Adv. Appl. Prob. **47,** 128–145 (2015)
Printed in Northern Ireland

VARIANCE OPTIMAL STOPPING FOR GEOMETRIC LÉVY PROCESSES

KAMILLE SOFIE TÅGHOLT GAD,* **
JESPER LUND PEDERSEN,* *University of Copenhagen*

Abstract

The main result of this paper is the solution to the optimal stopping problem of maximizing the variance of a geometric Lévy process. We call this problem the variance problem. We show that, for some geometric Lévy processes, we achieve higher variances by allowing randomized stopping. Furthermore, for some geometric Lévy processes, the problem has a solution only if randomized stopping is allowed. When randomized stopping is allowed, we give a solution to the variance problem. We identify the Lévy processes for which the allowance of randomized stopping times increases the maximum variance. When it does, we also solve the variance problem without randomized stopping.

Keywords: Variance criterion; variance optimal stopping; geometric Lévy process; quadratic optimal stopping

2010 Mathematics Subject Classification: Primary 60G40
Secondary 90C20

1. Introduction

In this paper we solve the optimal stopping problem of maximizing the variance of a geometric Lévy process. We call this problem the variance problem. It is distinguished from classical optimal stopping problems in that we maximize the variance and not the expectation. The nonlinear structure of the variance moves the problem outside the scope of classical optimal stopping problems, and, thus, we cannot directly rely on results from, e.g. [7] and [9].

As in Markowitz mean-variance analysis [3] we identify the variance of a stock price with a risk. In the context of risk management, where an investor wishes to sell an asset, the variance problem provides the worst-case scenario, that is, the value function is an upper bound for the risk (variance) for any strategy and the optimal strategy is the strategy at highest risk.

The results in this paper extend the results of [6], in which the variance problem is solved for various diffusions. However, we face different technical issues in working with geometric Lévy processes. Whereas the optimal stopping times for the diffusions in [6] are hitting times, this is not the case for all geometric Lévy processes. For some geometric Lévy processes, the solution is of another kind. Furthermore, for some of these processes, we achieve higher variances by allowing randomized stopping. For some processes the variance problem has a solution only if randomized stopping is allowed. This is in contrast to classical optimal stopping problems and variance problems for diffusions where randomized stopping times do not change the value function.

Received 7 August 2012; revision received 23 February 2014.
* Postal address: Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, 2100 Copenhagen, Denmark.
** Email address: kamille@math.ku.dk

https://doi.org/10.1239/aap/1427814584 Published online by Cambridge University Press

Mathematically, the problem addressed in this paper is the following. Let X be a Lévy process, let $\mathcal{F}$ be the augmented natural filtration satisfying the usual conditions, and let $\mathcal{T}$ denote the set of stopping times with respect to $\mathcal{F}$ (all terms are defined as in [2]). The main problem we consider is to find a stopping time $\tau^* \in \mathcal{T}$ such that

$$\sup_{\tau \in \mathcal{T}} \mathbb{V}[\mathrm{e}^{X_\tau}] = \mathbb{V}[\mathrm{e}^{X_{\tau^*}}]. \tag{1}$$

We call this problem the variance problem.

Initially, we identify the Lévy processes for which the variance problem is trivial to solve. If X is a deterministic process then $\mathcal{T}$ contains only almost surely (a.s.) deterministic times and the variance for any stopping time is 0. Let $\psi(\beta) = \log(\mathbb{E}[\mathrm{e}^{\beta X_1}])$ denote the Laplace exponent. If $\psi(2) > 0$ and the Lévy process is nondeterministic, then $\psi(2) > 2\psi(1)$ by Jensen's inequality. Therefore, $\mathbb{V}[\mathrm{e}^{X_t}] = \mathbb{E}[\mathrm{e}^{2X_1}]^t - \mathbb{E}[\mathrm{e}^{X_1}]^{2t} = \mathrm{e}^{\psi(2)t} - \mathrm{e}^{2\psi(1)t} \to \infty$ for $t \to \infty$ and the variance problem is unbounded.

In the following we consider Lévy processes with $\psi(2) < 0$. Lévy processes with $\psi(2) = 0$ are considered separately (see Theorem 2).

In [6], the variance problem was solved for various diffusions. This was achieved using a method of embedding the problem into the following classical optimal stopping problem, which we call the quadratic problem:

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}[(\mathrm{e}^{X_\tau} - c)^2] = \mathbb{E}[(\mathrm{e}^{X_{\tau^*}} - c)^2], \qquad c > 0. \tag{2}$$

The solution to the variance problem in this paper is also based on this embedding method, and the solution to the quadratic problem is presented in Theorem 1 below. As shown in [6], it holds that if τ^* is optimal for (2) and solves

$$\mathbb{E}[\mathrm{e}^{X_{\tau^*}}] = c \tag{3}$$

then it is also an optimal stopping time for variance problem (1).

The processes studied in [6] all have a combination of $\tau^* \in \mathcal{T}$ and $c \in \mathbb{R}$ that solve both (2) and (3). But some Lévy processes do not. Let $\bar{X}_\infty = \sup_{t \geq 0} X_t$. The problem of finding a combination of τ^* and c that solves both (2) and (3) arises from possible discontinuities in the distribution of $\bar{X}_\infty$. Discontinuities exist exactly when 0 is irregular for $(0, \infty)$ (see Lemma 1), and continuity of the distribution of $\bar{X}_\infty$ ensures that the variance problem has an excess boundary time solution. It is sometimes possible to use the embedding method even when the distribution of $\bar{X}_\infty$ has discontinuities. We derive two equations that each give sufficient conditions that the embedding method can be used to find an excess boundary time solution (see Theorem 2).

When there is no excess boundary time that solves both (2) and (3), we solve the variance problem by introducing randomized stopping times. The concept is to allow stopping decisions to depend not only on the Lévy process, but also on a random variable independent of the Lévy process. As argued in [9], this may be powerful when solving optimal stopping problems with constraints because it sometimes easily gives a wider class of solutions to the unconstrained problem. We see in Theorem 4 that the class of randomized optimal stopping times for the quadratic problem (2) is so wide that one of them also solves (3). Thus, for any Lévy process, it is possible to solve the variance problem with the embedding method if we maximize over the randomized stopping times.

We return to the original variance problem, where randomization is not allowed. In the case there is no combination of τ^* and c that solves both (2) and (3), the situation depends on the jump structure and the drift of the Lévy process. For compound Poisson processes, the randomized stopping times can be mimicked because the processes stay for a positive time at 0 before the first jump. This positive time is independent of the rest of the behavior of the process and in Theorem 6 we show how this is used as the independent random information needed. The processes which are not compound Poisson processes moves from 0 right away. In Theorem 7 we show that, for these processes, it holds that if there is no excess boundary time solution then the randomized solutions cannot be mimicked and there is no stopping time in $\mathcal{T}$ giving as high a variance as that obtained by the randomized solution. If the jump part is not a compound Poisson process then the filtration grows sufficiently fast that we may find a sequence of stopping times from $\mathcal{T}$ for which the variance converges to the variance of the randomized solution. But if the jump part is a compound Poisson process then the filtration does not generate sufficient information and there is a gap between the value function of the variance problem with and without randomized stopping times (see Theorem 7).

2. The quadratic optimal stopping problem

In this section we solve the quadratic optimal stopping problem (2) for Lévy processes with $\psi(2) < 0$. This problem has some resemblance to the optimal stopping problem presented in [1] and is solved by a similar method.

The quadratic problem is a classical optimal stopping problem for a Lévy process with gain function $G(x) = (\mathrm{e}^x - c)^2$. As G is continuous, and Lévy processes are Feller processes, then the state space can be divided into a stopping region and a continuation region (see [7]), with an optimal stopping time being the first time the process reaches the stopping region. To get a first idea of the stopping region, note that, from Jensen's inequality, a Lévy process with $\psi(2) < 0$ has $\mathbb{E}[X_1] < 0$ and, thus, it converges to $-\infty$ when t goes to ∞ (see [2, Theorem 7.2]). Hence, when maximizing $\mathbb{E}[(\mathrm{e}^{X_\tau} - c)^2]$, it is clear that the value c^2 may be obtained by never stopping the process. Therefore, it is never optimal to stop the process if $(\mathrm{e}^{X_t} - c)^2 < c^2$ and the stopping region has to be above $\log(2c)$.

