (2.1).

Thus, for any  $s \in \widehat{A}^\lambda$ ,

$$\text{Span}\{c_{st}^\lambda + A^{\triangleright\lambda} \mid t \in \text{Std}(\lambda)\} \subseteq A^{\triangleright\lambda}/A^{\triangleright\lambda}$$

is a model for the cell module  $\Delta(\lambda)$ . When we need to emphasize the algebra or the ground ring, we may write  $\Delta_A(\lambda)$  or  $\Delta^R(\lambda)$ . Note that  $\Delta^{\mathbb{F}}(\lambda) = \Delta(\lambda) \otimes_R \mathbb{F}$  is the cell module for  $A^{\mathbb{F}}$  corresponding to  $\lambda$ .

If  $A$  is an  $R$ -algebra with involution  $*$ , then  $*$  induces functors  $M \rightarrow M^*$  interchanging left and right  $A$ -modules, and taking  $A$ - $A$  bimodules to  $A$ - $A$  bimodules. We identify  $M^{**} M$  via  $x^{**} \mapsto x$  and, for modules  ${}_A M$  and  $N_A$ , have  $(M \otimes_R N)^* \cong N^* \otimes_R M^*$ , as  $A$ - $A$  bimodules, with the isomorphism determined by  $(m \otimes n)^* \mapsto n^* \otimes m^*$ . For a right  $A$ -module  $M_A$ , using both of these isomorphisms, we identify  $(M^* \otimes M)^*$  with  $M^* \otimes M^{**} = M^* \otimes M$ , via  $(x^* \otimes y)^* \mapsto y^* \otimes x$ . Now we apply these observations with  $A$  a cellular algebra and  $\Delta(\lambda)$  a cell module. The assignment

$$\alpha_\lambda : c_{st}^\lambda + A^{\triangleright\lambda} \mapsto (c_s^\lambda)^* \otimes (c_t^\lambda)$$

determines an  $A$ - $A$  bimodule isomorphism from  $A^{\triangleright\lambda}/A^{\triangleright\lambda}$  to  $(\Delta(\lambda))^* \otimes_R \Delta(\lambda)$ . Moreover,  $* \circ \alpha_\lambda = \alpha_\lambda \circ *$ , which reflects the cellular algebra axiom  $(c_{st}^\lambda)^* \equiv c_{ts}^\lambda \pmod{A^{\triangleright\lambda}}$ .

A certain bilinear form on the cell modules plays an essential role in the theory of cellular algebras. Let  $A$  be a cellular algebra over  $R$  and let  $\lambda \in \widehat{A}$ . The cell module  $\Delta(\lambda)$  can be regarded as an  $A/A^{\triangleright\lambda}$  module. For  $x, y, z \in \Delta(\lambda)$ , it follows from the definition of the cell module and the map  $\alpha_\lambda$  that  $x\alpha_\lambda^{-1}(y^* \otimes z) \in Rz$ . Define  $\langle x, y \rangle$  by

$$x\alpha_\lambda^{-1}(y^* \otimes z) = \langle x, y \rangle z. \tag{2.2}$$

Then  $\langle x, y \rangle$  is  $R$ -linear in each variable and  $\langle xa, y \rangle = \langle x, ya^* \rangle$  for  $x, y \in \Delta(\lambda)$  and  $a \in A$ . Note that

$$c_{st}^\lambda c_{uv}^\lambda = \langle c_t^\lambda, c_u^\lambda \rangle c_{sv}^\lambda,$$

which is the customary definition of the bilinear form.

**DEFINITION 2.3 [5].** A cellular algebra  $A$  is said to be *cyclic cellular* if every cell module is cyclic as an  $A$ -module.

If  $A$  is cyclic cellular,  $\lambda \in \widehat{A}$  and  $\delta(\lambda)$  is a generator of the cell module  $\Delta(\lambda)$ , let  $m_\lambda$  be a lift in  $A^{\triangleright\lambda}$  of  $\alpha_\lambda^{-1}(\delta(\lambda)^* \otimes \delta(\lambda))$ .

**LEMMA 2.4.** *The element  $m_\lambda$  has the following properties.*

- (1)  $m_\lambda \equiv m_\lambda^* \pmod{A^{\triangleright\lambda}}$ .
- (2)  $A^{\triangleright\lambda} = Am_\lambda A + A^{\triangleright\lambda}$ .
- (3)  $(m_\lambda A + A^{\triangleright\lambda})/A^{\triangleright\lambda} \cong \Delta(\lambda)$ , as right  $A$ -modules.

**PROOF.** Lemma 2.5 in [5]. □

In examples of interest to us, we can always choose  $m_\lambda$  to satisfy  $m_\lambda^* = m_\lambda$  (and, moreover,  $m_\lambda$  is given explicitly).

**2.2. Sequences of diagram algebras.** Here and in the remainder of the paper, we will consider an increasing sequence  $(A_r)_{r \geq 0}$  of cellular algebras over an integral domain  $R$  with field of fractions  $\mathbb{F}$ . We assume that all the inclusions are unital and that the involutions are consistent; that is, the involution on  $A_{r+1}$ , restricted to  $A_r$ , agrees with the involution on  $A_r$ . We will establish a list of assumptions (D1)–(D6). For convenience, we call an increasing sequence of cellular algebras satisfying these assumptions a *sequence of diagram algebras*.

Let  $(\widehat{A}_r, \triangleright)$  denote the partially ordered set in the cell datum for  $A_r$ . For  $\lambda \in \widehat{A}_r$ , let  $\Delta_r(\lambda)$  denote the corresponding cell module. If  $S$  is an integral domain with a unital homomorphism  $R \rightarrow S$ , write  $A_r^S = A_r \otimes_R S$  and  $\Delta_r^S(\lambda)$  for  $\Delta_r(\lambda) \otimes_R S$ . In particular, write  $A_r^{\mathbb{F}} = A_r \otimes_R \mathbb{F}$  and  $\Delta_r^{\mathbb{F}}(\lambda)$  for  $\Delta_r(\lambda) \otimes_R \mathbb{F}$ .

**DEFINITION 2.5.** Let  $A$  be a cellular algebra over  $R$ . If  $M$  is a right  $A$ -module, a *cell filtration* of  $M$  is a filtration by right  $A$ -modules

$$\{0\} = M_0 \subseteq M_1 \subseteq \dots \subseteq M_s = M,$$

such that  $M_i/M_{i-1} \cong \Delta(\lambda^{(i)})$  for some  $\lambda^{(i)} \in \widehat{A}$ . We say that the filtration is *order preserving* if  $\lambda^{(i)} \triangleright \lambda^{(i+1)}$  in  $\widehat{A}$  for all  $i \geq 1$ .

**DEFINITION 2.6 [6, 7].** Let  $(A_r)_{r \geq 0}$  be an increasing sequence of cellular algebras over an integral domain  $R$ .

- (1) The tower  $(A_r)_{r \geq 0}$  is *restriction-coherent* if, for each  $r \geq 0$  and each  $\mu \in \widehat{A}_{r+1}$ , the restricted module  $\text{Res}_{A_r}^{A_{r+1}}(\Delta_{r+1}(\mu))$  has an order-preserving cell filtration.
- (2) A tower  $(A_r)_{r \geq 0}$  is *induction-coherent* if, for each  $r \geq 0$  and each  $\lambda \in \widehat{A}_r$ , the induced module  $\text{Ind}_{A_r}^{A_{r+1}}(\Delta_r(\lambda))$  has an order-preserving cell filtration.
- (3) The tower  $(A_r)_{r \geq 0}$  is *coherent* if it is both restriction- and induction-coherent.

**REMARK 2.7.** We have changed the terminology from [4, 6, 7], as the weaker notion of coherence, in which the order-preserving requirement is omitted, plays no role here.

We now list the first of our assumptions for a sequence of diagram algebras.

(D1)  $A_0 = R$ .

(D2) the algebras  $A_r$  are cyclic cellular for all  $r \geq 0$ .

For all  $k$  and for all  $\lambda \in \widehat{A}_k$ , fix once and for all a bimodule isomorphism  $\alpha_\lambda : A_k^{\triangleright \lambda} / A_k^{\triangleright \lambda} \rightarrow (\Delta_k(\lambda))^* \otimes_R \Delta_k(\lambda)$ , a generator  $\delta_k(\lambda)$  of the cyclic  $A_k$ -module  $\Delta_k(\lambda)$  and an element  $m_\lambda \in A_k^{\triangleright \lambda}$  satisfying  $\alpha_\lambda(m_\lambda + A_k^{\triangleright \lambda}) = (\delta_k(\lambda))^* \otimes \delta_k(\lambda)$ . Recall the properties of  $m_\lambda$  from Lemma 2.4. We require the following mild assumption on the elements  $m_\lambda$ .

(D3)  $m_\lambda = m_\lambda^*$ .

Our list of assumptions continues as follows.

(D4)  $A_r^{\mathbb{F}}$  is split semisimple for all  $r \geq 0$ .

(D5) The sequence of algebras  $(A_r)_{r \geq 0}$  is restriction-coherent.

As discussed in [4, Section 3], under the assumptions (D1)–(D5) above, there exists a well-defined multiplicity-free branching diagram  $\widehat{A}$  associated with the sequence  $(A_r)_{r \geq 0}$ . The branching diagram is an infinite, graded, directed graph with vertices  $\widehat{A}_r$  at level  $k$  and edges determined as follows. For  $\lambda \in \widehat{A}_{r-1}$  and  $\mu \in \widehat{A}_r$ , there is an edge  $\lambda \rightarrow \mu$  in  $\widehat{A}$  if and only if  $\Delta_{r-1}(\lambda)$  appears as a subquotient of an order-preserving cell filtration of  $\text{Res}_{A_{r-1}}^{A_r}(\Delta_r(\mu))$ . Note that  $\widehat{A}_0$  is a singleton; we denote its unique element by  $\emptyset$ . We can choose  $\Delta_0(\emptyset) = R$ ,  $\delta_0(\emptyset) = 1$  and  $m_\emptyset = 1$ .

**DEFINITION 2.8.** Given  $\nu \in \widehat{A}_r$ , we define a *standard tableau* of shape  $\nu$  to be a directed path  $t$  on the branching diagram  $\widehat{A}$  from  $\emptyset \in \widehat{A}_0$  to  $\nu$ , given by

$$t = (\emptyset = t(0) \rightarrow t(1) \rightarrow t(2) \rightarrow \cdots \rightarrow t(r-1) \rightarrow t(r) = \nu).$$

We let  $\text{Std}_r(\nu)$  denote the set of all such paths and let  $\text{Std}_r = \bigcup_{\nu \in \widehat{A}_r} \text{Std}_r(\nu)$ .

It is shown in [4, Section 3] that there exist certain ‘branching factors’  $d_{\lambda \rightarrow \mu} \in A_r$  associated to each edge  $\lambda \rightarrow \mu$  in  $\widehat{A}$ , related to the cell filtration of  $\text{Res}_{A_{r-1}}^{A_r}(\Delta_r(\mu))$ . Given a path  $t \in \text{Std}_r(\nu)$ ,

$$\emptyset = t(0) \rightarrow t(1) \rightarrow t(2) \rightarrow \cdots \rightarrow t(r-1) \rightarrow t(r) = \nu,$$

define  $d_t = d_{t(r-1) \rightarrow t(r)} d_{t(r-2) \rightarrow t(r-1)} \cdots d_{t(0) \rightarrow t(1)}$ .

We say that two cellular bases of an algebra  $A$  with involution are *equivalent* if they determine the same two-sided ideals  $A^{\triangleright \lambda}$  and isomorphic cell modules.

