J. Austral. Math. Soc. (Series A) 69 (2000), 298–302

SUMS OF SQUARES OF INTEGRAL LINEAR FORMS

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(Received 10 December 1999; revised 18 March 2000)

Communicated by W. W. L. Chen

Abstract

In this paper we prove that every positive definite n-ary integral quadratic form with $12 \leq n \leq 13$ (respectively $14 \leq n \leq 20$) that can be represented by a sum of squares of integral linear forms is represented by a sum of $2 \cdot 3^n + n + 6$ (respectively $3 \cdot 4^n + n + 3$) squares. We also prove that every positive definite 7-ary integral quadratic form that can be represented by a sum of squares is represented by a sum of 25 squares.

2000 *Mathematics subject classification*: primary 11E08, 11E12, 11E20, 11E25, 15A63.

1. Introduction

For any integer $n \geq 1$ we define $g_{\mathbf{Z}}(n)$ to be the smallest positive integer such that every positive definite n-ary integral quadratic form that can be represented by a sum of squares is represented by a sum of $g_{\mathbf{Z}}(n)$ squares.

Mordell [7], and later Mordell [8] and Ko [6], proved that for $n \leq 5$, every positive definite integral quadratic form of rank n is a sum of $n + 3$ squares of integral linear forms, that is, $g_{\mathbf{Z}}(n) = n + 3$ for $1 \leq n \leq 5$. So it was naturally expected that every positive definite integral quadratic form of n variables would be represented by a sum of $n + 3$ squares. This, however, turned out to be false. Indeed, Mordell [9] showed that the integral quadratic form associated to the Dynkin diagram E_6, which we denote also by E_6 by abuse of notation, cannot be represented by a sum of squares. Kim and Oh [3, 4] proved that every positive definite 6-ary integral quadratic form that can be represented by a sum of squares is represented by a sum of 10 squares, and showed that $g_{\mathbf{Z}}(6) = 10$.

However, no explicit evaluation of $g_{\mathbb{Z}}(n)$ for all n is known. Icaza [2] has given an explicit function $f(n)$ such that $g_{\mathbb{Z}}(n) \le f(n)$ for all $n \ge 5$:

$$f(n) = 8 + \frac{2^{4(h(I_{n+3})-1)}}{b_n}\left[\frac{n(n+1)(2n+1)}{6} - 55\right],$$

where $b_n = n^{-(n-1)}(4/5)^{(n-3)(n-4)/2}(\pi/4)^n(\Gamma(n/2+1))^{-2}$, I_k is the $\mathbb{Z}$-lattice corresponding to the sum of k squares, and $h(I_k)$ is the class number of I_k. Oh [10] has given an upper bound of $g_{\mathbb{Z}}(n)$ for $7 \le n \le 11$: $g_{\mathbb{Z}}(n) \le n + 3 + n(n+1)/2$.

In this paper we give an upper bound for $g_{\mathbb{Z}}(n)$ for $12 \le n \le 20$ and for $n = 7$:

$$g_{\mathbb{Z}}(n) \le \begin{cases} 2\cdot 3^n + n + 6 & \text{for } 12 \le n \le 13, \\ 3 \cdot 4^n + n + 3 & \text{for } 14 \le n \le 20, \end{cases} \quad \text{and} \quad g_{\mathbb{Z}}(7) \le 25.$$

These bounds are better than those of Icaza [2] and Oh [10].

We use the terminology and notation of O'Meara [12]. Let l be a positive definite $\mathbb{Z}$-lattice of rank n equipped with a symmetric bilinear form B. Here, a $\mathbb{Z}$-lattice is a free $\mathbb{Z}$-module with $s(l) \subseteq \mathbb{Z}$, where $s(l)$ is the scale of l. We denote $l^{(a)}$ a scaling of l for $a \in \mathbb{Q}$. We denote the corresponding quadratic form by $f_l(x_1, x_2, \dots, x_n) = \sum_{i,j=1}^n f_{ij}x_ix_j$ and the corresponding matrix by $M_l = (f_{ij})$, where $f_{ij} = B(v_i, v_j) \in \mathbb{Z}$, for a fixed basis $\{v_1, v_2, \dots, v_n\}$ of l. Let $I_N = \perp_N \mathbb{Z}e_i = \perp_N \langle 1 \rangle$, where $\{e_1, \dots, e_N\}$ is a $\mathbb{Z}$-basis of I_N with $B(e_i, e_j) = \delta_{ij}$ for all $i, j = 1, \dots, N$. Thus I_N is the $\mathbb{Z}$-lattice corresponding to the sum of N squares.