We use the following notation for the excess boundary times. For $y \in \mathbb{R}$, define

$$\tau_y^+ = \inf\{t \geq 0 \mid X_t \geq y\} \quad \text{and} \quad \tau_y^{++} = \inf\{t \geq 0 \mid X_t > y\}.$$

Recall that

$$\bar{X}_\infty = \sup_{t \geq 0} X_t.$$

When we solve the quadratic and the variance problems, we repeatedly need the following fluctuation identity, which is a minor generalization of [1, Lemma 1]. If $x, y, \beta \in \mathbb{R}$ and $\mathbb{E}[\mathrm{e}^{\beta \bar{X}_\infty}] < \infty$, then

$$\mathbb{E}_x[\mathrm{e}^{\beta X_{\tau_y^+}} \mathbf{1}_{(\bar{X}_\infty \geq y)}] = \mathrm{e}^{\beta x} \frac{\mathbb{E}[\mathrm{e}^{\beta \bar{X}_\infty} \mathbf{1}_{(\bar{X}_\infty \geq y - x)}]}{\mathbb{E}[\mathrm{e}^{\beta \bar{X}_\infty}]}, \tag{4}$$

where the subscript on the expectation refers to the starting value of the process. The proof follows in the same way as the proof of [1, Lemma 1].

As in [2], we say that we have continuous fit if the value function is continuous at the boundary of the stopping region, and we say that we have smooth fit if the value function is differentiable at the boundary of the stopping region.

Theorem 1. *Let X be a Lévy process with $\psi(2) < 0$. Then $\tau_{y_c}^{+}$ and $\tau_{y_c}^{++}$ are both optimal stopping times of the quadratic problem (2), where* $y_c = \log(2c\mathbb{E}[\mathrm{e}^{2\bar{X}_\infty}]/\mathbb{E}[\mathrm{e}^{\bar{X}_\infty}])$.

(a) *If X is spectrally negative then* $y_c = \log(2c(\phi(0)-1)/(\phi(0)-2))$, *where ϕ is the right inverse of ψ.*

(b) *There is always continuous fit at y_c, and there is smooth fit at y_c exactly if the distribution of $\bar{X}_\infty$ is continuous at* 0.

Proof. We choose $\tau^* = \tau_{y_c}^{+}$ as a candidate for an optimal stopping time and define the corresponding function $v^*(x) = \mathbb{E}_x[G(X_{\tau^*})]$. By [5, Lemma 1], $\psi(2) < 0$ implies that $\mathbb{E}[\mathrm{e}^{2\bar{X}_\infty}] < \infty$ and $\mathbb{E}[\mathrm{e}^{\bar{X}_\infty}] < \infty$, and y_c and $v^*(x)$ are well defined. Then we use the well-known sufficient conditions (see [2, Lemma 9.1]) that τ^* is an optimal stopping time if the following conditions hold:

(i) $\mathbb{P}_x(\text{there exists } \lim_{t\to\infty} G(X_t) < \infty) = 1$,

(ii) $v^*(x) \geq G(x)$ for all $x \in \mathbb{R}$,

(iii) $(v^*(X_t))_{t\geq 0}$ is a right-continuous supermartingale.

We show that each of the three conditions are fulfilled.

(i) Whenever $\psi(2) < 0$, the Lévy process converges to $-\infty$ when t goes to ∞ and, thus, the requirement is fulfilled.

(ii) It follows from (4) and the definition of y_c that

$$\begin{aligned}
v^*(x) &= \mathbb{E}_x[(\mathrm{e}^{X_{\tau_{y_c}^{+}}} - c)^2] \\
&= c^2 - 2c\mathbb{E}_x[\mathrm{e}^{X_{\tau_{y_c}^{+}}}\mathbf{1}_{(\bar{X}_\infty \geq y_c)}] + \mathbb{E}_x[\mathrm{e}^{2X_{\tau_{y_c}^{+}}}\mathbf{1}_{(\bar{X}_\infty \geq y_c)}] \\
&= c^2 - 2c\mathrm{e}^x\frac{\mathbb{E}[\mathrm{e}^{\bar{X}_\infty}\mathbf{1}_{(\bar{X}_\infty \geq y_c - x)}]}{\mathbb{E}[\mathrm{e}^{\bar{X}_\infty}]} + \mathrm{e}^{2x}\frac{\mathbb{E}[\mathrm{e}^{2\bar{X}_\infty}\mathbf{1}_{(\bar{X}_\infty \geq y_c - x)}]}{\mathbb{E}[\mathrm{e}^{2\bar{X}_\infty}]} \qquad (5) \\
&= (c-\mathrm{e}^x)^2 + 2c\mathrm{e}^x\bigg(1 - \frac{\mathbb{E}[\mathrm{e}^{\bar{X}_\infty}\mathbf{1}_{(\bar{X}_\infty \geq y_c - x)}]}{\mathbb{E}[\mathrm{e}^{\bar{X}_\infty}]}\bigg) - \mathrm{e}^{2x}\bigg(1 - \frac{\mathbb{E}[\mathrm{e}^{2\bar{X}_\infty}\mathbf{1}_{(\bar{X}_\infty \geq y_c - x)}]}{\mathbb{E}[\mathrm{e}^{2\bar{X}_\infty}]}\bigg) \\
&= G(x) + \mathrm{e}^x\bigg(\mathrm{e}^{y_c}\frac{\mathbb{E}[\mathrm{e}^{\bar{X}_\infty}\mathbf{1}_{(\bar{X}_\infty < y_c - x)}]}{\mathbb{E}[\mathrm{e}^{2\bar{X}_\infty}]} - \mathrm{e}^x\frac{\mathbb{E}[\mathrm{e}^{2\bar{X}_\infty}\mathbf{1}_{(\bar{X}_\infty < y_c - x)}]}{\mathbb{E}[\mathrm{e}^{2\bar{X}_\infty}]}\bigg) \\
&\geq G(x) + \mathrm{e}^x\bigg(\mathrm{e}^{y_c}\frac{\mathbb{E}[\mathrm{e}^{\bar{X}_\infty}\mathbf{1}_{(\bar{X}_\infty < y_c - x)}]}{\mathbb{E}[\mathrm{e}^{2\bar{X}_\infty}]} - \mathrm{e}^x\frac{\mathrm{e}^{y_c - x}\mathbb{E}[\mathrm{e}^{\bar{X}_\infty}\mathbf{1}_{(\bar{X}_\infty < y_c - x)}]}{\mathbb{E}[\mathrm{e}^{2\bar{X}_\infty}]}\bigg) \qquad (6) \\
&= G(x).
\end{aligned}$$

