BULL. AUSTRAL. MATH. SOC. 54C08, 54C60
VOL. 37 (1988) [423–428]

QUASI-CONTINUITY WITH RESPECT TO SEMI-REGULARISATION TOPOLOGY

STANISŁAW PSYK

This paper gives some sufficient conditions under which upper, lower or both upper and lower quasi-continuity of multifunction in the process of semi-regularisation of a topological space are preserved. Analogous results for continuous maps are true.

A subset A of a topological space X is said to be *semi open* if there exists an open set U such that $U \subseteq A \subseteq \overline{U}$ [5]. It is easy to see that A is semi open if and only if $A \subseteq \overline{\text{Int} A}$. A subset B of a topological space (X, τ) is called *regular open* if $B = \text{Int}\overline{B}$ [2]. It is well known that the family of all regular open sets forms a base for a smaller topology τ_s on X, called the *semi-regularisation* of τ. The topological space (X, τ) is called *semi-regular* if $\tau = \tau_s$.

In the class of all topologies which have the same families of semi open sets as the topology τ, there exists the finest topology τ_α [7, Corollary 1]. All topologies from the above class determine the same families of regular open sets [7, Proposition 6], so they have the same semi-regularisations, but the topology τ_α and its semi-regularisation can have different families of semi open sets (see [7] or the topology τ in Example 1).

The symbols $\overline{A}^i$, $\text{Int}_i A$ are used to denote closure and interior of the set A in the space (X, τ_i). The symbol "i" may be replaced by "α" or "s" or may be omitted for topological spaces (X, τ_α), (X, τ_s) or (X, τ) respectively.

The topological spaces are not assumed to satisfy any separation axioms, and regularity does not include the T_1 axiom. All notions and symbols which are not defined in this paper are used as in Engelking [2].

Let F be a multivalued map which to each point $x \in X$ assigns a non-empty subset $F(x)$ of a topological space Y (for simplicity we will write $F\colon X \to Y$). For any set $A \subseteq X$, $B \subseteq Y$ we will write [1]

$$F(A) = \cup\{F(x)\colon x \in A\},$$
$$F^+(B) = \{x \in X : F(x) \subseteq B\},$$
$$F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

Received 13 August 1987

Copyright Clearance Centre, Inc. Serial-fee code: 0004-9729/88 $A2.00+0.00.

https://doi.org/10.1017/S0004972700027040 Published online by Cambridge University Press

A multivalued map $F\colon X \to Y$ is said to be:
- *upper (respectively lower) quasi-continuous at a point* $x_0 \in X$ if for each open set $V \subseteq Y$ such that $F(x_0) \subseteq V$ (respectively $F(x_0) \cap V \neq \emptyset$) and for each open neighbourhood U of x_0 there exists an open non-empty set $U_1 \subseteq U$ such that $F(x) \subset V$ (respectively $F(x) \cap V \neq \emptyset$) for each $x \in U_1$ [**9**],
- *upper (respectively lower) almost continuous at a point* $_0 \in X$ if for each open set $V \subseteq Y$ such that $F(x_0) \subseteq V$ (respectively $F(x_0) \cap V \neq \emptyset$) we have $x_0 \in \operatorname{Int} \overline{F^+(V)}$ (respectively $x_0 \in \operatorname{Int} \overline{F^-(V)}$) [**10, 11**].

A multivalued map is said to be *upper* or *lower quasi-continuous (almost continuous)* if it has this property at each point.

In the sequel we will use the following equivalent conditions:
- a multivalued map $F\colon X \to Y$ is *upper (respectively lower) quasi-continuous* if and only if for any open set $V \subseteq Y$ the set $F^+(V)$ (respectively $F^-(V)$) is semi open,
- a multivalued map $F\colon X \to Y$ is *upper (respectively lower) almost continuous* if and only if for any open set $V \subseteq Y$ we have $F^+(V) \subseteq \operatorname{Int} \overline{F^+(V)}$ (respectively $F^-(V) \subseteq \operatorname{Int} \overline{F^-(V)}$) [**10**].

Any single-valued map $f\colon X \to Y$ can be considered as a multivalued map with values $\{f(x)\}$. In this case the above definitions give the definitions of quasi-continuity in the sense of Kempisty [**4**] (sometimes called semi-continuity [**5**]) and almost continuity in the sense of Husain [**3**].