**THEOREM 2.9** [4, Section 3]. *Let  $(A_r)_{r \geq 0}$  be a sequence of algebras satisfying assumptions (D1)–(D5).*

- (1) *Let  $\lambda \in \widehat{A}_r$ . The set  $\{m_\lambda d_t + A_r^{\triangleright \lambda} \mid t \in \text{Std}_r(\lambda)\}$  is a basis of the cell module  $\Delta_r(\lambda)$ .*
- (2) *The set  $\{d_{\mathfrak{s}}^* m_\lambda d_t \mid \lambda \in \widehat{A}_r, \mathfrak{s}, t \in \text{Std}_r(\lambda)\}$  is a cellular basis of  $A_r$ , equivalent to the original cellular basis.*
- (3) *For a fixed  $\lambda \in \widehat{A}_r$ , we let  $\mu(1) \triangleright \mu(2) \triangleright \cdots \triangleright \mu(s)$  be a listing of the  $\mu \in \widehat{A}_{r-1}$  such that  $\mu \rightarrow \lambda$ . Let*

$$M_j = \text{Span}_R \{m_\lambda d_t + A_r^{\triangleright \lambda} \mid t \in \text{Std}_r(\lambda), t(k-1) \triangleright \mu(j)\}.$$

*Then*

$$(0) \subset M_1 \subset \cdots \subset M_s = \Delta_r(\lambda)$$

*is a filtration of  $\Delta_r(\lambda)$  by  $A_{r-1}$  submodules, and  $M_j/M_{j-1} \cong \Delta_{r-1}(\mu_j)$ .*

**NOTATION 2.10.** We write  $m_{\mathfrak{s}t}^\lambda = d_{\mathfrak{s}}^* m_\lambda d_t$ . Also write  $m_t = m_\lambda d_t + A_r^{\triangleright \lambda} \in \Delta_r(\lambda)$ . We refer to the cellular basis  $\{m_{\mathfrak{s}t}^\lambda \mid \lambda \in \widehat{A}_r \text{ and } \mathfrak{s}, t \in \text{Std}_r(\lambda)\}$  as the *Murphy cellular basis* of  $A_r$  and  $\{m_t^\lambda \mid t \in \text{Std}_r(\lambda)\}$  as the *Murphy basis* of the cell module  $\Delta_k(\lambda)$ .

We will now continue with our list of assumed properties of the sequence of algebra  $(A_r)_{r \geq 0}$  with one final axiom.

(D6) There exist ‘ $u$ -branching factors’  $u_{\mu \rightarrow \lambda} \in A_{r+1}^R$  such that

$$m_\lambda d_{\mu \rightarrow \lambda} = (u_{\mu \rightarrow \lambda})^* m_\mu. \tag{2.3}$$

**EXAMPLE 2.11.** It is shown in [4] that the Hecke algebras of type  $A$ , the symmetric group algebras, the Brauer algebras, the Birman–Wenzl–Murakami algebras, the partition algebras and the Jones–Temperley–Lieb algebras all are examples of sequences of algebras satisfying properties (D1)–(D6). For the Hecke algebras, the cellular basis of Theorem 2.9 agrees with the Murphy basis from [15], up to a normalization. In each case, the ground ring  $R$  can be taken to be the generic ground ring for the class of algebras. For example, for the Hecke algebras, this is  $\mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]$ , and for the Brauer algebras it is  $\mathbb{Z}[\delta]$ , where  $\mathbf{q}$  and  $\delta$  are indeterminants.

Let  $G_n$  be either the general linear group  $GL_n$ , the orthogonal group  $O_n$  or the symplectic group  $Sp_{2n}$ , and let  $V$  denote its natural module. The centralizer algebra  $\text{End}_{G_n}(V^{\otimes r})$  is a quotient of the symmetric group  $\mathfrak{S}_r$ , the Brauer algebra  $B_r(n)$  or the Brauer algebra  $B_r(-2n)$ , respectively. In each case, the ground ring  $R$  can be taken to be  $\mathbb{Z}$ . It is shown in [3] that the algebras  $\text{End}_G(V^{\otimes r})$  all satisfy axioms (D1)–(D6).

**DEFINITION 2.12.** Given  $0 \leq s \leq r$  and  $\lambda \in \widehat{A}_s, \nu \in \widehat{A}_r$ , we define a *skew standard tableau* of shape  $\nu \setminus \lambda$  and degree  $r - s$  to be a directed path  $t$  on the branching diagram  $\widehat{A}$  from  $\lambda$  to  $\nu$ , given by

$$t = (\lambda = t(s) \rightarrow t(s + 1) \rightarrow t(s + 2) \rightarrow \cdots \rightarrow t(r - 1) \rightarrow t(r) = \nu).$$

We let  $\text{Std}_{s,r}(\nu \setminus \lambda)$  denote the set of all such paths with given  $\lambda$  and  $\nu$ . Given  $0 \leq s \leq r$ , we set  $\text{Std}_{s,r} = \bigcup_{\lambda \in \widehat{A}_s, \nu \in \widehat{A}_r} \text{Std}_{s,r}(\nu \setminus \lambda)$ .

Given two paths  $\mathbf{s} \in \text{Std}_{q,s}(\mu \setminus \lambda)$  and  $\mathbf{t} \in \text{Std}_{s,r}(\nu \setminus \mu)$  such that the final point of  $\mathbf{s}$  is the initial point of  $\mathbf{t}$ , define  $\mathbf{s} \circ \mathbf{t}$  to be the obvious path obtained by concatenation.

**REMARK 2.13.** Given a path  $t \in \text{Std}_{s,r}(\nu \setminus \lambda)$  of the form

$$\lambda = t(s) \rightarrow t(s + 1) \rightarrow t(s + 2) \rightarrow \cdots \rightarrow t(r - 1) \rightarrow t(r) = \nu,$$

define

$$d_t = d_{t(r-1) \rightarrow t(r)} d_{t(r-2) \rightarrow t(r-1)} \cdots d_{t(s) \rightarrow t(s+1)}$$

and

$$u_t = u_{t(s) \rightarrow t(s+1)} \cdots u_{t(r-2) \rightarrow t(r-1)} u_{t(r-1) \rightarrow t(r)}.$$

Then it follows from the compatibility relation (2.3) and induction on  $r - s$  that

$$u_t^* m_\lambda = m_\nu d_t. \tag{2.4}$$

Because  $m_\emptyset$  can be chosen to be 1, this gives, in particular, for  $t \in \text{Std}_r(\nu)$ ,

$$u_t^* = m_\nu d_t. \tag{2.5}$$

Therefore the cellular basis  $\{m_{st}^\nu\}$  can also be written in the apparently asymmetric form

$$m_{st}^\nu = d_s^* m_\nu d_t = d_s^* u_t^*.$$

Using the symmetry of the cellular basis  $(m_{st}^\nu)^* = m_{ts}^\nu$  (which follows from the assumption (D3)), we also get

$$m_{st}^\nu = u_s d_t.$$

Using (2.5), the basis  $\{m_t^\nu \mid t \in \text{Std}_r(\nu)\}$  of the cell module  $\Delta_r(\nu)$  is given by

$$m_t^\nu = u_t^* + A_r^{\triangleright \nu}.$$

Now, for any  $0 \leq q \leq s \leq r$ , let  $t_{[q,s]}$  denote the truncated path

$$t(q) \rightarrow t(q+1) \rightarrow t(q+2) \rightarrow \dots \rightarrow t(s-1) \rightarrow t(s).$$

The representative  $u_t^*$  of  $m_t$  has the remarkable property that, for any  $0 \leq s \leq r$ ,

$$u_t^* = u_{t_{[s,r]}}^* u_{t_{[0,s]}}^*, \tag{2.6}$$

and

$$u_{t_{[0,s]}}^* = m_{t(s)} d_{t_{[0,s]}} \in m_{t(s)} A_s \subseteq A_s^{\triangleright t(s)}.$$

The compatibility relations (2.4), together with the factorizability (2.6) of representatives  $u_t^*$  of the Murphy basis, play a crucial role in this paper. They lead directly to our dominance triangularity results, which, in turn, lead to strong results about restriction of cell modules and allow the construction of skew cell modules. In our view, these are the distinguishing properties of the Murphy bases of diagram algebras and, even in the original context of the Hecke algebras [15], these properties provide new insight.

### 3. Seminormal basis and dominance triangularity

**3.1. Gelfand–Zeitlin idempotents.** Consider an increasing sequence of algebras  $(A_r)_{r \geq 0}$  satisfying assumptions (D1)–(D6) of Section 2.2. Let us recall the following notion pertaining to the tower  $(A_r^{\mathbb{F}})_{r \geq 0}$ . The terminology is from Okounkov and Vershik [16, 17].

**DEFINITION 3.1.** The *Gelfand–Zeitlin subalgebra*  $G_r$  of  $A_r^{\mathbb{F}}$  is the subalgebra generated by the centers of  $A_0^{\mathbb{F}}, A_1^{\mathbb{F}}, \dots, A_r^{\mathbb{F}}$ .

The Gelfand–Zeitlin subalgebra is a maximal abelian subalgebra of  $A_r^{\mathbb{F}}$  and contains a canonical family of idempotents indexed by paths on the branching diagram  $\widehat{A}$ . For each  $s$ , let  $\{z_s^\lambda \mid \lambda \in \widehat{A}_s\}$  denote the set of minimal central idempotents in  $A_s^{\mathbb{F}}$ . For  $r \geq 1$  and  $t$  a path on  $\widehat{A}$  of length  $r$ , let  $F_t = \prod_{s=1}^r z_s^{t(s)}$ . The elements  $F_t$  for  $t \in \text{Std}_r$  are mutually orthogonal minimal idempotents whose sum is the identity; moreover,  $\sum_{t \in \text{Std}_r(\lambda)} F_t = z_r^\lambda$ . If  $s \in \text{Std}_s$  and  $t \in \text{Std}_r$  with  $s \leq r$ , then  $F_s F_t = \delta_{s, t_{[0,s]}} F_t$ . Evidently, the set  $\{F_t \mid t \in \text{Std}_s, 0 \leq s \leq r\}$  generates  $G_r$ . Let us call this set of idempotents the family of *Gelfand–Zeitlin idempotents* for  $(A_r^{\mathbb{F}})_{r \geq 0}$ .



**3.2. Seminormal bases.** Let  $\{m_{st}^\lambda \mid \lambda \in \widehat{A}_r \text{ and } s, t \in \text{Std}_r(\lambda)\}$  be the Murphy cellular basis of  $A_r$  constructed in Section 2.2, and, for  $\lambda \in \widehat{A}_r$ , let  $\{m_t^\lambda \mid t \in \text{Std}_r(\lambda)\}$  be the Murphy basis of the cell module  $\Delta_r(\lambda)$ . For  $t \in \text{Std}_r$ , let  $F_t$  be the corresponding Gelfand–Zeitlin idempotent in  $A_r^{\mathbb{F}}$ .

**DEFINITION 3.2 (Seminormal bases).** Let  $r \geq 0$  and  $\lambda \in \widehat{A}_r$ . For  $s, t \in \text{Std}_r(\lambda)$ , define  $f_t^\lambda = m_t^\lambda F_t \in \Delta_r^{\mathbb{F}}(\lambda)$  and  $F_{st}^\lambda = F_s m_{st}^\lambda F_t \in A_r^{\mathbb{F}}$ .

We now define two partial orders on the set of paths  $\text{Std}_{s,r}$  in the branching graph.

**DEFINITION 3.3 (Dominance order).** For  $s, t \in \text{Std}_{s,r}$ , define  $s \triangleright t$  if  $s(j) \triangleright t(j)$  for all  $0 \leq j \leq r$ . We write  $s \triangleright t$  if  $s \neq t$  and  $s \triangleright t$ .

This is evidently a partial order, which we call the dominance order on paths. In particular, the dominance order is defined on  $\text{Std}_r$  and on  $\text{Std}_r(\lambda)$  for  $\lambda \in \widehat{A}_r$ .

**DEFINITION 3.4 (Reverse lexicographic order).** For  $s, t \in \text{Std}_{s,r}$ , define  $s \succcurlyeq t$  if  $s = t$  or if  $s(j) \triangleright t(j)$  for the last index  $j$  such that  $s(j) \neq t(j)$ . We write  $s \succ t$  if  $s \neq t$  and  $s \succcurlyeq t$ .

This is also a partial order on paths (and is defined on  $\text{Std}_r$  and on  $\text{Std}_r(\lambda)$  for  $\lambda \in \widehat{A}_r$ ). Evidently,  $s \triangleright t$  implies  $s \succ t$ .