(iii) Let Y be an independent copy of X, and let $\bar{X}_{[t,\infty)} = \sup_{s\in[t,\infty)} X_s$. Note that $\bar{X}_{[t,\infty)}$ is equal in law to $X_t + \bar{Y}_\infty$. Then use (5) and the definition of y_c to obtain

$$\begin{aligned}
\mathbb{E}[v^*(X_t + x)] = \mathbb{E}\bigg[c^2 - 2c\mathrm{e}^{x+X_t}&\frac{\mathbb{E}[\mathrm{e}^{\bar{Y}_\infty}\mathbf{1}_{(\bar{Y}_\infty \geq y_c - (x+X_t))}\mid X]}{\mathbb{E}[\mathrm{e}^{\bar{Y}_\infty}]} \\
&+ \mathrm{e}^{2(x+X_t)}\frac{\mathbb{E}[\mathrm{e}^{2\bar{Y}_\infty}\mathbf{1}_{(\bar{Y}_\infty \geq y_c - (x+X_t))}\mid X]}{\mathbb{E}[\mathrm{e}^{2\bar{Y}_\infty}]}\bigg]
\end{aligned}$$

https://doi.org/10.1239/aap/1427814584 Published online by Cambridge University Press

$$
\begin{aligned}
&= c^2 - \frac{2c}{\mathbb{E}[\mathrm{e}^{\bar{X}_\infty}]}\mathbb{E}[\mathrm{e}^{\bar{Y}_\infty+x+X_t}\,\mathbf{1}_{(\bar{Y}_\infty+x+X_t\ge y_c)}] \\
&\quad + \frac{1}{\mathbb{E}[\mathrm{e}^{2\bar{X}_\infty}]}\mathbb{E}[\mathrm{e}^{2\bar{Y}_\infty+x+X_t}\,\mathbf{1}_{(\bar{Y}_\infty+x+X_t\ge y_c)}] \\
&= c^2 - \frac{2c}{\mathbb{E}[\mathrm{e}^{\bar{X}_\infty}]}\mathrm{e}^{y_c}(\mathbb{E}_x[\mathrm{e}^{\bar{X}_{[t,\infty)}-y_c}\,\mathbf{1}_{(\bar{X}_{[t,\infty)}\ge y_c)}] - \mathbb{E}_x[\mathrm{e}^{2(\bar{X}_{[t,\infty)}-y_c)}\,\mathbf{1}_{(\bar{X}_{[t,\infty)}\ge y_c)}]) \\
&\le c^2 - \frac{2c}{\mathbb{E}[\mathrm{e}^{\bar{X}_\infty}]}\mathrm{e}^{y_c}(\mathbb{E}_x[\mathrm{e}^{\bar{X}_\infty-y_c}\,\mathbf{1}_{(\bar{X}_\infty\ge y_c)}] - \mathbb{E}_x[\mathrm{e}^{2(\bar{X}_\infty-y_c)}\,\mathbf{1}_{(\bar{X}_\infty\ge y_c)}]) \\
&= v^*(x).
\end{aligned}
$$

Hence, for $s \le t$ it follows that $\mathbb{E}[v^*(X_t) \mid \mathcal{F}_s] = \mathbb{E}[v^*((X_t - X_s) + X_s) \mid \mathcal{F}_s] \le v^*(X_s)$. Thus, $v^*(X_t)$ is a supermartingale. We then need to prove that $v^*(X_t)$ is right continuous in t. As X_t is right continuous in t, it is sufficient to prove that $x \mapsto v^*(x)$ is continuous. From (5), it follows that the jump size of v^* at x is

$$
\begin{aligned}
&\mathbb{P}(\bar{X}_\infty = y_c - x)\bigg(-2c\mathrm{e}^x\frac{\mathrm{e}^{y_c-x}}{\mathbb{E}[\mathrm{e}^{\bar{X}_\infty}]} + \mathrm{e}^{2x}\frac{\mathrm{e}^{2(y_c-x)}}{\mathbb{E}[\mathrm{e}^{2\bar{X}_\infty}]}\bigg) \\
&\qquad = \frac{\mathbb{P}(\bar{X}_\infty = y_c - x)\mathrm{e}^{y_c}}{\mathbb{E}[\mathrm{e}^{2\bar{X}_\infty}]}\bigg(-2c\frac{\mathbb{E}[\mathrm{e}^{2\bar{X}_\infty}]}{\mathbb{E}[\mathrm{e}^{\bar{X}_\infty}]} + \mathrm{e}^{y_c}\bigg) \\
&\qquad = 0.
\end{aligned}
$$

Thus, v^* is continuous and, hence, $\tau_{y_c}^+$ is an optimal stopping time for the quadratic problem.

We can prove that $\tau_{y_c}^{++}$ is also an optimal stopping time in the same way as for $\tau_{y_c}^+$. The only difference is that we need the following modification of (4): whenever $x, y, \beta \in \mathbb{R}$ and $\mathbb{E}[\mathrm{e}^{\beta\bar{X}_\infty}] < \infty$, we have

$$
\mathbb{E}_x[\mathrm{e}^{\beta X_{\tau_y^{++}}}\,\mathbf{1}_{(\bar{X}_\infty > y)}] = \mathrm{e}^{\beta x}\frac{\mathbb{E}[\mathrm{e}^{\beta\bar{X}_\infty}\,\mathbf{1}_{(\bar{X}_\infty > y - x)}]}{\mathbb{E}[\mathrm{e}^{\beta\bar{X}_\infty}]}. \tag{7}
$$

(a) When X is spectrally negative, $\bar{X}_\infty \sim \mathrm{Exp}(\phi(0))$ and it follows that $y_c = \log(2c(\phi(0) - 1)/(\phi(0) - 2))$.

(b) Finally, we present the statements about the continuous fit and the smooth fit. As v^* is continuous, we have continuous fit at y_c. It follows from (ii) that there is smooth fit at y_c exactly if $h(x) = \mathrm{e}^{y_c}\mathbb{E}[\mathrm{e}^{\bar{X}_\infty}\,\mathbf{1}_{(\bar{X}_\infty < y_c - x)}] - \mathrm{e}^x\mathbb{E}[\mathrm{e}^{2\bar{X}_\infty}\,\mathbf{1}_{(\bar{X}_\infty < y_c - x)}]$ is differentiable in y_c. For $\varepsilon > 0$, we have $h(y_c + \varepsilon) - h(y_c) = 0$ and

$$
h(y_c) - h(y_c - \varepsilon) = \mathrm{e}^{y_c}\mathbb{E}[\mathrm{e}^{\bar{X}_\infty}(1 - \mathrm{e}^{\bar{X}_\infty})\,\mathbf{1}_{(\bar{X}_\infty < \varepsilon)}](+\mathrm{e}^{y_c} - \mathrm{e}^{y_c - \varepsilon})(-\mathbb{E}[\mathrm{e}^{\bar{X}_\infty}\,\mathbf{1}_{(\bar{X}_\infty < \varepsilon)}]).
$$

Thus, $(1/\varepsilon)(h(y_c) - h(y_c - \varepsilon)) \to \mathrm{e}^{-y_c}\mathbb{P}(\bar{X}_\infty = 0)$, and, hence, we have smooth fit at y_c exactly if the distribution of $\bar{X}_\infty$ is continuous at 0. This completes the proof.

Remark 1. Recall that 0 is regular for $(0, \infty)$ if $\tau_0^{++} = 0$ a.s., and 0 is irregular for $(0, \infty)$ if $\tau_0^{++} > 0$ a.s. By [2, Theorem 6.5], the latter is a subclass of Lévy processes with bounded variation, and it contains compound Poisson processes and processes with strictly negative drift. Finally, by [8, Lemma 49.6] we obtain $\tau_y^+ = \tau_y^{++}$ a.s. for $y > 0$, if X is not a compound Poisson process.

https://doi.org/10.1239/aap/1427814584 Published online by Cambridge University Press

Lemma 1. *Let X be a Lévy process with $\psi(2) < 0$.*

(a) *If* 0 *is regular for* $(0, \infty)$ *then the distribution of* $\bar{X}_\infty$ *is continuous.*

(b) *If* 0 *is irregular for* $(0, \infty)$ *then the distribution of* $\bar{X}_\infty$ *has discontinuity points that includes* 0*. If, additionally, X is not a compound Poisson process then* 0 *is the only discontinuity point for the distribution of* $\bar{X}_\infty$*.*

Proof. (a) Assume that 0 is regular for $(0, \infty)$. Clearly, $\mathbb{P}(\bar{X}_\infty = 0) = 0$. From Remark 1, it follows that $\tau_y^+ = \tau_y^{++}$ a.s. for $y > 0$ and, thus, for $y > 0$,

$$\mathbb{P}(\bar{X}_\infty = y) = \mathbb{P}(\tau_y^+ < \infty) - \mathbb{P}(\tau_y^{++} < \infty) = 0.$$

Recall that $\psi(2) < 0$ implies that X converges to $-\infty$. Thus, $\bar{X}_\infty < \infty$, and it follows that the distribution of $\bar{X}_\infty$ is continuous.