The symbols: $Q_u(F, \tau)$, $Q_l(F,\tau)$, $A_u(F, \tau)$, $A_l(F, \tau)$ are used to denote the sets of all points at which a multivalued map $F\colon (X, \tau) \to Y$ is upper or lower quasi-continuous or almost continuous respectively.

REMARKS 1: For topological spaces (x, τ) and (X, τ_s) we have:

(a) $\operatorname{Int}_s U \subseteq \operatorname{Int} U$ for every set $U \subseteq X$;
(b) $\overline{U} \subseteq \overline{U}^s$ for every set $U \subseteq X$;
(c) $\operatorname{Int} B = \operatorname{Int}_s B$ for every τ-closed set $B \subseteq X$;
(d) $\overline{A}^s = \overline{A}$ for every τ-open set $A \subset X$;
(e) any τ_s-semi open set is τ-semi open;
(f) τ_s-quasi-continuity implies τ-quasi-continuity;
(g) $\operatorname{Int} \overline{V} \subseteq \operatorname{Int}_s \overline{V}^s$ for every set $V \subseteq X$;
(h) τ-almost continuity implies τ_s-almost continuity.

PROOFS: The properties (a) and (b) follow from $\tau_s \subseteq \tau$. Now we are going to prove (c). $\operatorname{Int} B$ is regular open, hence $\operatorname{Int} B = \operatorname{Int}_s(\operatorname{Int} B) \subseteq \operatorname{Int}_s B$. The converse inclusion $\operatorname{Int}_s B \subseteq \operatorname{Int} B$ follows from (a). Property (d) follows from (c) applied to the set $X \setminus A$. If V is τ_s-semi open then from (d) and (a) we have $V \subset \overline{\operatorname{Int}_s V}^s = \overline{\operatorname{Int}_s V} \subset \overline{\operatorname{Int} V}$. Thus (e) is shown. Moveover (e) implies (f). From (c) and (b) we have

https://doi.org/10.1017/S0004972700027040 Published online by Cambridge University Press

$\operatorname{Int}\overline{V} = \operatorname{Int}_s\overline{V} \subseteq \overline{\operatorname{Int}_s\overline{V}}^s$ which proves (g). Finally the property (h) is an immediate consequence of (g). ∎

THEOREM 1. *Let (X, τ) be a topological space and Y a regular one. If $F\colon X \to Y$ is a multivalued map with compact values, then $\operatorname{Int}_s[Q_u(F, \tau) \cap Q_l(F, \tau)] \subset Q_u(F, \tau_s)$.*

PROOF: Assume that $x_0 \in \operatorname{Int}_s[Q_u(F, \tau) \cap Q_l(F, \tau)] \setminus Q_u(F, \tau_s)$. Then there exist an open set $W_0 \supseteq F(x_0)$ and a τ_s-open set U_0 such that $x_0 \in U_0 \subseteq \operatorname{Int}_s[Q_u(F, \tau) \cap Q_l(F, \tau)]$ and the following condition holds:

(1) every non-empty τ_s-open set $U \subseteq U_0$ contains a point x_u such that $F(x_u) \not\subset W_0$.

Because $F(x_0)$ is compact and Y is a regular space, there exists an open set $W \subset Y$ satisfying $F(x_0) \subseteq W \subseteq \overline{W} \subseteq W_0$. The point $x_0 \in Q_u(F, \tau)$; hence for sets U_0 and W, there exists a non-empty, τ-open set $M \subseteq U_0$ such that:

(2) $F(x) \subseteq W$ for every point $x \in M$.

The set M can be represented in the form $M = U \setminus N$ [8, Theorem 4.5] where $U \subseteq U_0$ is τ-regular open and N is a τ-nowhere dense set. In view of (1) there exists a point $x_u \in U$ such that $F(x_u) \cap (Y \setminus \overline{W}) \neq \emptyset$. Because $x_u \in Q_l(F, \tau)$ for the sets $Y \setminus \overline{W}$ and U there exists a non-empty, τ-open set $A \subseteq U$ such that:

(3) $F(x) \cap (Y \setminus \overline{W}) \neq \emptyset$ for every point $x \in A$.