**THEOREM 3.5 (Dominance triangularity).** Fix  $\lambda \in \widehat{A}_r$ . For all  $t \in \text{Std}_r(\lambda)$ , there exist coefficients  $r_s, r'_s \in \mathbb{F}$  such that

$$m_t^\lambda = f_t^\lambda + \sum_{\substack{s \in \text{Std}_r(\lambda) \\ s \triangleright t}} r_s f_s^\lambda \quad f_t^\lambda = m_t^\lambda + \sum_{\substack{s \in \text{Std}_r(\lambda) \\ s \triangleright t}} r'_s m_s^\lambda.$$

In particular,  $\{f_t^\lambda \mid t \in \text{Std}_r(\lambda)\}$  is a basis of  $\Delta_r^{\mathbb{F}}(\lambda)$ .

**PROOF.** The element  $\sum_{\mu \triangleright \lambda} z_s^\mu$  acts as the identity on the ideal  $A_s^{\triangleright \lambda}$ . For  $\lambda \in \widehat{A}_r$  and  $t \in \text{Std}_r(\lambda)$ ,  $u_t^*$  has the property that  $u_{t_{[0,s]}}^* \in A_s^{\triangleright t(s)}$  for all  $0 \leq s \leq r$ , and  $u_t^* = u_{t_{[s,r]}}^* u_{t_{[0,s]}}^*$ . Therefore

$$u_t^* = u_t^* \sum_{\mu(s) \triangleright t(s)} z_s^{\mu(s)}.$$

Applying this at each  $0 \leq s \leq r$  gives

$$u_t^* = u_t^* \prod_{1 \leq s \leq r} \sum_{\mu(s) \triangleright t(s)} z_s^{\mu(s)} = u_t^* \sum_{(\mu(1), \mu(2), \dots, \mu(r))} \prod_{1 \leq s \leq r} z_s^{\mu(s)},$$

where the sum is over all sequences  $(\mu(1), \mu(2), \dots, \mu(r))$  such that  $\mu(s) \in \widehat{A}_s$  and  $\mu(s) \triangleright t(s)$  for all  $0 \leq s \leq r$ . If such a sequence is not an element of  $\text{Std}_r$ , that is, it is not a path on  $\widehat{A}$ , then the product  $\prod_{1 \leq s \leq r} z_s^{\mu(s)}$  is zero. On the other hand, if

$$s = (\mu(1) \rightarrow \mu(2) \rightarrow \dots \rightarrow \mu(r))$$

is an element of  $\text{Std}_r$ , then  $\prod_{1 \leq s \leq r} z_s^{\mu(s)} = F_s$ . Thus

$$u_t^* = u_t^* \sum_{\substack{s \in \text{Std}_r \\ s \triangleright t}} F_s = u_t^* F_t + \sum_{\substack{s \in \text{Std}_r(\lambda) \\ s \triangleright t}} u_t^* F_s + y,$$

where  $y \in (A_r^{\mathbb{F}})^{\triangleright \lambda}$ . Passing to the cell module  $\Delta_r^{\mathbb{F}}(\lambda)$ ,

$$m_t^\lambda = f_t^\lambda + \sum_{\substack{s \in \text{Std}_r(\lambda) \\ s \triangleright t}} m_t^\lambda F_s.$$

But the range of  $F_s$  acting on the simple module  $\Delta_r^{\mathbb{F}}(\lambda)$  is of dimension one, spanned by  $f_s^\lambda$ , so this gives

$$m_t^\lambda = f_t^\lambda + \sum_{\substack{s \in \text{Std}_r(\lambda) \\ s \triangleright t}} r_s f_s^\lambda,$$

for appropriate  $r_s \in \mathbb{F}$ . This shows that the tuple  $[m_t^\lambda]_{t \in \text{Std}_r(\lambda)}$  is related to the tuple  $[f_t^\lambda]_{t \in \text{Std}_r(\lambda)}$  by an (invertible) matrix which is unitriangular with respect to dominance order of paths. Hence  $\{f_t^\lambda \mid t \in \text{Std}_r(\lambda)\}$  is a basis of  $\Delta_r^{\mathbb{F}}(\lambda)$  and the inverse change of basis matrix is also unitriangular.  $\square$

**COROLLARY 3.6.** *For  $r \geq 0$ :*

- (1)  $\{f_t^\lambda \mid t \in \text{Std}_r(\lambda)\}$  is a basis of  $\Delta_r^{\mathbb{F}}(\lambda)$  for all  $\lambda \in \widehat{A}_r$ ; and
- (2)  $\{F_{st}^\lambda \mid \lambda \in \widehat{A}_r \text{ and } s, t \in \widehat{A}_r\}$  is a cellular basis of  $A_r^{\mathbb{F}}$ .

**PROOF.** The first statement was verified in the proof of Theorem 3.5. The second statement follows from [7, Lemma 2.3], since  $F_{st}^\lambda$  is a lift of  $\alpha_\lambda^{-1}(f_s^\lambda \otimes_{\mathbb{F}} f_t^\lambda)$ .  $\square$

We note that the cell module  $\Delta_r^{\mathbb{F}}(\lambda)$  embeds in the algebra  $A_r^{\mathbb{F}}$  as a right ideal.

**LEMMA 3.7.** *Let  $r \geq 0$  and let  $\lambda \in \widehat{A}_r$ .*

- (1)  $\text{Span}_{\mathbb{F}}\{u_t^* F_t \mid t \in \text{Std}_r(\lambda)\}$  is a right ideal of  $A_r^{\mathbb{F}}$  and  $f_t^\lambda \mapsto u_t^* F_t$  determines an isomorphism of  $\Delta_r^{\mathbb{F}}(\lambda)$  onto this right ideal.
- (2) Likewise, for any fixed  $s \in \text{Std}_r(\lambda)$ ,

$$\Delta_r^{\mathbb{F}}(\lambda) \cong \text{Span}_{\mathbb{F}}\{m_{st}^\lambda F_t \mid t \in \text{Std}_r(\lambda)\} \cong \text{Span}_{\mathbb{F}}\{F_{st}^\lambda \mid t \in \text{Std}_r(\lambda)\},$$

with isomorphisms determined by  $f_t^\lambda \mapsto m_{st}^\lambda F_t$ , respectively,  $f_t^\lambda \mapsto F_{st}^\lambda$ .

**PROOF.** Recall that  $\delta_\lambda$  is a generator of the cell module  $\Delta_r^{\mathbb{F}}(\lambda)$  and  $m_\lambda$  is a lift in  $(A_r^{\mathbb{F}})^{\triangleright \lambda}$  of  $\alpha_\lambda^{-1}((\delta_\lambda)^* \otimes_{\mathbb{F}} \delta_\lambda)$ . For any fixed  $x \in A_r^{\mathbb{F}}$  such that  $\delta_\lambda x \neq 0$ ,

$$f_t^\lambda \mapsto \alpha_\lambda^{-1}(x^*(\delta_\lambda)^* \otimes_{\mathbb{F}} f_t^\lambda) = x^* m_\lambda d_t F_t + (A_r^{\mathbb{F}})^{\triangleright \lambda}$$

determines an isomorphism of  $\Delta_r^{\mathbb{F}}(\lambda)$  onto a submodule of  $(A_r^{\mathbb{F}})^{\triangleright \lambda} / (A_r^{\mathbb{F}})^{\triangleright \lambda}$ . Therefore, for  $b \in A_r^{\mathbb{F}}$ , if  $f_t^\lambda b = \sum_s \beta_s f_s^\lambda$ , then

$$x^* m_\lambda d_t F_t b = \sum_s \beta_s x^* m_\lambda d_s F_s + y,$$

where  $y \in (A_r^{\mathbb{F}})^{\triangleright \lambda}$ . Now for all  $\mathbf{s} \in \text{Std}_r(\lambda)$ ,  $F_{\mathbf{s}} z_k^\lambda = F_{\mathbf{s}}$ , but  $y z_k^\lambda = 0$ . Thus multiplying by  $z_k^\lambda$  on the right gives

$$x^* m_\lambda d_t F_t b = \sum_{\mathbf{s}} \beta_{\mathbf{s}} x^* m_\lambda d_{\mathbf{s}} F_{\mathbf{s}},$$

which shows that  $\text{Span}_{\mathbb{F}}\{x^* m_\lambda d_t F_t \mid t \in \text{Std}_r(\lambda)\}$  is a right ideal and  $f_t \mapsto x^* m_\lambda d_t F_t$  determines an isomorphism of  $\Delta_k^{\mathbb{F}}(\lambda)$  onto this right ideal. Taking  $x = 1$  yields statement (1), and taking  $x = d_{\mathbf{s}}$ , respectively,  $x = d_{\mathbf{s}} F_{\mathbf{s}}$ , gives the isomorphisms in part (2). □

**LEMMA 3.8.** *Let  $r \geq 0$ ,  $\lambda, \mu \in \widehat{A}_r$ ,  $\mathbf{s}, \mathbf{t} \in \text{Std}_r(\lambda)$  and  $\mathbf{u}, \mathbf{v} \in \text{Std}_r(\mu)$ .*

- (1)  $f_{\mathbf{t}}^\lambda F_{\mathbf{uv}}^\mu = \delta_{\lambda, \mu} \langle f_{\mathbf{t}}^\lambda, f_{\mathbf{u}}^\lambda \rangle f_{\mathbf{v}}^\lambda$ .
- (2)  $F_{\mathbf{st}}^\lambda F_{\mathbf{uv}}^\mu = \delta_{\lambda, \mu} \langle f_{\mathbf{t}}^\lambda, f_{\mathbf{u}}^\lambda \rangle F_{\mathbf{sv}}^\lambda$ .
- (3)  $\langle f_{\mathbf{t}}^\lambda, f_{\mathbf{u}}^\lambda \rangle \neq 0$  if and only if  $\mathbf{t} = \mathbf{u}$ .
- (4)  $\langle f_{\mathbf{t}}^\lambda, f_{\mathbf{t}}^\lambda \rangle^{-1} F_{\mathbf{tt}}^\lambda = F_{\mathbf{t}}$ .
- (5) *The set of elements  $E_{\mathbf{st}}^\lambda = \langle f_{\mathbf{s}}^\lambda, f_{\mathbf{s}}^\lambda \rangle^{-1} F_{\mathbf{st}}^\lambda$  for  $\lambda \in \widehat{A}_r$  and  $\mathbf{s}, \mathbf{t} \in \text{Std}_r(\lambda)$  is a complete family of matrix units with*

$$E_{\mathbf{st}}^\lambda E_{\mathbf{uv}}^\mu = \delta_{\lambda, \mu} \delta_{\mathbf{t}, \mathbf{u}} E_{\mathbf{sv}}^\lambda \quad \text{and} \quad E_{\mathbf{st}}^\lambda E_{\mathbf{ts}}^\lambda = E_{\mathbf{ss}}^\lambda = F_{\mathbf{s}}.$$

**PROOF.** If  $\lambda \neq \mu$ , then  $f_{\mathbf{t}}^\lambda F_{\mathbf{uv}}^\mu = f_{\mathbf{t}}^\lambda F_{\mathbf{t}} F_{\mathbf{u}} F_{\mathbf{v}}^\mu = 0$  and, similarly,  $F_{\mathbf{st}}^\lambda F_{\mathbf{uv}}^\mu = 0$ . We have  $f_{\mathbf{t}}^\lambda F_{\mathbf{uv}}^\lambda = \langle f_{\mathbf{t}}^\lambda, f_{\mathbf{u}}^\lambda \rangle f_{\mathbf{v}}^\lambda$  by the definition of the bilinear form (2.2), since  $F_{\mathbf{uv}}^\lambda$  is a lift of  $\alpha_\lambda^{-1}((f_{\mathbf{u}}^\lambda)^* \otimes f_{\mathbf{v}}^\lambda)$ . This proves part (1) and part (2) follows from Lemma 3.7 part (2). If  $\mathbf{t} \neq \mathbf{u}$ , then

$$\langle f_{\mathbf{t}}^\lambda, f_{\mathbf{u}}^\lambda \rangle = \langle f_{\mathbf{t}}^\lambda F_{\mathbf{t}}, f_{\mathbf{u}}^\lambda F_{\mathbf{u}} \rangle = \langle f_{\mathbf{t}}^\lambda F_{\mathbf{t}} F_{\mathbf{u}}, f_{\mathbf{u}}^\lambda \rangle = 0.$$