(b) Assume that 0 is irregular for $(0, \infty)$. We prove the statement by contradiction and, thus, assume that $\mathbb{P}(\bar{X}_\infty = 0) = 0$. Then $\tau_0^{++} < \infty$ a.s. Let $Y_1 = X_{\tau_0^{++}}$ and $X^{(1)} = X$, and, for $n \in \{2, 3, \ldots\}$, let $Y_n = X^{(n)}_{\tau_0^{++}}$ and $X^{(n)}_t = X^{(n-1)}_{\tau_0^{++}+t} - X^{(n-1)}_{\tau_0^{++}}$. It follows from induction, the strong Markov property, and the fact that $\tau_0^{++} < \infty$ a.s. that, for all $n \in \mathbb{N}$, the Lévy processes $X^{(n)}$ are identically distributed and a.s. well defined. Let $\tau_0^{++}(X^{(n)}) = \inf\{t \geq 0 \colon X^{(n)}_t > 0\}$. Then the sequences $(Y_n)_{n\in\mathbb{N}}$ and $(\tau_0^{++}(X^{(n)}))_{n\in\mathbb{N}}$ are both independent and indentically distributed(i.i.d.) For all $N \in \mathbb{N}$,

$$X_{\sum_{n=1}^{N} \tau_0^{++}(X^{(N)})} = \sum_{n=1}^{N} Y_n.$$

From the law of large numbers, it follows that $\sum_{n=1}^{N} \tau_0^{++}(X^{(n)}) \to \infty$ a.s. when $N \to \infty$. Since $Y_n \geq 0$ for $n \in \mathbb{N}$, then, for X, there will a.s. exist arbitrarily large $t \in (0, \infty)$ with $X_t \geq 0$. But this is in contradiction to $X_t \to -\infty$ a.s. when $t \to \infty$.

Lemma 2. *Let X be a Lévy process with $\psi(2) < 0$ which is not a compound Poisson process. If $y_c = 0$ from Theorem 1 and τ is an optimal stopping time for the quadratic problem such that $\mathbb{P}(\tau > 0) > 0$, then $\tau \geq \tau_0^{++}$.*

Proof. If 0 is irregular for $(0, \infty)$, the statement follows easily. If 0 is irregular for $(0, \infty)$, we prove the statement by contradiction and assume that $\mathbb{P}(\tau < \tau_0^{++}) > 0$. First note, by Blumenthal's 0–1 law, that $\tau > 0$ a.s. It holds that $v^*(x) = \mathbb{E}_x[G(X_{\tau_0^{++}})]$ and, by (6), it follows that $G(x) < v^*(x)$ for $x < 0$ because $\mathbb{P}(\bar{X}_\infty = 0) > 0$. By [8, Exercise 50.4], the assumption that $\mathbb{P}(\tau < \tau_0^{++}) > 0$ implies that $\mathbb{P}(X_\tau < 0) > 0$. Hence, $\mathbb{P}(v^*(X_\tau) > G(X_\tau)) > 0$ and it follows that $\mathbb{E}[G(X_\tau)] < \mathbb{E}[v^*(X_\tau)] \leq v^*(x)$ since $v^*(X_t)$ is a supermartingale, leading to a contradiction with the fact that τ is optimal.

Example 1. Let X be a Lévy process such that $-X$ is given by the Cramér–Lundberg model with exponential jumps, that is, $X_t = -dt + \sum_{n=1}^{N_t} Z_n$, where $d > 0$, N is a Poisson process with parameter $\lambda > 0$, and $(Z_n)_{n\in\mathbb{N}}$ is an i.i.d. sequence of random variables independent of N with $Z_1 \sim \operatorname{Exp}(\alpha)$ for some $\alpha > 0$. From [4, Chapter 4.2], it follows that $\mathbb{P}(\bar{X}_\infty = 0) = 1 - \lambda/\alpha d$ and, for $(0, \infty)$, the density of $\bar{X}_\infty$ is $f(x) = (\lambda/\alpha d)(\alpha - \lambda/d)\mathrm{e}^{-(\alpha-\lambda/d)x}$. Therefore, $\psi(2) < 0$ corresponds to $\alpha > 2 + \lambda/d$. When this condition is fulfilled, we find that, for $\beta = \alpha - \lambda/d$,

$$\mathbb{E}[\mathrm{e}^{\bar{X}_\infty}] = \frac{\alpha}{\lambda d}\left(\frac{2\beta - 1}{\beta - 1}\right) \quad \text{and} \quad \mathbb{E}[\mathrm{e}^{2\bar{X}_\infty}] = \frac{\alpha}{\lambda d}\left(\frac{2\beta - 2}{\beta - 2}\right).$$

https://doi.org/10.1239/aap/1427814584 Published online by Cambridge University Press

From Theorem 1 we find that the optimal stopping point is $y_c = \log(4c(\beta-1)^2/((\beta-2)(2\beta-1)))$. By Remark 1, it follows that $\tau^+_{y_c} = \tau^{++}_{y_c}$.

Example 2. Let X be a compound Poisson process given by $X_t = \sum_{n=1}^{N_t} Z_n$, where N is a Poisson process with parameter $\lambda > 0$, and $(Z_n)_{n\in\mathbb{N}}$ is an i.i.d. sequence of random variables which is independent of N and for which $\mathbb{P}(Z_1 = \alpha) = 1 - \mathbb{P}(Z_1 = -\alpha) = p$ for some $p \in (0, 1)$. When $p < \frac{1}{2}$, the distribution of $\bar{X}_\infty/\alpha$ is geometric with $\mathbb{P}(\bar{X}_\infty \geq k\alpha) = (p/(1-p))^k$ and, when $p \geq \frac{1}{2}$, then $\bar{X}_\infty = \infty$ a.s. Therefore, assume that $p < \frac{1}{2}$, and we then obtain $\mathbb{E}[\mathrm{e}^{\bar{X}_\infty}] = (1-2p)/(1-p-\mathrm{e}^\alpha p)$ and $\mathbb{E}[\mathrm{e}^{2\bar{X}_\infty}] = (1-2p)/(1-p-\mathrm{e}^{2\alpha} p)$. By Theorem 1, it follows that the optimal stopping point is $y_c = \log(2c(1-p-\mathrm{e}^\alpha p)/(1-p-\mathrm{e}^{2\alpha}p))$. Both $\tau^+_{y_c}$ and $\tau^{++}_{y_c}$ are solutions.

3. The variance optimal stopping problem

In this section we solve variance problem (1) for those processes where the method of embedding can be applied. Recall that if there exists some stopping time τ^* and some constant c such that both (2) and (3) are fulfilled, then τ^* solves the variance problem. Indeed, for all τ, we have

$$\mathbb{V}[\mathrm{e}^{X_\tau}] \leq \mathbb{E}[(\mathrm{e}^{X_{\tau^*}} - c)^2] - (\mathbb{E}[\mathrm{e}^{X_\tau}] - c)^2 \tag{8}$$

$$= \mathbb{V}[\mathrm{e}^{X_{\tau^*}}] - (\mathbb{E}[\mathrm{e}^{X_\tau}] - \mathbb{E}[\mathrm{e}^{X_{\tau^*}}])^2. \tag{9}$$

The existence of a combination of τ^* and c that fulfills both of the two requirements is not certain. We show that it depends on whether at least one of the following equations has a solution:

$$\frac{\mathbb{E}[\mathrm{e}^{\bar{X}_\infty}]}{2\mathbb{E}[\mathrm{e}^{2\bar{X}_\infty}]}\mathrm{e}^y = \frac{\mathbb{E}[\mathrm{e}^{\bar{X}_\infty}\mathbf{1}_{(\bar{X}_\infty \geq y)}]}{\mathbb{E}[\mathrm{e}^{\bar{X}_\infty}]}, \tag{10}$$

$$\frac{\mathbb{E}[\mathrm{e}^{\bar{X}_\infty}]}{2\mathbb{E}[\mathrm{e}^{2\bar{X}_\infty}]}\mathrm{e}^y = \frac{\mathbb{E}[\mathrm{e}^{\bar{X}_\infty}\mathbf{1}_{(\bar{X}_\infty > y)}]}{\mathbb{E}[\mathrm{e}^{\bar{X}_\infty}]}. \tag{11}$$

We call these the embedding equations.