The sets A and U are non-empty, τ-open and $A \subseteq U$. So we have $A \cap (U \setminus N) = A \cap M \neq \emptyset$. This means that $A \cap M$ contains a point satisfying (2) and (3). This is the contradiction which finishes the proof. ∎

THEOREM 2. *Let (X, τ) be a topological space and Y a regular one. If $F\colon X \to Y$ is a multivalued map then $\operatorname{Int}_s[Q_u(F, \tau) \cap Q_l(F, \tau)] \subseteq Q_l(F, \tau_s)$.*

PROOF: Assume that $x_0 \in \operatorname{Int}_s[Q_u(F, \tau) \cap Q_l(F, \tau)] \setminus Q_l(F, \tau_s)$. Then there exist an open set $W_0 \subseteq Y$ such that $W_0 \cap F(x_0) \neq \emptyset$ and a τ_s-open set U_0 such that $x_0 \in U_0 \subseteq \operatorname{Int}_s[Q_u(F, \tau) \cap Q_l(F, \tau)]$ and:

(1) every non-empty τ_s-open set $U \subseteq U_0$ contains a point x_u such that $F(x_u) \cap W_0 = \emptyset$.

Let us take any point $y_0 \in F(x_0) \cap W_0$ and an open set W for which $y_0 \in W \subseteq \overline{W} \subseteq W_0$ holds. Because $x_0 \in Q_l(F, \tau)$ there exists a τ-open non-empty set $M \subseteq U_0$ satisfying:

(2) $F(x) \cap W \neq \emptyset$ for every point $x \in M$.

Let $M = U \setminus N$, where $U \subseteq U_0$ is a τ-regular open set and N is τ-nowhere dense [8, Theorem 4.5]. From (1) we have a point $x_u \in U$ such

https://doi.org/10.1017/S0004972700027040 Published online by Cambridge University Press

that $F(x_u) \subset Y \setminus \overline{W}$. The point $x_u \in Q_u(F, \tau)$, so there exists a τ-open set $A \subseteq U$ such that:

(3) $F(x) \subseteq Y \setminus \overline{W}$ for every point $x \in A$.

The sets A and U are non-empty τ-open and $A \subseteq U$. So we have $A \cap (U \setminus N) = A \cap M \neq \emptyset$, this means that $A \cap M$ contains a point satisfying (2) and (3) and this is the contradiction completing the proof. ∎

From Theorems 1, 2 and Remarks 1(f) we immediately obtain:

Corollary 1. *Let (X, τ) be a topological space and Y a regular one. A multivalued map $F: X \to Y$ with compact values is lower and upper quasi-continuous if and only if it is simultaneously lower and upper τ_s-quasi-continuous.*

Corollary 2. *Let τ_1 and τ_2 be topologies on X which have the same semi-regularisations. If Y is a regular space, then a single valued map $f: X \to Y$ is τ_1-quasi-continuous if and only if it is τ_2-quasi-continuous.*

Observe that regularity of the space Y in Theorems 1 and 2 cannot be replaced by semi-regularity.

Example 1.. *Let us take on the real line $\mathbf{R}$ the following topologies:*

$$\tau = \{U \subseteq \mathbf{R} : 0 \in U\} \cup \{\emptyset\},$$
$$\tau_2 = \{\emptyset, \{0\}, \{2\}, \{0,2\}, \mathbf{R}\}$$

It is easy to see that topology τ_2 is semi-regular and the semi-regularisation τ_s of τ is the indiscrete topology. The multifunction $F: (\mathbf{R}, \tau) \to (\mathbf{R}, \tau_2)$ given by the formula $F(x) = \{0\}$ for $x = 2$ and $F(x) = \{x\}$ for $x \neq 2$ is upper and lower τ-quasi-continuous but it is neither upper nor lower τ_s-quasi-continuous.