Suppose that  $\langle f_{\mathbf{t}}^\lambda, f_{\mathbf{t}}^\lambda \rangle = 0$  for some  $\mathbf{t} \in \text{Std}_r(\lambda)$ . Then it follows from part (2) and the orthogonality of the elements  $f_{\mathbf{u}}^\lambda$ , which was just established, that  $f_{\mathbf{t}}^\lambda F_{\mathbf{uv}}^\mu = 0$  for all  $\mu$  and all  $\mathbf{u}, \mathbf{v} \in \text{Std}_r(\mu)$ . But, by Corollary 3.6, the identity element of  $A_r^{\mathbb{F}}$  is in the span of the set of  $F_{\mathbf{uv}}^\mu$ , so it follows that  $f_{\mathbf{t}}^\lambda = 0$ , which is a contradiction. This proves part (3). By part (2),  $G_{\mathbf{t}} = \langle f_{\mathbf{t}}^\lambda, f_{\mathbf{t}}^\lambda \rangle^{-1} F_{\mathbf{tt}}^\lambda$  is an idempotent such that  $G_{\mathbf{t}} F_{\mathbf{t}} = G_{\mathbf{t}}$ . Since  $F_{\mathbf{t}}$  is a minimal idempotent, it follows that  $G_{\mathbf{t}} = F_{\mathbf{t}}$ , which proves part (4). Part (5) follows from parts (2), (3) and (4). □

**3.3. Restriction of the seminormal representations.** For  $0 \leq s < r$ , write  $A_r^{\mathbb{F}} \cap (A_s^{\mathbb{F}})'$  for the set of  $x \in A_r^{\mathbb{F}}$  that commute pointwise with  $A_s^{\mathbb{F}}$ .

**PROPOSITION 3.9.** *Let  $1 \leq s < r$ . Let  $\nu \in \widehat{A}_r$  and  $\mathbf{t} \in \text{Std}_r(\nu)$ . Write  $\lambda = \mathbf{t}(s)$ ,  $\mathbf{t}_1 = \mathbf{t}_{[0, s]} \in \text{Std}_s(\lambda)$  and  $\mathbf{t}_2 = \mathbf{t}_{[s, r]} \in \text{Std}_{s, r}(\nu \setminus \lambda)$ .*

- (1) *Let  $x \in A_s^{\mathbb{F}}$  and suppose that  $f_{\mathbf{t}_1}^\lambda x = \sum_{\mathbf{s} \in \text{Std}_s(\lambda)} \alpha_{\mathbf{s}} f_{\mathbf{s}}^\lambda$ . Then*

$$f_{\mathbf{t}}^\nu x = \sum_{\mathbf{s} \in \text{Std}_s(\lambda)} \alpha_{\mathbf{s}} f_{\mathbf{s} \circ \mathbf{t}_2}^\nu.$$

(2) In particular, for  $\mu \in \widehat{A}_s$  and  $u, v \in \text{Std}_s(\mu)$ ,

$$f_t^v F_{uv}^\mu = \delta_{\mu, \lambda} \delta_{t_1, u} \langle f_{t_1}^\lambda, f_{t_1}^\lambda \rangle f_{v \circ t_2}^v.$$

(3) For  $x \in A_r^{\mathbb{F}} \cap (A_s^{\mathbb{F}})'$ ,  $f_t^v x = \sum_{\mathbf{s}} r_{\mathbf{s}} f_{t_1 \circ \mathbf{s}}^v$ , where the sum is over  $\mathbf{s} \in \text{Std}_{s,r}(v \setminus \lambda)$  and the coefficients depend only on  $x$  and  $t_2$  and are independent of  $t_1$ .

**PROOF.** We can embed the cell module  $\Delta_r^{\mathbb{F}}(v)$  in  $A_r^{\mathbb{F}}$ , identifying  $f_t^v$  with  $u_t^* F_t$ , using Lemma 3.7. We can write  $F_t = F_{t_1} F_{t_2}$ , where  $F_{t_2} \in A_r^{\mathbb{F}} \cap (A_s^{\mathbb{F}})'$ . Thus

$$f_t^v x = u_t^* F_t x = u_{t_2}^* F_{t_2} u_{t_1}^* F_{t_1} x.$$

Applying Lemma 3.7 part (1) to  $\Delta_s^{\mathbb{F}}(\lambda)$  we find that this equals

$$u_{t_2}^* F_{t_2} \sum_{\mathbf{s}} \alpha_{\mathbf{s}} u_{\mathbf{s}}^* F_{\mathbf{s}} = \sum_{\mathbf{s}} \alpha_{\mathbf{s}} u_{t_2}^* u_{\mathbf{s}}^* F_{\mathbf{s}} F_{t_2} = \sum_{\mathbf{s}} \alpha_{\mathbf{s}} u_{\mathbf{s} \circ t_2}^* F_{\mathbf{s} \circ t_2} = \sum_{\mathbf{s}} \alpha_{\mathbf{s}} f_{\mathbf{s} \circ t_2}^v.$$

This proves part (1) and part (2) is an immediate consequence. We now consider part (3). Given  $x \in A_r^{\mathbb{F}} \cap (A_s^{\mathbb{F}})'$ , we let  $f_t^v x = \sum_{u \in \text{Std}_r(v)} r_u f_u^v$ . Since  $x$  commutes with  $A_s^{\mathbb{F}}$ ,

$$f_t^v x = f_t^v F_{t_1} x = f_t^v x F_{t_1} = \sum_u r_u f_u^v F_{t_1},$$

which shows that  $r_u = 0$  unless  $u_{[0,s]} = t_1$ . Thus, we can rewrite this as  $f_t x = \sum_{\mathbf{s}} r_{\mathbf{s}} f_{t_1 \circ \mathbf{s}}$ , where now the sum is over  $\mathbf{s} \in \text{Std}_{s,r}(v \setminus \lambda)$ . It remains to show that the coefficients are independent of  $t_1$ . If  $v \in \text{Std}_s(\lambda)$ , then

$$f_{v \circ t_2}^v x = f_{t_1 \circ t_2}^v E_{t_1 v}^\lambda x = f_t^v E_{t_1 v}^\lambda x = f_t^v x E_{t_1 v}^\lambda = \sum_{\mathbf{s}} r_{\mathbf{s}} f_{t_1 \circ \mathbf{s}}^v E_{t_1 v}^\lambda = \sum_{\mathbf{s}} r_{\mathbf{s}} f_{v \circ \mathbf{s}}^v,$$

where we have applied part (2). □

**3.4. Dominance triangularity and restriction rules for the Murphy basis.** For  $0 \leq s < r$ , we now show that the Murphy basis is compatible with restriction to the subalgebra  $A_s \subseteq A_r$ , and also to the subalgebra  $A_r \cap A'_s$ . This is a first step towards constructing skew cell modules in the next section.

**LEMMA 3.10.** Let  $1 \leq s < r$ ,  $v \in \widehat{A}_r$ ,  $\lambda \in \widehat{A}_s$  and  $t \in \text{Std}_{s,r}(v \setminus \lambda)$ . Suppose that  $y \in m_\lambda A_s \cap A_s^{\triangleright \lambda}$ . Then there exist coefficients  $r_z \in R$  such that

$$u_t^* y \equiv \sum_{\substack{z \in \text{Std}_r(v) \\ z_{[s,r]} \triangleright t \\ z(s) \triangleright \lambda}} r_z u_z^* \pmod{A_r^{\triangleright v}}.$$

**PROOF.** Recall that the element  $\sum_{\lambda(s) \triangleright \lambda} z_s^{\lambda(s)}$  acts as the identity on the ideal  $A_s^{\triangleright \lambda}$  and that  $y \in A_s^{\triangleright \lambda}$ . By the assumption that  $y \in m_\lambda A_s$  and  $t(s) = \lambda$ ,

$$u_{t_{[s,j]}}^* y \in m_{t(j)} A_j \subseteq A_j^{\triangleright t(j)}$$

for each  $j$  with  $s < j \leq r$ . In fact, if  $y = m_\lambda x$ , then  $u_{\mathfrak{t}(s,j)}^* y = u_{\mathfrak{t}(s,j)}^* m_\lambda x = m_{\mathfrak{t}(j)} d_{\mathfrak{t}(s,j)} x$ , using (2.4). It follows that  $u_{\mathfrak{t}}^* y = u_{\mathfrak{t}}^* y(\sum_{\lambda(j) \geq \mathfrak{t}(j)} z_j^{\lambda(j)})$ . Finally, for  $j < s$ , we can write  $1 = \sum_{\lambda(j) \in \widehat{A}_j} z_j^{\lambda(j)}$ . Arguing as in the proof of Theorem 3.5,

$$u_{\mathfrak{t}}^* y = \sum_{\substack{z \in \text{Std}_r \\ \mathfrak{z}(s,r) \triangleright \mathfrak{t} \\ \mathfrak{z}(s) \triangleright \lambda}} u_{\mathfrak{t}}^* y F_z \equiv \sum_{\substack{z \in \text{Std}_r(\nu) \\ \mathfrak{z}(s,r) \triangleright \mathfrak{t} \\ \mathfrak{z}(s) \triangleright \lambda}} u_{\mathfrak{t}}^* y F_z \pmod{(A_r^{\mathbb{F}})^{\triangleright \nu}}.$$

Now, because the range of  $F_z$  on the cell module  $\Delta_r^{\mathbb{F}}(\nu)$  is  $\mathbb{F}\{f_z\} = \mathbb{F}\{u_z^* F_z\}$ , we have  $u_{\mathfrak{t}}^* y F_z = \alpha_z u_z^* F_z$  for some  $\alpha_z \in \mathbb{F}$ . Thus

$$u_{\mathfrak{t}}^* y \equiv \sum_{\substack{z \in \text{Std}_r(\nu) \\ \mathfrak{z}(s,r) \triangleright \mathfrak{t} \\ \mathfrak{z}(s) \triangleright \lambda}} \alpha_z u_z^* F_z \pmod{(A_r^{\mathbb{F}})^{\triangleright \nu}}.$$

By dominance triangularity (Theorem 3.5),

$$u_{\mathfrak{t}}^* y = \sum_{\substack{z \in \text{Std}_r(\nu) \\ \mathfrak{z}(s,r) \triangleright \mathfrak{t} \\ \mathfrak{z}(s) \triangleright \lambda}} r_z u_z^* + y',$$

with coefficients  $r_z \in \mathbb{F}$ , and with  $y' \in (A_r^{\mathbb{F}})^{\triangleright \nu}$ . But, since  $u_{\mathfrak{t}}^* y \in A_r^R$ , we have that  $r_z \in R$  and  $y' \in A_r^{\triangleright \nu}$ . □

The following proposition is an immediate consequence of the lemma.

**PROPOSITION 3.11.** *Let  $1 \leq s < r$ ,  $\nu \in \widehat{A}_r$ ,  $\lambda \in \widehat{A}_s$  and  $\mathfrak{t} \in \text{Std}_{s,r}(\nu \setminus \lambda)$ . Let  $x \in m_\lambda A_s$  and write*

$$x = \sum_{s \in \text{Std}_s(\lambda)} \alpha_s u_s^* + y,$$

with  $y \in A_s^{\triangleright \lambda}$ . Then there exist coefficients  $r_z \in R$ , such that

$$u_{\mathfrak{t}}^* x \equiv \sum_{s \in \text{Std}_s(\lambda)} \alpha_s u_{\mathfrak{t}}^* u_s^* + \sum_{\substack{z \in \text{Std}_r(\nu) \\ \mathfrak{z}(s,r) \triangleright \mathfrak{t} \\ \mathfrak{z}(s) \triangleright \lambda}} r_z u_z^* \pmod{A_r^{\triangleright \nu}}.$$

The following corollary is a restriction rule for the Murphy-type basis of a tower of diagram algebras.