Theorem 2. *Let X be a Lévy process with $\psi(2) < 0$. If the embedding equation (10) has a solution $\hat{y}$ then $\tau^+_{\hat{y}}$ is an optimal stopping time for variance problem (1). If embedding equation (11) has a solution $\hat{y}$ then $\tau^{++}_{\hat{y}}$ is an optimal stopping time for variance problem (1).*

(a) *Assume that 0 is regular for $(0, \infty)$. Then the embedding equations coincide and there exists a solution $\hat{y}$. If, additionally, X is spectrally negative then*

$$\hat{y} = \frac{1}{\phi(0)} \log\left(2\frac{\phi(0)-1}{\phi(0)-2}\right).$$

(b) *Assume that 0 is irregular for $(0, \infty)$. Then if*

$$\mathbb{E}[\mathrm{e}^{\bar{X}_\infty}]^2 > 2\mathbb{E}[\mathrm{e}^{2\bar{X}_\infty}]\mathbb{E}[\mathrm{e}^{\bar{X}_\infty}\mathbf{1}_{(\bar{X}_\infty > 0)}], \tag{12}$$

the embedding equations have no solutions. Assume, additionally, that X is not a compound Poisson process and that (12) is not satisfied, then at least one of the embedding equations has a solution.

(c) *Assume that $\psi(2) = 0$. Then, for all $\tau \in \mathcal{T}$, we have $\mathbb{V}[\mathrm{e}^{X_\tau}] < 1$, but $\mathbb{V}[\mathrm{e}^{X_t}] \to 1$ as $t \to \infty$.*

Proof. From Theorem 1, both $\tau_{\hat{y}}^{+}$ and $\tau_{\hat{y}}^{++}$ are solutions to the quadratic problem with parameter $c = (\mathbb{E}[e^{\bar{X}_\infty}]/2\mathbb{E}[e^{2\bar{X}_\infty}])e^{\hat{y}}$. Thus, the left-hand side of the embedding equations give the parameter value of c needed for $\tau_{\hat{y}}^{+}$ and $\tau_{\hat{y}}^{++}$ to solve the quadratic problem. From (4) we deduce that the right-hand side of the embedding equations give the respective values $\mathbb{E}[e^{X_{\tau_{\hat{y}}^{+}}}]$ and $\mathbb{E}[e^{X_{\tau_{\hat{y}}^{++}}}]$. This proves that, when $\hat{y}$ solves as least one of the embedding equations, then $\tau_{\hat{y}}^{+}$ and $\tau_{\hat{y}}^{++}$ respectively solve the variance problem.

Next, we investigate the existence of a solution to the embedding equation (10). First, note that $\mathbb{E}[e^{\bar{X}_\infty}] \le \mathbb{E}[e^{2\bar{X}_\infty}]$ and, thus, for $y = 0$, the left-hand side is less than or equal to $\frac{1}{2}$, whereas the right-hand side is 1. For $y \to \infty$, the left-hand side increases continuously to ∞, whereas the right-hand side converges to 0.

(a) As 0 is regular for $(0, \infty)$, then, by Lemma 1, the distribution of $\bar{X}_\infty$ is continuous. Hence, the right-hand side of (10) is also continuous and, hence, the embedding equation (10) has a solution. As the distribution of $\bar{X}_\infty$ is continuous, the embedding equations coincide and, thus, they both have a solution. In the special case where X is a spectrally negative Lévy process, we recall that $\bar{X}_\infty$ is exponentially distributed with parameter $\phi(0)$ and, hence, the result is straightforward.

(b) The left-hand side of the embedding equations are equal. As a function of y, the right-hand side of (10) is left continuous, and the right-hand side of (11) is the right-continuous version. Thus, if the right-hand side of (11) is smaller than the left-hand side of (11) for $y = 0$ then the embedding equations have no solutions. This corresponds to (12). The right-hand sides of (10) and (11) have discontinuities only at points where the distribution of $\bar{X}_\infty$ has discontinuities. If X is not a compound Poisson process then 0 is the only discontinuity point for the distribution of $\bar{X}_\infty$ and, thus, if (12) does not hold then the embedding equations must have a solution.

(c) Consider a $\tau \in \mathcal{T}$. If $\tau = \infty$ a.s. then $\mathbb{V}[e^{X_\tau}] = 0$. If $\mathbb{P}(\tau < \infty) > 0$ then $\mathbb{E}[e^{X_\tau}] > 0$ and, thus,

$$\mathbb{V}[e^{X_\tau}] < \mathbb{E}[e^{2X_\tau}] \le \mathbb{E}[e^{2X_\tau} \mathbf{1}_{(\tau<\infty)}] \le \lim_{t\to\infty} \mathbb{E}[e^{2X_{\tau\wedge t}}] = 1.$$

Next, recall that $\psi(2) = 0$ implies that e^{2X_t} is a martingale and recall that X is not deterministic. Thus, $\mathbb{V}[e^{X_1}] > 0$ and we obtain $\psi(1) = \log \mathbb{E}[e^{X_1}] < 0$. Therefore, $\mathbb{V}[e^{X_t}] = e^{\psi(2)t} - e^{\psi(1)t} \to 1$ as $t \to \infty$.

Remark 2. Let $Y_t = X_t + x$ be a Lévy process starting at x. Then, for any stopping time τ,

$$\mathbb{V}[e^{Y_\tau}] = \mathbb{V}[e^{X_\tau + x}] = e^{2x}\mathbb{V}[e^{X_\tau}].$$

Therefore, a stopping time is optimal for the variance problem of process Y, if it is optimal for the variance problem of process X. Whenever the variance problem for X is solved by a hitting time, then so is the variance problem for Y. However, the stopping region will be shifted by the starting value x.

Remark 3. When X is a spectrally negative process, it is usually possible to calculate $\phi(0)$ and, therefore, we may determine the optimal stopping time for the quadratic problem given in Theorem 1. Furthermore, as the distribution of $\bar{X}_\infty$ is $\text{Exp}(\phi(0))$, we can solve the embedding equations and find a solution in order to determine the optimal stopping time for the variance given in Theorem 2. However, if X is not spectrally negative, we need to determine $\mathbb{E}[e^{\bar{X}_\infty}]$ and $\mathbb{E}[e^{2\bar{X}_\infty}]$ in order to apply Theorem 1, and to solve the embedding equations in order to apply Theorem 2. These are, in most cases, impossible to calculate as the distribution of $\bar{X}_\infty$ is often not known.