If F is upper or lower quasi-continuous only, then Corollary 1 may be false, as the following shows:

Example 2. *Let N be the natural topology on $\mathbf{R}$ and let τ be the topology τ from Example 1. The multifunction $F_1: (\mathbf{R}, \tau) \to (\mathbf{R}, N)$ given by the formula $F_1(x) = \{0\}$ for $x \in (-\infty, 0\rangle$ and $F_1(x) = \langle 0, 1 \rangle$ for $x \in (0, +\infty)$ has compact values and it is easy to verify that F_1 is upper τ-quasi-continuous but it is not upper τ_s-quasi-continuous. However the formula $F_2(x) = \{0\}$ for $x \in (-\infty, 0)$ and $F_2(x) = \langle 0, 1 \rangle$ for $x \in \langle 0, +\infty)$ gives a multifunction which is lower τ-quasi-continuous but is not lower τ_s-quasi-continuous.*

Stronger than Corollary 1 is the following:

https://doi.org/10.1017/S0004972700027040 Published online by Cambridge University Press

THEOREM 3. *Let* (X, τ) *be a topological space and* Y *a regular one. Let, for* $i = 1,2,3, \tau_i$ *be a topology on* X *such that* $\tau_s \subseteq \tau_i \subseteq \tau_\alpha$ *(in general* $(\tau_i)_\alpha \neq (\tau_j)_\alpha \neq (\tau)_\alpha = \tau_\alpha$ *and* $\tau_i \not\subset \tau_j$ *for* $i,j = 1,2,3$*). A multivalued map* $F\colon X \to Y$ *with compact values is lower* τ_1*-quasi-continuous and upper* τ_2*-quasi-continuous if and only if it is simultaneously upper and lower* τ_3*-quasi-continuous.*

PROOF: Because the semi-regularisation of τ_α is the topology τ_s ([7] Proposition 6), hence from the inclusion $\tau_s \subseteq \tau_i \subseteq \tau_\alpha$ and from Remark 1(d) we have $\overline{\mathrm{Int}_i A}^i \subset \overline{\mathrm{Int}_\alpha A}^s = \overline{\mathrm{Int}_\alpha A}^\alpha$ for any set $A \subseteq X$. This means that any τ_i-semiopen set is τ_α-semiopen, so F is simultaneously lower and upper τ_α-quasi-continuous and in virtute of Corollary 1 it is τ_s-quasi-continuous. To this end we shall prove that the semi-regularisation of τ_i is the topology τ_s and then apply Remark 1(f). If the set B is τ_i-regular open, then from $\tau_s \subseteq \tau_i \subseteq \tau_\alpha$ and from Remark 1(c) we have: $\mathrm{Int}_\alpha \overline{B}^\alpha = \mathrm{Int}_s \overline{B}^\alpha \subseteq \mathrm{Int}_i \overline{B}^i = B = \mathrm{Int}_\alpha B \subseteq \mathrm{Int}_\alpha \overline{B}^\alpha$. This means $B = \mathrm{Int}_\alpha \overline{B}^\alpha$ is τ_α (hence τ_s) regular open. On the other hand if B is τ_s-regular open, then $B = \mathrm{Int}_i B \subseteq \mathrm{Int}_i \overline{B}^i \subset \mathrm{Int}_\alpha \overline{B}^s = \mathrm{Int}_s \overline{B}^s = B$, so $B = \mathrm{Int}_i \overline{B}^i$ and the proof is complete. ∎

Let us observe that if $(\tau_1)_s = (\tau_2)_s = (\tau_3)_s$ only, then Theorem 3 may be false as Example 2 shows together with the topology $\tau_3 = \{U \subset \mathbf{R} : 3 \in U\} \cup \{\emptyset\}$. The multifunction F_1 is upper τ-quasi-continuous and lower τ_3-quasi-continuous, but it is neither lower nor upper τ_3-quasi-continuous.

Now we will formulate some sufficient conditions under which upper (respectively lower) τ-quasi-continuity implies upper (respectively lower) τ_s-quasi-continuity.

THEOREM 4. *Let* (X, τ) *be a topological space and* Y *a regular one. If* $F\colon X \to Y$ *is a multivalued map with compact values, then* $Int_s[Q_u(F,\tau) \cap A_l(F,\tau)] \subset Q_u(F,\tau_s)$.