**COROLLARY 3.12.** *Let  $r > 1$ ,  $\nu \in \widehat{A}_r$  and  $\mathfrak{t} \in \text{Std}_r(\nu)$ . Let  $1 \leq s < r$  and write  $\lambda = \mathfrak{t}(s)$ . Let  $a \in A_s$  and suppose that*

$$u_{\mathfrak{t}(0,s)}^* a \equiv \sum_{s \in \text{Std}_s(\lambda)} r_s u_s^* \pmod{A_s^{\triangleright \lambda}}.$$

Then there exist coefficients  $r_z \in R$  such that

$$u_{\mathfrak{t}}^* a \equiv \sum_{s \in \text{Std}_s(\lambda)} r_s u_{\mathfrak{t}(s,r)}^* u_s^* + \sum_{\substack{z \in \text{Std}_r(\nu) \\ \mathfrak{z}(s,r) \triangleright \mathfrak{t} \\ \mathfrak{z}(s) \triangleright \lambda}} r_z u_z^* \pmod{A_r^{\triangleright \nu}}.$$

**PROOF.** Apply the previous proposition with  $x = u_{\mathfrak{t}_{[0,s]}}^* a$ . □

**REMARK 3.13.** Corollary 3.12 is an improvement of the restriction rule for ‘path bases’ obtained in [7, Proposition 2.18].

We now consider restriction to  $A_r \cap A'_s \subseteq A_r$ .

**PROPOSITION 3.14.** *Let  $0 \leq s < r$ ,  $v \in \widehat{A}_r$  and  $\mathfrak{t} \in \text{Std}_r(v)$ . Write  $\lambda = \mathfrak{t}(s)$ . Let  $x \in A_r \cap A'_s$ . Then there exist coefficients  $r_w \in R$  such that*

$$u_{\mathfrak{t}}^* x \equiv \sum_{\substack{w \in \text{Std}_r(v) \\ w_{[0,s]} \triangleright \mathfrak{t}_{[0,s]}}} r_w u_w^* \pmod{A_r^{\triangleright v}}$$

**PROOF.** Using Theorem 3.5, write  $m_{\mathfrak{t}}^v = f_{\mathfrak{t}}^v + \sum_{u \triangleright \mathfrak{t}} \alpha_u f_u^v$ , with coefficients  $\alpha_u \in \mathbb{F}$ , so that  $m_{\mathfrak{t}}^v x = f_{\mathfrak{t}}^v x + \sum_{u \triangleright \mathfrak{t}} \alpha_u f_u^v x$ . Now applying Proposition 3.9 part (3), we see that  $m_{\mathfrak{t}}^v x$  is an  $\mathbb{F}$ -linear combination of seminormal basis elements  $f_v^v$  with  $v_{[0,s]} \triangleright \mathfrak{t}_{[0,s]}$ . Using Theorem 3.5 again, each such  $f_v^v$  is an  $\mathbb{F}$ -linear combination of Murphy basis elements  $m_w^v$ , with  $w \triangleright v$ , and thus  $w_{[0,s]} \triangleright v_{[0,s]} \triangleright \mathfrak{t}_{[0,s]}$ . Therefore

$$u_{\mathfrak{t}}^* x = \sum_{\substack{w \in \text{Std}_r(v) \\ w_{[0,s]} \triangleright \mathfrak{t}_{[0,s]}}} r_w u_w^* + y,$$

with coefficients  $r_w \in \mathbb{F}$  and with  $y \in A_r^{\triangleright v} \otimes_R \mathbb{F}$ . Since  $u_{\mathfrak{t}}^* x \in A_r^R$ , the coefficients are necessarily in  $R$  and  $y \in A_r^{\triangleright v}$ . □

### 4. Jucys–Murphy elements

We recall the definition and first properties of families of Jucys–Murphy elements for diagram algebras. The action of Jucys–Murphy elements on cell modules for diagram algebras was first considered systematically in Goodman and Graber [7], motivated by work of Mathas [12]. In this section, we use Theorem 3.5 to strengthen the results of [7] by replacing the reverse lexicographic order on tableaux with the dominance order on tableaux.

The treatment of Jucys–Murphy and the seminormal basis in [7] proceeds as follows. From the definition of Jucys–Murphy elements (Definitions 4.1 and 4.3), one concludes that the Jucys–Murphy elements act triangularly with respect to reverse lexicographic order on the Murphy basis. Therefore, Jucys–Murphy elements in the sense of [7] are also Jucys–Murphy elements in the sense of Mathas [12, Definition 2.4]. When the ground ring is a field  $\mathbb{F}$  and Mathas’ separation condition [12, Definition 2.8] is satisfied—which is true for standard examples of diagram algebras over the field of fractions of the generic ground ring—then, by [12, Corollary 2.9], the algebras  $A_r^{\mathbb{F}}$  are split semisimple. Moreover, following [12, Section 3], one can define a family of orthogonal idempotents  $F_{\mathfrak{t}}'$  labelled by  $\mathfrak{t} \in \text{Std}_r$ , using interpolation formulas for the Jucys–Murphy elements. By [7, Proposition 3.11], the idempotents  $F_{\mathfrak{t}}'$  coincide with the Gelfand–Zeitlin idempotents

$F_t$  for the tower  $(A_r^{\mathbb{F}})_{r \geq 0}$ . Finally, one defines the seminormal bases as in Definition 3.2. Since the  $F_t$  are polynomials in the Jucys–Murphy elements, it follows that the transition matrix between the Murphy basis and the seminormal basis is unitriangular with respect to reverse lexicographic order.

Here we reverse the logic. We consider a tower of diagram algebras, that is, a tower  $(A_r)_{r \geq 0}$  satisfying (D1)–(D6). Then we already have the Gelfand–Zeitlin idempotents at our disposal and can define the seminormal bases as in Definition 3.2. Moreover, the transition matrix between the Murphy basis and the seminormal basis is dominance unitriangular by Theorem 3.5. In cases where Jucys–Murphy elements in the sense of Definitions 4.1 and 4.3 exist, we will show that they act diagonally on the seminormal basis. It then follows from Theorem 3.5 that the Jucys–Murphy elements act triangularly on the Murphy basis with respect to dominance order. Assuming Mathas’ separation condition holds—as it does in standard examples—one can then take up the theory in [12, Section 3], defining orthogonal idempotents by interpolation formulas using the Jucys–Murphy elements, and these must coincide with the Gelfand–Zeitlin idempotents by [7, Proposition 3.11].

For the rest of this section, assume that  $(A_r)_{r \geq 0}$  is a tower of diagram algebras over an integral domain  $R$  with field of fractions  $\mathbb{F}$ , satisfying (D1)–(D6).

**DEFINITION 4.1.** We say that a family of elements  $\{L_r \mid r \geq 1\}$ , is an *additive family of Jucys–Murphy elements* if the following conditions hold.

- (1) For all  $r \geq 1$ ,  $L_r \in A_r$ ,  $L_r$  is invariant under the involution of  $A_r$ , and  $L_r$  commutes with  $A_{r-1}$ . In particular,  $L_i L_j = L_j L_i$  for all  $1 \leq i \leq j \leq r$ .
- (2) For all  $r \geq 1$  and  $\lambda \in \widehat{A}_r$ , there exists  $d(\lambda) \in R$  such that  $L_1 + \dots + L_r$  acts as the scalar  $d(\lambda)$  on the cell module  $\Delta_r^R(\lambda)$ . For  $\lambda \in \widehat{A}_0$ , we let  $d(\lambda) = 0$ .

**EXAMPLE 4.2.** The group algebras of symmetric groups, Temperley–Lieb, Brauer, walled Brauer and partition algebras all possess additive families of Jucys–Murphy elements.

**DEFINITION 4.3.** We say that a family of elements  $\{L_r \mid r \geq 1\}$ , is a *multiplicative family of Jucys–Murphy elements* if the following conditions hold.

- (1) For all  $r \geq 1$ ,  $L_r$  is an invertible element of  $A_r$ ,  $L_r$  is invariant under the involution  $*$  and  $L_r$  commutes with  $A_{r-1}$ . In particular,  $L_i L_j = L_j L_i$  for all  $1 \leq i \leq j \leq r$ .
- (2) For all  $r \geq 1$  and  $\lambda \in \widehat{A}_r$ , there exists  $d(\lambda) \in R$  such that  $L_1 \cdots L_r$  acts as the scalar  $d(\lambda)$  on the cell module  $\Delta_r^R(\lambda)$ . For  $\lambda \in \widehat{A}_0$ , we let  $d(\lambda) = 1$ .

**EXAMPLE 4.4.** The Hecke algebras of finite type  $A$  and the Birman–Murakami–Wenzl algebra possess a multiplicative family of Jucys–Murphy elements.

**PROPOSITION 4.5.** Assume that the tower  $(A_r)_{r \geq 0}$  has additive or multiplicative Jucys–Murphy elements  $L_i$ . Then the Jucys–Murphy elements act diagonally on the seminormal bases. More precisely, there exist scalars  $\kappa_{\mu \rightarrow \lambda} \in R$  associated to edges  $\mu \rightarrow \lambda$  in  $\widehat{A}$  such that, for all  $r$ ,  $\lambda \in \widehat{A}_r$ ,  $t \in \text{Std}_r(\lambda)$  and  $i \leq r$ ,

$$f_t^\lambda L_i = \kappa_{t(i-1) \rightarrow t(i)} f_t^\lambda.$$

**PROOF.** We consider the case of additive Jucys–Murphy elements. The proof for multiplicative Jucys–Murphy elements is nearly identical. Let  $\kappa_{\mu \rightarrow \lambda} = d(\lambda) - d(\mu)$ , where  $d(\cdot)$  is as in Definition 4.1. For  $i \geq 1$  and  $\mu \in \widehat{A}_i$ , the sum  $L_1 + \dots + L_i$  acts as the scalar  $d(\mu)$  on  $\Delta_i(\mu)$ . It follows from the restriction rule for the seminormal representations (Proposition 3.9) that, for all  $r \geq i$ ,  $\lambda \in \widehat{A}_r$  and  $\mathfrak{t} \in \text{Std}_r(\lambda)$ ,  $f_{\mathfrak{t}}^\lambda(L_1 + \dots + L_i) = d(\mathfrak{t}(i))f_{\mathfrak{t}}^\lambda$ . Hence  $f_{\mathfrak{t}}^\lambda L_i = (d(\mathfrak{t}(i)) - d(\mathfrak{t}(i - 1)))f_{\mathfrak{t}}^\lambda$ .  $\square$

**NOTATION 4.6.** We will also write  $\kappa_{\mathfrak{t}}(i)$  for  $\kappa_{\mathfrak{t}(i-1) \rightarrow \mathfrak{t}(i)}$ .

**THEOREM 4.7.** Assume that the tower  $(A_r)_{r \geq 0}$  has additive or multiplicative Jucys–Murphy elements  $L_i$ . Then the Jucys–Murphy elements act triangularly with respect to dominance order on the Murphy basis of the cell modules, that is, for  $r \geq 1$ ,  $\lambda \in \widehat{A}_r$  and  $\mathfrak{t} \in \text{Std}_r(\lambda)$ ,

$$m_{\mathfrak{t}}^\lambda L_i = \kappa_{\mathfrak{t}}(i)m_{\mathfrak{t}}^\lambda + \sum_{\mathfrak{s} > \mathfrak{t}} r_{\mathfrak{s}} m_{\mathfrak{s}}^\lambda, \tag{4.1}$$

where the coefficients are in  $R$ .

**PROOF.** Using Proposition 4.5 and Theorem 3.5, one obtains (4.1) with coefficients in  $\mathbb{F}$ . But, since  $L_i \in A_r^R$ , the coefficients are necessarily in  $R$ .  $\square$

The dominance triangularity results Theorems 3.5 and 4.7 apply to the following examples.

- (1) The symmetric group algebras and the Hecke algebras of finite type A. This example is discussed at length in Section 5 below.