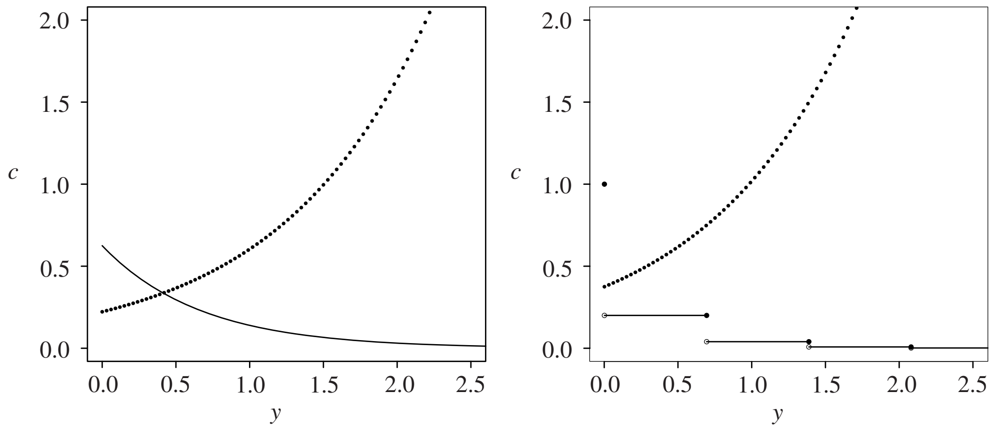

FIGURE 1: The dotted lines indicate the (y, c) values for which τ_y^+ solves the quadratic problem with parameter c, and the solid lines indicate the (y, c) values for which $c = \mathbb{E}[\exp(X_{\tau_y^+})]$. When the two lines intersect, the y-value of the intersection gives a value for which τ_y^+ solves the variance problem. *Left:* the Cramér–Lundberg process of Example 3 with $\alpha = 3$, $d = 4$, and $\lambda = 2$. *Right:* The compound Poisson process of Example 4 with $p = \frac{1}{11}$ and $\alpha = \log(2)$.

Example 3. Consider the negative Cramér–Lundberg Lévy process given in Example 1. Again, assume that $\alpha > 2 + \lambda/d$ and recall the distribution of $\bar{X}_\infty$ given in Example 1. For $y > 0$, we compute $\mathbb{E}[\mathrm{e}^{\bar{X}_\infty} \mathbf{1}_{(\bar{X}_\infty \geq y)}] = (\alpha/\lambda d)((\beta/(\beta-1))\mathrm{e}^{-(\beta-1)y})$ and write embedding equation (10) as

$$\mathrm{e}^{-\beta y} = \frac{(2\beta-1)^2(\beta-2)}{4\beta(\beta-1)^2}. \tag{13}$$

This equation has a solution, and it follows from Theorem 2 that the variance problem has solutions $\tau_{\hat{y}}^+$ and $\tau_{\hat{y}}^{++}$ with $\hat{y}$ solving (13). For example, if $\alpha = 3$, $\lambda = 2$, and $d = 4$, then $\hat{y} = \frac{2}{5}\log(\frac{45}{16}) \approx 0.413\,630$. See Figure 1.

Example 4. Consider the compound Poisson process given in Example 2 with $p = \frac{1}{11}$ and $\alpha = \log(2)$. Recall from Section 2 that $\bar{X}_\infty$ has a geometric distribution with $\mathbb{P}(\bar{X}_\infty \geq k\log(2)) = (\frac{1}{10})^k$. Thus, the left-hand side of (12) equals $\frac{27}{64}$, whereas the right-hand side equals $\frac{9}{40}$. As $\frac{9}{40} < \frac{27}{64}$, it follows that the embedding equations have no solution and, therefore, Theorem 2 does not give a solution for the variance problem of this process. See Figure 1.

4. Randomized stopping

Theorem 2 only offers a solution to the variance problem if one of the embedding equations has a solution. In this section we introduce randomized stopping in order to overcome this problem. When the embedding equations have no solution, an immediate complication arises because it is not easy to find a constant c and stopping time τ^* such that both (2) and (3) are solved, and so the embedding method cannot be applied. By taking the supremum over a wider class of stopping times we overcome this problem.

We introduce randomized stopping times for an optimal stopping problem by expanding the filtration without removing the Markov property of the process. We create this expansion by introducing a random variable U, which is defined on the same probability space as X, and which is uniformly distributed on $[0, 1]$ and independent of X. We note that this may require that we augment the probability space on which X is defined. We then define the new filtration $(\hat{\mathcal{F}}_t)_{t\geq 0}$ by $\hat{\mathcal{F}}_t = \sigma(U) \vee \mathcal{F}_t$. We let $\hat{\mathcal{T}}$ be the set of stopping times with respect to the new filtration $\hat{\mathcal{F}}_t$, and we refer to these as *randomized stopping times*. Randomized stopping times are discussed in a discrete-time setup in [9], where it was shown that the value function of a classical optimal stopping problem does not change when randomized stopping times are introduced. This result carries over to the quadratic problem, but it does not carry over to the variance problem. The verification theorem in the proof of Theorem 1 is based on the optional sampling theorem, and, hence, it also holds for a classical optimal stopping problem with randomized stopping. This is due to the fact that the Lévy process remains a Markov process with the augmented filtration. In particular, the stopping times of Theorem 1 also solve the quadratic problem with randomized stopping.

Theorem 3. *Let X be a Lévy process with $\psi(2) < 0$, and consider the quadratic optimal stopping problem with randomized stopping. Let U be a random variable uniformly distributed on $[0, 1]$ and independent of X. Let $y_c = \log(2c\mathbb{E}[\mathrm{e}^{2\bar{X}_\infty}]/\mathbb{E}[\mathrm{e}^{\bar{X}_\infty}])$, $p \in [0, 1]$, and $Y = \mathbf{1}_{(U<p)}$. Then $\tau^* = Y\tau^+_{y_c} + (1-Y)\tau^{++}_{y_c}$ is also an optimal stopping time.*

Proof. The result follows from the facts that τ^* is a stopping time with respect to the augmented filtration, and the expected gain at τ^* equals the expected gain at $\tau^+_{y_c}$ and $\tau^{++}_{y_c}$.

By Remark 1, we see that, for some Lévy processes, $\tau^+_y = \tau^{++}_y$ for all $y \geq 0$ and the stopping times τ^* in Theorem 3 do not introduce a new class of optimal stopping times for the quadratic problem. However, if 0 is irregular for $(0, \infty)$ then, for some discontinuity points y of the distribution of $\bar{X}_\infty$, we have $\tau^+_y < \tau^{++}_y$. In this case the introduction of randomized stopping times creates a new family of optimal stopping times for the quadratic problem such that $\tau^+_{y_c} \leq \tau^* \leq \tau^{++}_{y_c}$. With the wider family of stopping times we can solve the variance problem in all cases.

Theorem 4. *Let X be a Lévy process with $\psi(2) < 0$.*

(a) *If at least one of the embedding equations (10) and (11) has a solution, then the optimal stopping times given in Theorem 2 are also optimal for the variance problem with randomized stopping.*

(b) *If the embedding equations (10) and (11) have no solutions, let*

$$\hat{y} \equiv \inf\left\{ y \geq 0 \colon \frac{\mathbb{E}[\mathrm{e}^{\bar{X}_\infty}]}{2\mathbb{E}[\mathrm{e}^{2\bar{X}_\infty}]}\mathrm{e}^y > \frac{\mathbb{E}[\mathrm{e}^{\bar{X}_\infty}\mathbf{1}_{(\bar{X}_\infty \geq y)}]}{\mathbb{E}[\mathrm{e}^{\bar{X}_\infty}]} \right\}. \tag{14}$$

Let U be uniformly distributed on $[0, 1]$ and independent of X, and let $Y = \mathbf{1}_{(U<p)}$, where

$$p = \frac{(\mathbb{E}[\mathrm{e}^{\bar{X}_\infty}]/2\mathbb{E}[\mathrm{e}^{2\bar{X}_\infty}])\mathrm{e}^{\hat{y}} - \mathbb{E}[\mathrm{e}^{X_{\tau^{++}_{\hat{y}}}}]}{\mathbb{E}[\mathrm{e}^{X_{\tau^{+}_{\hat{y}}}}] - \mathbb{E}[\mathrm{e}^{X_{\tau^{++}_{\hat{y}}}}]}. \tag{15}$$

Then $\tau^* = Y\tau^+_{\hat{y}} + (1-Y)\tau^{++}_{\hat{y}}$ is optimal for the variance problem with randomized stopping. Moreover, the distribution of $\bar{X}_\infty$ has a discontinuity in $\hat{y}$, and if 0 is irregular for $(0, \infty)$ and X is not a compound Poisson process, then $\hat{y} = 0$.

Proof. (a) The optimal stopping times given in Theorem 2 solve the corresponding problems with randomized stopping. This result follows in the same way as in the proof of Theorem 2.