2. Upper bound for $g_{\mathbb{Z}}(n)$

LEMMA 2.1. *Every positive definite $\mathbb{Z}$-lattice l of rank n is represented by the genus of I_{n+3}, that is, $l \to L$ for some $L \in \text{gen}(I_{n+3})$.*

PROOF. See the proof of Kim-Oh [4, Theorem 2.1] or Icaza [2, Theorem 1]. □

We now give an upper bound for $g_{\mathbb{Z}}(n)$ for $12 \le n \le 20$.

THEOREM 2.2. $g_{\mathbb{Z}}(n) \le \begin{cases} 2\cdot 3^n + n + 6 & \textit{for } 12 \le n \le 13, \\ 3 \cdot 4^n + n + 3 & \textit{for } 14 \le n \le 20. \end{cases}$

PROOF. We set $k = k(n) = 3$ for $12 \le n \le 13$, and $k = 4$ for $14 \le n \le 20$. Let $l = l_n$ be a positive definite n-ary $\mathbb{Z}$-lattice with $12 \le n \le 20$. Suppose that $l \to I_N$ for some N. Then we can write $f_l = \sum_{i=1}^N (a_{1i}x_1 + \cdots + a_{ni}x_n)^2$.

If there exist k linear forms $(a_{1i_j}x_1 + \cdots + a_{ni_j}x_n)$ $(j = 1, \dots, k)$ with

$$(a_{1i_j}x_1, \dots, a_{ni_j}x_n) \equiv (a_{1i_m}x_1, \dots, a_{ni_m}x_n) \bmod k$$

for $1 \le j, m \le k$, then the $\mathbb{Z}$-lattice K corresponding to

$$f_K = \sum_{j=1}^{k} \left(a_{1i_j}x_1 + \cdots + a_{ni_j}x_n\right)^2$$

is (semi-)positive definite and $s(K) \subseteq k\mathbb{Z}$.

Then we have

$$f_l = \sum_i (a_{1i}x_1 + \cdots + a_{ni}x_n)^2 + f_K, \tag{1}$$

where

$$f_K = \sum_j \left(\sum_{r=1}^{k} \left(a_{1i_{(j,r)}}x_1 + \cdots + a_{ni_{(j,r)}}x_n\right)^2 \right)$$

with

$$\left(a_{1i_{(j,r)}}, \ldots, a_{ni_{(j,r)}}\right) \equiv \left(a_{1i_{(j,s)}}, \ldots, a_{ni_{(j,s)}}\right) \bmod k$$

for $1 \le r, s \le k$ and the corresponding $\mathbb{Z}$-lattice K is of scale in $k\mathbb{Z}$. Thus we may assume that there exist at most $k-1$ linear forms with $(a_{1i}, \ldots, a_{ni}) \equiv (a_{1j}, \ldots, a_{nj}) \bmod k$ in the summation of the first term in (1). So we see that the number of the squares of the linear forms in the summation of the first term in (1) is at most $(k-1)k^n$.

Applying Lemma 2.1 and, Conway-Sloane [1, Theorem 14, Appendix] for $k = 3$, Conway-Sloane [1, Theorem 18] for $k = 4$, respectively, K is represented by I_{n+6} for $k = 3$, and I_{n+3} for $k = 4$. Thus we have

$$g_{\mathbb{Z}}(n) \le (k-1) \cdot k^n + \phi(k),$$

where $\phi(k) = n + 6$ for $k = 3$, and $\phi(k) = n + 3$ for $k = 4$. □

Next we give a better upper bound for $g_{\mathbb{Z}}(7)$. Let l_n be semi-positive definite $\mathbb{Z}$-lattice with rank n. Let k be an integer with $0 < k \le n$. If there exists a basis of l_n such that

$$l_n \cong \begin{pmatrix} A_{11} & A_{12} \\ {}^tA_{12} & A_{22} \end{pmatrix}$$

with $A_{11} \in M_k(\mathbb{Z})$, $A_{12} \in M_{k,n-k}(\mathbb{Z})$, $A_{22} \in M_{n-k}(\mathbb{Z})$ satisfying $A_{11} \equiv 0_{k,k} \bmod 2$ then we say that l_n has *a k-null space over $\mathbb{F}_2$*. Moreover, we say that l_n has *an orthogonal k-null space over $\mathbb{F}_2$* if $A_{12} \equiv 0_{k,n-k} \bmod 2$.