For the remaining examples, dominance triangularity of the Jucys–Murphy elements, and of the transition matrix from a path basis or a Murphy-type basis to the seminormal basis, are new results.

- (2) The towers of Temperley–Lieb algebras, partition algebras, Brauer algebras and Birman–Murakami–Wenzl algebras [4].
- (3) The towers of centralizer algebras on tensor space discussed in Example 2.11, [3].

### 5. Application to the Hecke algebras of the symmetric groups

The arguments of Sections 3 and 4, applied to the Hecke algebras of the symmetric groups, result in some modest simplifications of the theory of these algebras, as presented, for example, in [11].

Let  $S$  be an integral domain and  $q \in S$  a unit. The Hecke algebra  $H_r(S; q)$  is the unital  $S$ -algebra with generators  $T_1, \dots, T_{r-1}$  satisfying the braid relations and the quadratic relation  $(T_i - q)(T_i + 1) = 0$ . For any  $S$ , the specialization  $H_r(S; 1)$  is isomorphic to  $S \mathfrak{S}_r$ . The generic ground ring for the Hecke algebras is the Laurent polynomial ring  $R = \mathbb{Z}[q, q^{-1}]$ , where  $q$  is an indeterminate. Let  $\mathbb{F}$  denote  $\mathbb{Q}(q)$ , the field of fractions of  $R$ .

We will write  $H_r(q)$  for  $H_r(R; q)$ . The algebra  $H_r(q)$  has an  $R$ -basis  $\{T_w \mid w \in \mathfrak{S}_r\}$ , defined as follows. If  $w = s_{i_1} s_{i_2} \dots s_{i_l}$  is a reduced expression for  $w$  in the usual



generators of  $\mathfrak{S}_r$ , then  $T_w = T_{i_1} T_{i_2} \cdots T_{i_l}$ , independent of the reduced expression. Define  $T_w^* = T_{w^{-1}}$ ; then  $*$  is an algebra involution on  $H_r(\mathbf{q})$ .

Let  $\widehat{\mathfrak{S}}_r$  denote the set of Young diagrams or partitions of size  $r$ . *Dominance order*  $\trianglerighteq$  on  $\widehat{\mathfrak{S}}_r$  is determined by  $\lambda \trianglerighteq \mu$  if  $\sum_{i=1}^j \lambda_i \geq \sum_{i=1}^j \mu_i$  for all  $1 \leq j \leq r$ . *Young's graph* or *lattice*,  $\widehat{\mathfrak{S}}$ , is the branching diagram with vertices  $\widehat{\mathfrak{S}}_r$  on level  $r$  and a directed edge  $\lambda \rightarrow \mu$  if  $\mu$  is obtained from  $\lambda$  by adding one box. We can identify *standard tableaux* of shape  $\lambda$  with directed paths on  $\widehat{\mathfrak{S}}$  from  $\emptyset$  to  $\lambda$ . For  $\lambda \in \widehat{\mathfrak{S}}_r$ , denote the set of standard tableaux of shape  $\lambda$  by  $\text{Std}_r(\lambda)$ .

Let  $\lambda$  be a Young diagram and let  $\alpha = (i, j)$  be a box in  $\lambda$ . The *content* of  $\alpha$  is  $c(\alpha) = j - i$ . If  $\mathfrak{t} \in \text{Std}_r(\nu)$  and  $k \leq r$ , define  $c_{\mathfrak{t}}(k) = c(\alpha)$ , where  $\alpha = \mathfrak{t}(k) \setminus \mathfrak{t}(k-1)$ . When  $\mathfrak{t}$  is regarded as an array with the boxes of the Young diagram  $\nu$  filled with the numbers from one to  $r$ ,  $c_{\mathfrak{t}}(k)$  is the content of the box containing the entry  $k$ .

For  $\lambda$  a partition of  $r$ , let  $x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} T_w$ . Let  $\mathfrak{t}^\lambda$  be the row reading tableaux of shape  $\lambda$ . For any  $\lambda$ -tableau  $\mathfrak{t}$ , there is a unique  $w(\mathfrak{t}) \in \mathfrak{S}_r$  with  $\mathfrak{t} = \mathfrak{t}^\lambda w(\mathfrak{t})$ . For  $\lambda$  a partition of  $r$  and  $\mathfrak{s}, \mathfrak{t} \in \text{Std}_r(\lambda)$ , let

$$x_{\mathfrak{st}}^\lambda = (T_{w(\mathfrak{s})})^* x_\lambda T_{w(\mathfrak{t})}.$$

**THEOREM 5.1 (The Murphy basis, [15]).** *The set  $\mathcal{X} = \{x_{\mathfrak{st}}^\lambda \mid \lambda \in \widehat{\mathfrak{S}}_r \text{ and } \mathfrak{s}, \mathfrak{t} \in \text{Std}_r(\lambda)\}$  is a cellular basis of  $H_r(\mathbf{R}; \mathbf{q})$ , with respect to the involution  $*$  and the partially ordered set  $(\widehat{\mathfrak{S}}_r, \trianglerighteq)$ .*

We let  $x_{\mathfrak{t}}^\lambda$  denote the basis element of the cell module  $\Delta_r(\lambda)$  corresponding to  $\mathfrak{t} \in \text{Std}_r(\lambda)$ ,  $x_{\mathfrak{t}}^\lambda = x_\lambda T_{w(\mathfrak{t})} + H_r^{\triangleright \lambda}$ . If  $1 \leq a \leq i$ , define

$$T_{a,i} = T_a T_{a+1} \cdots T_{i-1},$$

and  $T_{i,a} = T_{a,i}^*$ . If  $\lambda \vdash i - 1$  and  $\mu \vdash i$ , with  $\mu = \lambda \cup \{(j, \mu_j)\}$ , let  $a = \sum_{k=1}^j \mu_k$ . Define

$$d_{\lambda \rightarrow \mu} = T_{a,i} \quad \text{and} \quad u_{\lambda \rightarrow \mu} = T_{i,a} \sum_{k=0}^{\lambda_j} T_{a,a-k}.$$

Given a path  $\mathfrak{t} \in \text{Std}_r(\nu)$ ,

$$\emptyset = \mathfrak{t}(0) \rightarrow \mathfrak{t}(1) \rightarrow \mathfrak{t}(2) \rightarrow \cdots \rightarrow \mathfrak{t}(r-1) \rightarrow \mathfrak{t}(r) = \nu,$$

define

$$d_{\mathfrak{t}} = d_{\mathfrak{t}(r-1) \rightarrow \mathfrak{t}(r)} d_{\mathfrak{t}(r-2) \rightarrow \mathfrak{t}(r-1)} \cdots d_{\mathfrak{t}(0) \rightarrow \mathfrak{t}(1)}.$$

**PROPOSITION 5.2 [4].**

- (1) For  $\mathfrak{t} \in \text{Std}_r(\nu)$ ,  $d_{\mathfrak{t}} = T_{w(\mathfrak{t})}$ .
- (2) For  $\lambda \rightarrow \mu$  in  $\widehat{\mathfrak{S}}$ ,  $x_\mu d_{\lambda \rightarrow \mu} = u_{\lambda \rightarrow \mu}^* x_\lambda$ .

Thus the Murphy cellular basis is recovered from the ordered products of branching factors:  $x_{\mathfrak{st}}^\lambda = (d_{\mathfrak{s}})^* x_\lambda d_{\mathfrak{t}}$ . Moreover, Remark 2.13 regarding factorization of representatives of the Murphy basis of cell modules applies to the Hecke algebra. We want to stress that Proposition 5.2 is established by computation and does not rely on

the connection, established in [4], between the branching factors  $d_{\lambda \rightarrow \mu}$  and  $u_{\lambda \rightarrow \mu}$  and cell filtrations of restricted and induced cell modules.

Next, we recall the Jucys–Murphy elements for the Hecke algebras (see [11, Section 3.3 and Exercise 6, page 49]). The Jucys–Murphy elements in  $H_n(\mathbf{q})$  are defined by

$$L_1 = 1 \quad \text{and} \quad L_k = \mathbf{q}^{1-k} T_{k-1} \cdots T_1 T_1 \cdots T_{k-1} \quad \text{for } k > 1.$$

**PROPOSITION 5.3.**

- (1)  $L_1 L_2 \cdots L_r$  is in the center of  $H_r(\mathbf{q})$ .
- (2) The elements  $L_k$  are multiplicative Jucys–Murphy elements in the sense of Definition 4.3.

See [11, Section 3.3] for the proof. Again, the proof is computational and does not depend on deeper results on the Hecke algebras. We will now list several properties of the Hecke algebras and afterwards sketch two logical routes through this material, both using the ideas of Sections 3 and 4.

- (H1) The Hecke algebras  $H_r(\mathbb{F}, \mathbf{q})$  are split semisimple.
- (H2) The branching diagram for the sequence  $(H_r(\mathbb{F}, \mathbf{q}))_{r \geq 0}$  of split semisimple algebras is Young’s lattice.
- (H3) The sequence of Hecke algebras over  $R$  is restriction-coherent, with  $\Delta_{r-1}^R(\lambda)$  appearing as a subquotient of  $\text{Res}_{H_{r-1}(\mathbf{q})}^{H_r(\mathbf{q})}(\Delta_r^R(\mu))$  if and only if  $\lambda \subset \mu$ .
- (H4) More precisely, statement (3) of Theorem 2.9 holds.

Since the sequence  $(H_r(\mathbb{F}, \mathbf{q}))_{r \geq 0}$  is a multiplicity free sequence of split semisimple algebras over  $\mathbb{F}$ , one can define Gelfand–Zeitlin idempotents as in Section 3.1 and seminormal bases as in Definition 3.2 using the Gelfand–Zeitlin idempotents and the Murphy basis.

- (H5) The set  $\{f_t^\lambda \mid t \in \text{Std}_r(\lambda)\}$  is a basis of  $\Delta_r^{\mathbb{F}}(\lambda)$  and the transition matrix between this basis and the Murphy basis is dominance unitriangular.
- (H6) The Jucys–Murphy elements act diagonally on the seminormal basis,  $f_t^\lambda L_k = \mathbf{q}^{c_t(k)} f_t^\lambda$ .
- (H7) The Jucys–Murphy elements act triangularly on the Murphy basis,

$$x_t^\lambda L_k = \mathbf{q}^{c_t(k)} x_t^\lambda + \sum_{s > t} r_s x_s^\lambda.$$

We now sketch two logical paths through statements (H1)–(H7). Of course, there are many logical arrangements of this material, and these are just two possibilities.

*First path.* One can first establish that the Hecke algebras  $H_r(\mathbb{F}, \mathbf{q})$  are split semisimple. One easy way to do this is as follows. Let  $\langle \cdot, \cdot \rangle$  denote the bilinear form on each cell module arising from the Murphy basis. Let  $\phi_\lambda$  be the determinant of the Gram matrix  $[\langle m_s^\lambda, m_t^\lambda \rangle]_{s,t}$ . Then  $\phi_\lambda \in R = \mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]$ . Since the specialization of the Hecke algebra at  $\mathbf{q} = 1$  is the symmetric group algebra over  $\mathbb{Q}$ , which is semisimple, it follows that

$\phi_\lambda(1) \neq 0$  and hence  $\phi_\lambda \neq 0$ . Now it follows from the general theory of cellular algebras that  $H_r(\mathbb{F}, \mathbf{q})$  is split semisimple (see [11, Corollary 2.21]).

The next step, which is more substantial, is to show that (H3) holds. This is proved in [8] or [13]. This implies that (H2) holds as well, using [6, Lemma 2.2], which is elementary.

Taking into account Proposition 5.2, we now have verified axioms (D1) through (D6) for the Hecke algebras, as well as statements (1) and (2) of Theorem 2.9. A subtle point here is that we have not verified and do not need to verify at this point that the  $d$ -branching coefficients actually arise from the cell filtrations of restricted cell modules, that is, that statement (3) of Theorem 2.9 holds.