(b) From the proof of Theorem 2(b), it follows that if the embedding equations (10) and (11) have no solution, then the distribution of $\bar{X}_\infty$ has a discontinuity in the value $\hat{y}$, defined in (14). The inequality in (14) holds for every $y > \hat{y}$. Thus, from (7), it follows that

$$\frac{\mathbb{E}[e^{\bar{X}_\infty}]}{2\mathbb{E}[e^{2\bar{X}_\infty}]}e^{y} > \mathbb{E}[e^{X_{\tau_y^{++}}}]$$

for every $y > \hat{y}$. As both sides are right-continuous in y and as the embedding equations have no solution, then the same holds for $y = \hat{y}$. On the other hand, the inequality in (14) does not hold for any $y < \hat{y}$. As $\mathbb{E}[\exp(X_{\tau_{\hat{y}}^{+}})]$ is the left-continuous version of $\mathbb{E}[\exp(X_{\tau_{\hat{y}}^{++}})]$ then

$$\frac{\mathbb{E}[e^{\bar{X}_\infty}]}{2\mathbb{E}[e^{2\bar{X}_\infty}]}e^{\hat{y}} \leq \mathbb{E}[e^{X_{\tau_{\hat{y}}^{+}}}].$$

Hence, $p \in [0, 1]$ and it follows from Theorem 3 that τ^* is an optimal stopping time for the quadratic problem with parameter $c = (\mathbb{E}[e^{\bar{X}_\infty}]/2\mathbb{E}[e^{2\bar{X}_\infty}])e^{\hat{y}}$. Then we need only to prove that $\mathbb{E}[e^{X_{\tau^*}}] = c$:

$$\begin{aligned}
\mathbb{E}[e^{X_{\tau^*}}] &= p\mathbb{E}[e^{X_{\tau_{\hat{y}}^{+}}}] + (1-p)\mathbb{E}[e^{X_{\tau_{\hat{y}}^{++}}}] \\
&= \left(\frac{(\mathbb{E}[e^{\bar{X}_\infty}]/2\mathbb{E}[e^{2\bar{X}_\infty}])e^{\hat{y}} - \mathbb{E}[e^{X_{\tau_{\hat{y}}^{++}}}]}{\mathbb{E}[e^{X_{\tau_{\hat{y}}^{+}}}] - \mathbb{E}[e^{X_{\tau_{\hat{y}}^{++}}}]}\right)\mathbb{E}[e^{X_{\tau_{\hat{y}}^{+}}}] \\
&\quad + \left(\frac{\mathbb{E}[e^{X_{\tau_{\hat{y}}^{+}}}] - (\mathbb{E}[e^{\bar{X}_\infty}]/2\mathbb{E}[e^{2\bar{X}_\infty}])e^{\hat{y}}}{\mathbb{E}[e^{X_{\tau_{\hat{y}}^{+}}}] - \mathbb{E}[e^{X_{\tau_{\hat{y}}^{++}}}]}\right)\mathbb{E}[e^{X_{\tau_{\hat{y}}^{++}}}] \\
&= \frac{\mathbb{E}[e^{\bar{X}_\infty}]}{2\mathbb{E}[e^{2\bar{X}_\infty}]}e^{\hat{y}} \\
&= c.
\end{aligned}$$

If the Lévy process is not a compound Poisson process then, by Lemma 1, the distribution of $\bar{X}_\infty$ has a discontinuity point only at 0. Therefore, $\hat{y} = 0$ and the optimal stopping time is a combination of $\tau_0^+ = 0$ and τ_0^{++}.

Example 5. Let X denote the compound Poisson process considered in Example 4 with $p = \frac{1}{11}$ and $\alpha = \log(2)$. Recall that we cannot solve the variance problem for this process by Theorem 2. From (15) we calculate $p = \frac{7}{32}$. Let Y be a random variable independent of X and with $\mathbb{P}(Y = 1) = 1 - \mathbb{P}(Y = 0) = p$. By Theorem 4, it follows that $\tau^* = Y\tau_0^+ + (1-Y)\tau_0^{++} = (1-Y)\tau_0^{++}$ is an optimal stopping time for the variance problem with randomized stopping. See Figure 2. The variance for this stopping time is

$$\mathbb{V}[e^{X_{\tau^*}}] = p\mathbb{E}[e^{2X_{\tau_0^+}}] + (1-p)\mathbb{E}[e^{2X_{\tau_0^{++}}}] - (p\mathbb{E}[e^{X_{\tau_0^+}}] + (1-p)\mathbb{E}[e^{X_{\tau_0^{++}}}])^2 = 0.390\,625.$$

We determine the best stopping time of the form τ_y^+ and τ_y^{++} for $y \geq 0$. It is sufficient to consider stopping times of the form $\tau_{k\log(2)}^+$ for $k \in \mathbb{N}$. For these, $\mathbb{V}[\exp(X_{\tau_{k\log(2)}^+})] = 0.4^k - 0.04^k$. Inspection of this function reveals that the maximal value is obtained for $k = 1$ with $\mathbb{V}[e^{X_{\tau_{\log(2)}^+}}] = 0.36$. This is a smaller variance than that obtained from our solution τ^* from Theorem 4. However, the stopping time τ^* is not a stopping time with respect to the filtration generated from X.

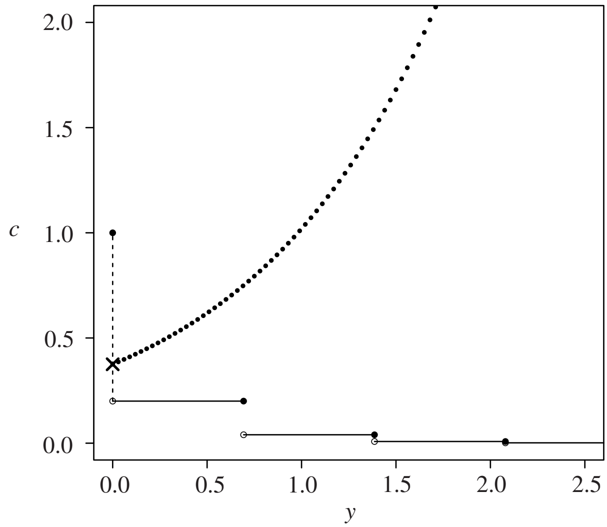

FIGURE 2: An illustration of Example 5, with $p = \frac{1}{11}$ and $\alpha = \log(2)$. The dotted line indicates the (y, c) values for which τ_y^+ and τ_y^{++} solve the quadratic problem with parameter c, and the solid line indicates the (y, c) values for $c = \mathbb{E}[\exp(X_{\tau_y^+})]$. The dashed line and the cross illustrate how, by randomizing between τ_0^+ and τ_0^{++}, we may obtain a (τ, c) such that both requirements are fulfilled.

5. The variance problem without randomized stopping revisited

As demonstrated in Example 2, the embedding equations do not always have a solution. In this section we want to solve the variance problem without randomized stopping when the embedding equations have no solution. At first, one may hope that some new approach reveals an excess boundary solution for the variance problem when the embedding equations have no solution. However, Theorem 5 below reveals that this is not possible.