LEMMA 2.3. *Let l_7 be a positive definite $\mathbb{Z}$-lattice and suppose l_7 has a 5-null space over $\mathbb{F}_2$. Then l_7 is represented by I_{10}.*

PROOF. By assumption we fix a basis of l_7 such that l_7 has a 5-null space over $\mathbb{F}_2$.

Since l_7 is represented by the lattice in the genus of I_{10}, l_7 is represented by $\mathrm{cls}(I_{10})$ or $\mathrm{cls}(E_8 \perp I_2)$.

Suppose that l_7 is represented by $\mathrm{cls}(E_8 \perp I_2)$. Then there exists a semi-positive definite $\mathbb{Z}$-lattice l_7' corresponding to the quadratic form

$$f_{l_7'} = f_{l_7} - (a_{11}x_1 + \cdots + a_{71}x_7)^2 - (a_{12}x_1 + \cdots + a_{72}x_7)^2$$

such that l_7' is represented by E_8. Since the norm ideal of $s(l_7')$ is in $2\mathbb{Z}$ and by the assumption, we have $(a_{11}, \ldots, a_{51}) \equiv (a_{12}, \ldots, a_{52}) \bmod 2$ and l_7' has also a 5-null space over $\mathbb{F}_2$. Then we can easily see that l_7' has an orthogonal 3-null space $\mathbb{F}_2$ by changing the basis of l_7' if necessary. This implies that the rank of the unimodular Jordan component of $\mathbb{Z}_2$-lattice $(l_7')_2$ is less than 5 and by O'Meara [11, Theorem 2], we have $(l_7')_2 \to (I_8)_2$. Since $(l_7')_p \to (I_8)_p$ for all odd primes p (including ∞) and $\mathrm{gen}(I_8) = \mathrm{cls}(I_8)$, l_7' is represented by I_8. Thus we conclude $l_7 \to I_{10}$. □

Let L and M be $\mathbb{Z}$-lattices. For a sublattice l of $L \perp M$ of the form

$$l = \mathbb{Z}(x_1 + y_1) + \cdots + \mathbb{Z}(x_n + y_n)$$

with $x_i \in L$, $y_i \in M$, we define

$$l(L) := \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_n \quad \text{and} \quad l(M) := \mathbb{Z}y_1 + \cdots + \mathbb{Z}y_n.$$

Even when $\phi : l \to L \perp M$, we simply write $l(L)$ instead of $\phi(l)(L)$ if no confusion arises.

Let $I_N = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_N$. For a subset S of $\{1, 2, \ldots, N\}$, we define

$$I_N^S := \mathbb{Z}e_{k_1} + \cdots + \mathbb{Z}e_{k_r},$$

where $\{k_1, \ldots, k_r\}$ is the complement of S in $\{1, 2, \ldots, N\}$.

LEMMA 2.4. *For a sublattice $L \subseteq I_N$ of rank n with $N > n(n+1)/2$, there exists a subset $S \subseteq \{1, 2, \ldots, N\}$ such that $|S| \le n(n+1)/2$ and $s(L(I_N^S)) \subseteq 2\mathbb{Z}$.*

PROOF. See Oh [10, Proposition 3.3.2]. □

THEOREM 2.5. $g_{\mathbb{Z}}(7) \le 25$.

PROOF. Let l be a $\mathbb{Z}$-lattice of rank 7. Assume that l is represented by I_N and put $l = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_7$, where $v_i = (a_{i1}, \ldots, a_{i7}) \in I_N$. Put $l' := \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_5 \subseteq l$. Let S be the set satisfying the property in Lemma 2.4 for l'. Then we see that $l(I_N^S)$ has 5-null space and $l(I_N^S) \to I_{10}$ by Lemma 2.3. Thus we conclude that

$$g_{\mathbb{Z}}(7) \le \frac{5 \cdot 6}{2} + 10 = 25. \qquad \square$$

https://doi.org/10.1017/S1446788700002470 Published online by Cambridge University Press

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