We are now entitled to plug into the arguments of Sections 3 and 4, which give us the conclusions (H5) through (H7), but without revealing that the eigenvalues of the Jucys–Murphy elements are  $\mathbf{q}^{c_i(k)}$ . This additional information must come from an additional analysis, for example, by extending the analysis of Okounkov and Vershik [16] to the Hecke algebras. Moreover, we obtain (H4) from Corollary 3.12.

*Second path.* The second path follows the analysis in [11, Ch. 3] and its abstraction in [12]. Start with [11, Theorem 3.32], which proves (H7). This implies that the Jucys–Murphy elements  $L_k$  are Jucys–Murphy elements in the sense of [12] and, moreover, satisfy the ‘separating condition’ of [12, Section 3] over  $\mathbb{F}$ . One can therefore define idempotents  $F_t$ , indexed by standard tableaux using interpolation formulas involving the Jucys–Murphy elements as in [11, Section 3.3] or [12, Section 3], and the seminormal basis of the cell modules by  $f_t^\lambda = x_t^\lambda F_t$ . From the general theory in [12, Section 3], one obtains (H5) and (H6). Moreover, the separating condition also implies (H1). The restriction rule (H2) results from actually computing the seminormal representations as in [11, Theorem 3.36]. We are now left with the task of verifying (H3) and (H4).

Note that we have (D1)–(D4) as well as (D6) at our disposal in addition to statements (1) and (2) of Theorem 2.9. Moreover, by [7, Proposition 3.11], the idempotents  $F_t$  obtained from the Jucys–Murphy elements have to coincide with the Gelfand–Zeitlin idempotents. This is all we need to follow through the arguments of Section 3, and we end up with Corollary 3.12, which implies (H3) and (H4).

**REMARK 5.4.** The net result of this discussion is that if one uses the result from the literature on the restriction-coherence of the tower of Hecke algebras, one can avoid some of the work involved with showing dominance triangularity of the Jucys–Murphy elements; on the other hand, if one uses the results on dominance triangularity, one can avoid some of the work involved in proving restriction-coherence (which was only recently proved in [8, 13]).

## 6. Skew cell modules for diagram algebras

In this section, we construct skew cell modules for diagram algebras and provide integral bases of these modules indexed by skew tableaux. We begin by constructing

skew cell modules for  $A_r \cap A'_s$  when  $0 \leq s \leq r$ , and afterwards we introduce a final axiom which allows us to view these modules as  $A_{r-s}$  modules.

Let  $0 \leq s < r$ ,  $\nu \in \widehat{A}_r$  and  $\lambda \in \widehat{A}_s$ . Let  $\mathfrak{t}^\lambda \in \text{Std}_s(\lambda)$  be maximal in  $\text{Std}_s(\lambda)$  with respect to the dominance order on paths. Define

$$\Delta(\nu; \triangleright \lambda) = \text{Span}_R\{m_{\mathfrak{t}}^\nu \mid \mathfrak{t} \in \text{Std}_r(\nu) \text{ and } \mathfrak{t}(s) \triangleright \lambda\}$$

and

$$\Delta(\nu; \mathfrak{t}^\lambda) = \text{Span}_R\{m_{\mathfrak{t}^\lambda \circ \mathfrak{t}}^\nu \mid \mathfrak{t} \in \text{Std}_{r-s}(\nu \setminus \lambda)\},$$

which are both  $R$ -submodules of  $\Delta_r^R(\nu)$ .

**LEMMA 6.1.**  $\Delta(\nu; \triangleright \lambda)$  and  $\Delta(\nu; \mathfrak{t}^\lambda) + \Delta(\nu; \triangleright \lambda)$  are  $A_r \cap A'_s$  submodules of  $\Delta_r^R(\nu)$ .

**PROOF.** This follows from Proposition 3.14 and maximality of  $\mathfrak{t}^\lambda$ . □

Let  $\Delta(\nu \setminus \lambda)$  denote the  $A_r \cap A'_s$  module

$$\Delta(\nu \setminus \lambda) = (\Delta(\nu; \mathfrak{t}^\lambda) + \Delta(\nu; \triangleright \lambda)) / \Delta(\nu; \triangleright \lambda).$$

**LEMMA 6.2.**

- (1)  $\Delta(\nu \setminus \lambda)$  is, up to isomorphism, independent of the choice of the maximal element  $\mathfrak{t}^\lambda$ .
- (2)  $\Delta(\nu; \mathfrak{t}^\lambda)F_{\mathfrak{t}^\lambda} \subseteq \Delta_r^{\mathbb{F}}(\nu)$  is an  $A_r \cap A'_s$  submodule, and  $\Delta(\nu \setminus \lambda) \cong \Delta(\nu; \mathfrak{t}^\lambda)F_{\mathfrak{t}^\lambda}$ .
- (3) For any choice of  $\mathfrak{s} \in \text{Std}_s(\lambda)$ ,

$$\Delta^{\mathbb{F}}(\nu \setminus \lambda) := \Delta(\nu \setminus \lambda) \otimes_R \mathbb{F} \cong \Delta_r^{\mathbb{F}}(\nu)F_{\mathfrak{s}}$$

as  $A_r^{\mathbb{F}} \cap (A'_s)^{\mathbb{F}}$ -modules.

**PROOF.** The first statement in part (2) follows from Lemma 6.1 because  $\Delta(\nu; \triangleright \lambda)F_{\mathfrak{t}^\lambda} = 0$ . Now there is an  $A_r \cap A'_s$  module homomorphism  $m \mapsto mF_{\mathfrak{t}^\lambda}$  from  $\Delta(\nu; \mathfrak{t}^\lambda) + \Delta(\nu; \triangleright \lambda)$  to  $\Delta(\nu; \mathfrak{t}^\lambda)F_{\mathfrak{t}^\lambda}$  with kernel  $\Delta(\nu; \triangleright \lambda)$ , which gives the isomorphism in part (2). If  $\mathfrak{t}_1$  and  $\mathfrak{t}_2$  are two maximal elements in  $\text{Std}_s(\lambda)$ , then right multiplication by  $E_{\mathfrak{t}_1 \mathfrak{t}_2}^\lambda$  is an  $A_r \cap A'_s$ -module isomorphism from  $\Delta(\nu; \mathfrak{t}_1)F_{\mathfrak{t}_1}$  to  $\Delta(\nu; \mathfrak{t}_2)F_{\mathfrak{t}_2}$ , which proves part (1).

For part (3), using Theorem 3.5 and Proposition 3.9 part (3), first note that given any  $\mathfrak{s}, \bar{\mathfrak{s}} \in \text{Std}_s(\lambda)$ , we have that  $\Delta_r^{\mathbb{F}}(\nu)F_{\mathfrak{s}} \cong \Delta_r^{\mathbb{F}}(\nu)F_{\bar{\mathfrak{s}}}$  as  $R$ -modules, with the isomorphism realized by right multiplication by  $E_{\mathfrak{s}\bar{\mathfrak{s}}}^\lambda$ . This map preserves the  $A_r^{\mathbb{F}} \cap (A'_s)^{\mathbb{F}}$ -module structure, since  $A_r^{\mathbb{F}} \cap (A'_s)^{\mathbb{F}}$  commutes with  $E_{\mathfrak{s}\bar{\mathfrak{s}}}^\lambda$ . In particular, we can set  $\mathfrak{s} = \mathfrak{t}^\lambda$  without loss of generality. Now, it follows from Theorem 3.5 that

$$\Delta^{\mathbb{F}}(\nu; \triangleright \lambda) := \Delta(\nu; \triangleright \lambda) \otimes_R \mathbb{F} = \text{Span}_{\mathbb{F}}\{f_{\mathfrak{t}}^\nu \mid \mathfrak{t} \in \text{Std}_r(\nu) \text{ and } \mathfrak{t}(s) \triangleright \lambda\}$$

and

$$\Delta(\nu; \mathfrak{t}^\lambda) \otimes_R \mathbb{F} + \Delta^{\mathbb{F}}(\nu; \triangleright \lambda) = \text{Span}_{\mathbb{F}}\{f_{\mathfrak{t}^\lambda \circ \mathfrak{t}}^\nu \mid \mathfrak{t} \in \text{Std}_{r-s}(\nu \setminus \lambda)\} + \Delta^{\mathbb{F}}(\nu; \triangleright \lambda).$$

Arguing as for part (2),

$$\Delta^{\mathbb{F}}(\nu \setminus \lambda) \cong \text{Span}_{\mathbb{F}}\{f_{\mathfrak{t}^\lambda \circ \mathfrak{t}}^\nu \mid \mathfrak{t} \in \text{Std}_{r-s}(\nu \setminus \lambda)\}F_{\mathfrak{t}^\lambda}.$$

Now one easily verifies that  $\text{Span}_{\mathbb{F}}\{f_{\mathfrak{t}^\lambda \circ \mathfrak{t}}^\nu \mid \mathfrak{t} \in \text{Std}_{r-s}(\nu \setminus \lambda)\}F_{\mathfrak{t}^\lambda} = \Delta_r^{\mathbb{F}}(\nu)F_{\mathfrak{t}^\lambda}$ . □

Now we introduce one final axiom for towers of diagram algebras, which allows us to regard  $\Delta(\nu \setminus \lambda)$  as an  $A_{r-s}$ -module.

(D7) There is an automorphism,  $f_r$ , of order two of each  $A_r$  such that, for each  $s$  with  $0 \leq s \leq r$ ,  $f_r(A_{r-s})$  commutes with  $A_s \subset A_r$ .

Using this final axiom, we can define a homomorphism  $\varphi_{r,s} : A_s \otimes_R A_{r-s} \rightarrow A_r$  by

$$\varphi_{r,s}(a \otimes b) = af_r(f_{r-s}(b)),$$

where both  $A_s$  and  $A_{r-s}$  are regarded as subalgebras of  $A_r$  via the usual embeddings. We can restrict any  $A_r$ -module to  $A_s \otimes_R A_{r-s}$ , by composing with  $\varphi_{r,s}$ .

**REMARK 6.3.** Condition (D7) is satisfied by the group algebras of symmetric groups, the Hecke algebras, and the Brauer, BMW, partition and Temperley–Lieb algebras. For each of these examples, the involution  $f_r$  is given by flipping a diagram or tangle through its vertical axis (whereas  $*$  is given by flipping a diagram through its horizontal axis). Also, in each of these genuine diagram or tangle algebras, one can define a tensor product operation, that is, a homomorphism from  $A_s \otimes_R A_{r-s}$  to  $A_r$ , which, on the level of diagrams or tangles, is just placing diagrams side by side. Taking  $f_r$  to be the flip through a vertical axis, the homomorphism  $\varphi_{r,s}$  defined above agrees with the homomorphism determined by placing diagrams side by side. Note that  $f_r \circ f_{r-s}$  is the shift operation on  $A_{r-s}$  determined on the level of diagrams by adding  $s$  vertical strands to the left of a diagram in  $A_{r-s}$ .

Denote by  $\Delta_{r-s}(\nu \setminus \lambda)$  the  $A_r \cap A'_s$  module  $\Delta(\nu \setminus \lambda)$ , when regarded as an  $A_{r-s}$ -module by composing with the homomorphism  $f_r \circ f_{r-s} : A_{r-s} \rightarrow A_r \cap A'_s$ .

**DEFINITION 6.4.** Let  $(A_r)_{r \geq 0}$  denote a tower of algebras satisfying conditions (D1)–(D7). Given  $\lambda \in \widehat{A}_s$  and  $\nu \in \widehat{A}_r$ , the  $A_{r-s}$ -module  $\Delta_{r-s}^R(\nu \setminus \lambda)$  described above is called the *skew cell module* associated to  $\lambda$  and  $\nu$ .