PROOF: Just as in Theorem 4 we can prove that there exist sets $W, M = U \setminus N$ (where $W \subset Y$ is open $M \subseteq X$ is open, $U \subseteq X$ is regular open and $N \subseteq X$ is a nowhere dense set) and the point $x_u \in U$ for which $F(x_u) \cap (Y \setminus \overline{W}) \neq \emptyset$ and:
(1) $F(x) \subseteq W$ for every point $x \in M$.
Because $x_u \in A_l(F,\tau)$, $F^-(Y \setminus \overline{W})$ is τ-dense in the set $\mathrm{Int}\, \overline{F^-(Y \setminus \overline{W})}$ and consequently the set $G = U \cap F^-(Y \setminus \overline{W})$ is non-empty and dense in the set $H = U \cap \mathrm{Int}\, \overline{F^-(Y \setminus \overline{W})}$. Because $G \cap (H \setminus N) \neq \emptyset$ and $H \cap M = H \cap (U \setminus N) = (H \cap U) \setminus N = H \setminus N$ we have $G \cap M \supseteq G \cap H \cap M \neq \emptyset$. But $F(x) \not\subset \overline{W}$ for every point $x \in G$, hence we have a contradiction to (1), so the proof is completed. ∎

Using arguments similar to those in the proofs of Theorem 2 and 4 we can prove:

THEOREM 5. *Let* (x, τ) *be a topological space and* Y *a regular one. If* $F\colon X \to Y$ *is a multivalued map then* $Int_s[A_u(F,\tau) \cap Q_l(F,\tau)] \subseteq Q_l(F,\tau_s)$.

https://doi.org/10.1017/S0004972700027040 Published online by Cambridge University Press

COROLLARY 3. *Let* (X, τ) *be a topological space and* Y *a regular one. If a multivalued map* $F: X \to Y$ *with compact values is upper quasi-continuous and lower almost continuous, then it is upper* τ_s*-quasi-continuous.*

COROLLARY 4. *Let* (X, τ) *be a topological space and* Y *a regular one. If a multivalued map* $F: X \to Y$ *is lower quasi-continuous and upper almost continuous, then it is lower* τ_s*-quasi-continuous.*

Observe that in Theorems 4 and 5 the assumption of lower or upper almost continuity cannot be weakened by lower or upper τ_s-almost continuity, respectively. In Example 2 both multifunctions F_1, F_2 are upper and lower τ_s-almost continuous but are neither upper nor lower τ_s-quasi-continuous. Moreover Example 1 shows that the *assumption of reqularity of the space* Y *is essential.*

REMARK 2: Theorems 1, 3, 4 will be true if we consider multifunctions with closed values in a normal space. This same applies to Corollaries 1 and 3.

REMARK 3: If in all our considerations, beginning from Theorem 1, the term "*quasi-continuity*" is replaced by "*continuity*", similar results for continuous maps are obtained. In this case (for example) from Corollary 2 we have a theorem of Katetov [**6**, Lemma 2].

REFERENCES

[1] C. Berge, *Espaces topologiques. Functions multivoques* (Paris, 1966).

[2] R. Engelking, *General topology* (Warszawa, 1977).

[3] T. Husain, 'Almost continuous mappings', *Prace Mat.* **10** (1966), 1–7.

[4] S. Kempisty, 'Sur les functions quasicontinues', *Fund. Math.* **19** (1932), 184–197.

[5] N. Levine, 'Semi-open sets and semi-continuity in topological spaces', *Amer. Math. Monthly* **70** (1963), 36–41.

[6] M. Mršević, I.L. Reilly and M.K. Vamanamurthy, 'On semi-regularization topologies', *J. Austral. Math. Soc. Ser. A* **38** (1985), 40–54.

[7] O. Njåstad, 'On some classes of nearly open sets', *Pacific J. Math.* **15** (1965), 961–970.

[8] J.C. Oxtoby, *Measure and category* (Springer-Verlag, New York Heidelberg Berlin, 1971).

[9] V. Popa, 'Asupra unei descompuneri a cvasicontinuitati multifunctiilor', *Stud. Cerc. Mat.* **27** (1975), 325–328.

[10] V. Popa, 'Asupra unor proprietătii ale multifunctiilor cvasicontinue si aproape continue', *Stud. Cerc. Mat.* **30** (1978), 441–446.

[11] R.E. Smithson, 'Almost and weak continuity for multifunctions', *Bull. Calcutta Math. Soc.* **70** (1978), 383–390.

Department of Mathematics
Pedagogical University
Arciszewskiego 22
76–200 Słupsk
Poland

https://doi.org/10.1017/S0004972700027040 Published online by Cambridge University Press