**REMARK 6.5.** The definition of skew cell modules generalizes the definition of skew Specht modules for the tower of symmetric group algebras  $(\mathbb{Z}\mathfrak{S}_r)_{r \geq 0}$ . The skew Specht module  $S^{\nu \setminus \lambda}$  for a skew shape  $\nu \setminus \lambda$  is defined as the span of polytabloids of shape  $\nu \setminus \lambda$  (see [10, Section 4] for the definition of polytabloids and [19] for the definition of skew Specht modules). Let us regard  $S^{\nu \setminus \lambda}$  as an  $\mathfrak{S}_{\{s+1, \dots, r\}}$ -module, where  $|\lambda| = s$  and  $|\nu| = r$ . Let  $t_\lambda$  be the column reading standard tableau of shape  $\lambda$ , so  $t_\lambda$  is the unique maximal tableau in column dominance order  $\triangleright_{\text{col}}$  on standard tableaux of shape  $\lambda$ . Consider the following subsets of the Specht module  $S^\nu$ ,

$$S(\nu; \triangleright_{\text{col}}\lambda) = \text{Span}_{\mathbb{Z}}\{e_t \mid t \in \text{Std}_r(\nu) \text{ and } t(s) \triangleright_{\text{col}} \lambda\}$$

and

$$S(\nu; t_\lambda) = \text{Span}_{\mathbb{Z}}\{e_{t_\lambda \circ s} \mid s \in \text{Std}_{r-s}(\nu \setminus \lambda)\},$$

where  $e_t$  denotes a polytabloid. Using the Garnir relations [10], one can verify that  $S(\nu; \triangleright_{\text{col}}\lambda)$  and  $S(\nu; t_\lambda) + S(\nu; \triangleright_{\text{col}}\lambda)$  are  $\mathfrak{S}_{\{s+1, \dots, r\}}$  submodules of  $S^\nu$ , and the quotient

of these modules is isomorphic to the skew cell module  $S^{\nu \setminus \lambda}$ . A more general statement is proved in [19, Theorem 3.1]. Modulo the identification of cell modules of  $\mathfrak{S}_r$  with classical Specht modules—which involves both transpose of diagrams and twisting by the automorphism  $s_i \mapsto -s_i$ ; see [15, Section 5]—this shows that the classical skew Specht modules agree with the skew cell modules defined here.

**PROPOSITION 6.6.** *Given  $\lambda \in \widehat{A}_s$ ,  $\nu \in \widehat{A}_r$  and an  $A_{r-s}^{\mathbb{F}}$ -module  $M$ ,*

$$\text{Hom}_{A_s^{\mathbb{F}} \otimes A_{r-s}^{\mathbb{F}}}(\Delta_s^{\mathbb{F}}(\lambda) \otimes_{\mathbb{F}} M, \text{Res}_{A_s^{\mathbb{F}} \otimes A_{r-s}^{\mathbb{F}}}^{A_r^{\mathbb{F}}}(\Delta_r^{\mathbb{F}}(\nu))) \cong \text{Hom}_{A_{r-s}^{\mathbb{F}}}(M, \Delta_{r-s}^{\mathbb{F}}(\nu \setminus \lambda)).$$

**PROOF.** Identify  $\Delta_{r-s}^{\mathbb{F}}(\nu \setminus \lambda)$  with  $\Delta_r^{\mathbb{F}}(\nu)F_{\uparrow \lambda}$ , using Lemma 6.2 part (3). For  $\varphi \in \text{Hom}_{A_s^{\mathbb{F}} \otimes A_{r-s}^{\mathbb{F}}}(\Delta_s^{\mathbb{F}}(\lambda) \otimes_{\mathbb{F}} M, \text{Res}_{A_s^{\mathbb{F}} \otimes A_{r-s}^{\mathbb{F}}}^{A_r^{\mathbb{F}}}(\Delta_r^{\mathbb{F}}(\nu)))$ , define  $\bar{\varphi}$  by

$$\bar{\varphi}(m) = \varphi(f_{\uparrow \lambda}^{\lambda} \otimes m) = \varphi(f_{\uparrow \lambda}^{\lambda} F_{\uparrow \lambda} \otimes m) = \varphi(f_{\uparrow \lambda}^{\lambda} \otimes m)F_{\uparrow \lambda}.$$

It follows that  $\bar{\varphi} \in \text{Hom}_{A_{r-s}^{\mathbb{F}}}(M, \Delta_{r-s}^{\mathbb{F}}(\nu \setminus \lambda))$ . We have to check that the map  $\varphi \mapsto \bar{\varphi}$  is an isomorphism. For injectivity, suppose that  $\bar{\varphi} = 0$ . Then, for all  $m \in M$  and all  $x \in A_s^{\mathbb{F}}$ ,  $\varphi(f_{\uparrow \lambda}^{\lambda} x \otimes m) = \bar{\varphi}(m)x = 0$ . Hence  $\varphi = 0$ , since  $f_{\uparrow \lambda}^{\lambda} A_s^{\mathbb{F}} = \Delta_s^{\mathbb{F}}(\lambda)$ . For surjectivity, let  $\psi \in \text{Hom}_{A_{r-s}^{\mathbb{F}}}(M, \Delta_{r-s}^{\mathbb{F}}(\nu)F_{\uparrow \lambda})$ . Define  $\varphi$  by  $\varphi(f_{\uparrow \lambda}^{\lambda} x \otimes m) = \psi(m)x = \psi(m)F_{\uparrow \lambda} x$  for  $x \in A_s^{\mathbb{F}}$ . Then  $\varphi$  is well defined because  $f_{\uparrow \lambda}^{\lambda} x = 0 \Leftrightarrow F_{\uparrow \lambda} x = 0$ . Now one can easily check that  $\varphi$  is an  $A_s^{\mathbb{F}} \otimes A_{r-s}^{\mathbb{F}}$ -homomorphism and that  $\bar{\varphi} = \psi$ . □

Finally, let  $(A_r)_{r \geq 0}$  denote a tower of algebras satisfying (D1)–(D7). Let  $\lambda \in \widehat{A}_s$ ,  $\nu \in \widehat{A}_r$ ,  $\mu \in \widehat{A}_{r-s}$  and define associated multiplicities

$$A_{\lambda, \mu}^{\nu} = \dim_{\mathbb{F}} \text{Hom}_{A_{r-s}^{\mathbb{F}}}(m_{\mu} A_{r-s}^{\mathbb{F}}, \Delta_{r-s}^{\mathbb{F}}(\nu \setminus \lambda)), \tag{6.1}$$

$$a_{\lambda, \mu}^{\nu} = \dim_{\mathbb{F}} \text{Hom}_{A_{r-s}^{\mathbb{F}}}(\Delta_{r-s}^{\mathbb{F}}(\mu), \Delta_{r-s}^{\mathbb{F}}(\nu \setminus \lambda)). \tag{6.2}$$

We recall that  $m_{\mu} A_{r-s} / (m_{\mu} A_{r-s} \cap A_{r-s}^{\triangleright \mu})$  is isomorphic to the cell module  $\Delta_{r-s}(\mu)$  and therefore  $a_{\lambda, \mu}^{\nu} \leq A_{\lambda, \mu}^{\nu}$ , by definition.

**REMARK 6.7.** Consider the tower of the group algebras of symmetric groups  $(\mathfrak{S}_r)_{r \geq 0}$ . Recall that a *partition*  $\mu$  is defined to be a finite weakly decreasing sequence of nonnegative integers. We define the *degree* of the partition  $\mu$  to be the sum,  $|\mu|$ , over all nonzero terms in this sequence. Recall that  $\widehat{\mathfrak{S}}_{r-s}$  is the set of partitions of degree  $r - s$  (see [4]). Given  $\mu$  a partition of  $r - s$ , the module  $m_{\mu} \mathfrak{S}_{r-s}^{\mathbb{F}}$  is isomorphic to the so-called Young permutation module, that is, the module obtained by induction from the subgroup  $\mathfrak{S}_{\mu}$  which stabilizes the set  $\{1, \dots, \mu_1\} \times \{\mu_1 + 1, \dots, \mu_1 + \mu_2\} \times \dots \subseteq \{1, \dots, r - s\}$ . Therefore, the coefficients defined in (6.1) and (6.2) above are the skew-Kostka and Littlewood–Richardson coefficients, respectively [20, pages 311 and 338].

**6.1. The stable Kronecker coefficients.** Given  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\ell})$  a partition and  $n \in \mathbb{N}$  sufficiently large, we set

$$\lambda_{[n]} = (n - |\lambda|, \lambda_1, \lambda_2, \dots, \lambda_{\ell}).$$

Given  $n \in \mathbb{N}$ , we recall that  $\mathfrak{S}_n$  denotes the symmetric group on  $n$  letters. The tower of algebras  $(\mathbb{Z}\mathfrak{S}_n)_{n \geq 0}$  satisfy conditions (D1)–(D7). Given  $r \in \frac{1}{2}\mathbb{N}$ , we let  $P_{2r}(n)$  denote the partition algebra on  $r$  strands with parameter  $n \in \mathbb{N}$ . The tower of algebras  $(P_r(n))_{r \geq 0}$  satisfy conditions (D1)–(D7). The representation theories of the symmetric groups and partition algebras are intimately related via a generalization of classical Schur–Weyl duality. Through this duality, we obtain the following theorem.

**THEOREM 6.8.** *Let  $\lambda$  be a partition of degree  $s$ , let  $\mu$  be a partition of degree  $r - s$  and let  $\nu$  be a partition of degree less than or equal to  $r$ .*

$$\begin{aligned} & \text{Hom}_{\mathbb{Q}\mathfrak{S}_n}(\Delta_{\mathbb{Q}\mathfrak{S}_n}(\lambda_{[n]}) \otimes \Delta_{\mathbb{Q}\mathfrak{S}_n}(\mu_{[n]}), \Delta_{\mathbb{Q}\mathfrak{S}_n}(\nu_{[n]})) \\ & \cong \text{Hom}_{P_{r-s}^{\mathbb{Q}}(n)}(\Delta_{P_{r-s}^{\mathbb{Q}}(n)}(\mu), \Delta_{P_{r-s}^{\mathbb{Q}}(n)}(\nu \setminus \lambda)), \end{aligned}$$

for  $n$  sufficiently large ( $n \geq 2r$  will suffice).

**PROOF.** This follows immediately from Proposition 6.6 and [2, Corollary 3.4]. □

Given  $\lambda \vdash r - s, \mu \vdash s$  and  $\nu \vdash r$  and  $n \geq 2r$ , we are interested in the multiplicities

$$\begin{aligned} P_{\lambda,\mu}^{\nu} &= \dim_{\mathbb{Q}} \text{Hom}_{P_{r-s}^{\mathbb{Q}}(n)}(m_{\mu} P_{r-s}^{\mathbb{Q}}(n), \Delta_{r-s}^{\mathbb{Q}}(\nu \setminus \lambda)), \\ p_{\lambda,\mu}^{\nu} &= \dim_{\mathbb{Q}} \text{Hom}_{P_{r-s}^{\mathbb{Q}}(n)}(\Delta_{r-s}^{\mathbb{Q}}(\mu), \Delta_{r-s}^{\mathbb{Q}}(\nu \setminus \lambda)) \end{aligned}$$

for a cell quasi-idempotent  $m_{\mu} \in P_{r-s}^{\mathbb{Q}}(n)$ . By Theorem 6.8 and [2], the coefficients  $p_{\lambda,\mu}^{\nu}$  are equal to the *stable Kronecker coefficients*. These coefficients have been described as ‘perhaps the most challenging, deep and mysterious objects in algebraic combinatorics’ [18]. On the other hand, the coefficients  $P_{\lambda,\mu}^{\nu}$  do not seem to have been studied anywhere in the literature. Motivated by the classical case, we ask the following questions.

- Can one interpret the coefficients  $P_{\lambda,\mu}^{\nu}$  and  $p_{\lambda,\mu}^{\nu}$  in terms of the combinatorics of skew tableaux for the partition algebra?
- Do there exist natural generalizations of the *semistandard* and *lattice permutation* conditions in this setting?
- Do the coefficients  $P_{\lambda,\mu}^{\nu}$  provide a first step towards understanding the stable Kronecker coefficients  $p_{\lambda,\mu}^{\nu}$ ?

The first two authors shall address these questions in an upcoming series of papers with Maud De Visscher. In particular, we use the above interpretation to provide a positive combinatorial description of the stable Kronecker coefficients labelled by an infinite family of triples of partitions (including the Littlewood–Richardson coefficients and Kronecker coefficients indexed by two two-row partitions as important examples).

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