Theorem 5. *Let X be a Lévy process with $\psi(2) < 0$. Assume that the embedding equations have no solution. Then the variance problem with randomized stopping does not have an optimal stopping time of the form τ_y^+ or τ_y^{++} for any $y \in \mathbb{R}$.*

Proof. Let τ^* be an optimal stopping time for the variance problem given in Theorem 4, and let $\hat{y}$ be as given by (14). It follows from (9) that for some τ to solve the variance problem, we need $\mathbb{E}[e^{X_\tau}] = \mathbb{E}[e^{X_{\tau^*}}]$. However, from the proof of Theorem 4, it follows that

$$\mathbb{E}[e^{X_{\tau^*}}] = \frac{\mathbb{E}[e^{\bar{X}_\infty}]}{2\mathbb{E}[e^{2\bar{X}_\infty}]} e^{\hat{y}} \quad \text{and} \quad \mathbb{E}[e^{X_{\tau_{\hat{y}}^+}}] > \frac{\mathbb{E}[e^{\bar{X}_\infty}]}{2\mathbb{E}[e^{2\bar{X}_\infty}]} e^{\hat{y}} > \mathbb{E}[e^{X_{\tau_{\hat{y}}^{++}}}].$$

The inequalities are strict because the embedding equations have no solution. From (4) and (7), it follows that $\mathbb{E}[e^{X_{\tau^*}}] \neq \mathbb{E}[e^{X_{\tau_y^+}}]$ and $\mathbb{E}[e^{X_{\tau^*}}] \neq \mathbb{E}[e^{X_{\tau_y^{++}}}]$ for all y. Thus, there will not be any solutions of the form τ_y^+ or τ_y^{++} for $y \in \mathbb{R}$.

5.1. Compound Poisson processes

Let X be a compound Poisson process with jump intensity λ. That is, the Poisson process N that counts the number of jumps has parameter λ. Assume that the embedding equations have no solution, and consider the optimal stopping time of the variance problem with randomized

https://doi.org/10.1239/aap/1427814584 Published online by Cambridge University Press

stopping, given in Theorem 4. From Theorem 6 below, it follows that one may mimic τ^* by a stopping time in $\mathcal{T}$. The idea of the proof is based on the observation that $\mathcal{F}$ contains information about the process N, and N is independent of both $X_{\tau_{\hat{y}}^{+}}$ and $X_{\tau_{\hat{y}}^{++}}$. Hence, we may use the process N to choose between the two stopping times τ_y^+ and τ_y^{++} instead of using the random variable Y.

Theorem 6. *Let X be a compound Poisson process with $\psi(2) < 0$ and jump intensity $\lambda > 0$. Let T be the first jump time of X. Assume that the embedding equations have no solution, and let $\hat{y}$ and p be given as in Theorem 4. Then*

$$\tilde{\tau}^* = \tilde{Y}\tilde{\tau}_{\hat{y}}^{+} + (1-\tilde{Y})\tau_{\hat{y}}^{++}$$

is an optimal stopping time of the variance problem without randomized stopping where $\tilde{\tau}_{\hat{y}}^{+} = t_0 \vee \tau_{\hat{y}}^{+}$, $t_0 = (-1/\lambda)\log(p)$, and $\tilde{Y} = \mathbf{1}_{(t_0<T)}$.

Proof. First note that τ^* is a stopping time of $\mathcal{T}$. For every $t < t_0$, we have

$$\{\tilde{\tau}^* \le t\} = \{\tilde{\tau}_{\hat{y}}^{+} \le t,\ \tilde{Y} = 1\} \cup \{\tau_{\hat{y}}^{++} \le t,\ \tilde{Y} = 0\} = \{\tau_{\hat{y}}^{++} \le t,\ \tilde{Y} = 0\} = \{\tau_{\hat{y}}^{++} \le t\} \in \mathcal{F}_t,$$

and, for $t \ge t_0$,

$$\{\tilde{\tau}^* \le t\} = (\{\tau_{\hat{y}}^{+} \vee t_0 \le t\} \cap \{\tilde{Y} = 1\}) \cup (\{\tau_{\hat{y}}^{++} \le t\} \cap \{\tilde{Y} = 0\}) \in \mathcal{F}_t.$$

Thus, $\tau^* \in \mathcal{T}$.

Recall that, on the event $\{\tilde{Y} = 1\}$, $X_{\tilde{\tau}_{\hat{y}}^{+}}$ and $X_{\tau_{\hat{y}}^{+}}$ have the same distribution. Let τ^* be the optimal randomized stopping time given in Theorem 4(b). Then, for $\beta = 1, 2$, we have

$$\begin{aligned}
\mathbb{E}[\mathrm{e}^{\beta X_{\tilde{\tau}^*}}] &= p\mathbb{E}[\mathrm{e}^{\beta X_{\tilde{\tau}_{\hat{y}}^{+}}} \mid \tilde{Y} = 1] + (1-p)\mathbb{E}[\mathrm{e}^{\beta X_{\tau_{\hat{y}}^{++}}} \mid \tilde{Y} = 0] \\
&= p\mathbb{E}[\mathrm{e}^{\beta X_{\tau_{\hat{y}}^{+}}} \mid \tilde{Y} = 1] + (1-p)\mathbb{E}[\mathrm{e}^{\beta X_{\tau_{\hat{y}}^{++}}} \mid \tilde{Y} = 0] \\
&= p\mathbb{E}[\mathrm{e}^{\beta X_{\tau_{\hat{y}}^{+}}}] + (1-p)\mathbb{E}[\mathrm{e}^{\beta X_{\tau_{\hat{y}}^{++}}}] \\
&= \mathbb{E}[\mathrm{e}^{\beta X_{\tau^*}}].
\end{aligned}$$

Hence, we see that $\mathbb{V}[\mathrm{e}^{X_{\tilde{\tau}^*}}] = \mathbb{V}[\mathrm{e}^{X_{\tau^*}}]$ and we conclude that $\tilde{\tau}^*$ is an optimal stopping time of the variance problem without randomized stopping.

Example 6. Let X denote the compound Poisson process considered in Example 4, with $p = \frac{1}{11}$ and $\alpha = \log(2)$. Recall that, for this process, we cannot use Theorem 2 to solve the variance problem. Using Theorem 4, we obtain a randomized solution, τ^*, and we have seen that no stopping time of the form τ_y^+ or τ_y^{++} gives as high a variance as τ^*. However, by the use of Theorem 6, we may actually find a stopping time of $\mathcal{T}$, giving the same variance as the randomized solution τ^*. Let $\tilde{Y} = \mathbf{1}_{\{N_{t_0}=0\}}$ and $t_0 = -\log(\frac{25}{32})$, and define $\tilde{\tau}^* = \tilde{Y}t_0 + (1-\tilde{Y})\tau_0^{++}$. Then it follows that $\tilde{\tau}^* \in \mathcal{T}$ and $\tilde{\tau}^*$ solve the variance problem without randomized stopping with as high a variance as that obtained by using τ^*.

5.2. Lévy processes which are not compound Poisson processes

In this section we consider Lévy processes of bounded variation with $\psi(2) < 0$ which are not compound Poisson processes.

Theorem 7. *Let X be a Lévy process of bounded variation with $\psi(2) < 0$ which is not a compound Poisson process, and let d denote the drift of X. Assume that the embedding equations have no solution. Let $\tau^* \in \hat{\mathcal{T}}$ be the randomized optimal stopping time given in Theorem 4. Then $\mathbb{V}[\mathrm{e}^{X_\tau}] < \mathbb{V}[\mathrm{e}^{X_{\tau^*}}]$ for all $\tau \in \mathcal{T}$.*

(a) *If $X_t - dt$ is not a compound Poisson process then* $\sup_{\tau\in\mathcal{T}} \mathbb{V}[\mathrm{e}^{X_\tau}] = \mathbb{V}[\mathrm{e}^{X_{\tau^*}}]$.

(b) *If $X_t - dt$ is a compound Poisson process then* $\sup_{\tau\in\mathcal{T}} \mathbb{V}[\mathrm{e}^{X_\tau}] < \mathbb{V}[\mathrm{e}^{X_{\tau^*}}]$.

Proof. For $\tau = 0$, we have the inequality $\mathbb{V}[\mathrm{e}^{X_\tau}] < \mathbb{V}[\mathrm{e}^{X_{\tau^*}}]$, and, by Blumenthal's 0–1 law, it is enough to consider $\tau > 0$ in $\mathcal{T}$. Assume that $\mathbb{V}[\mathrm{e}^{X_\tau}] = \mathbb{V}[\mathrm{e}^{X_{\tau^*}}]$. It follows from Theorem 4 that $\hat{y} = 0$, and from (8) and (9) that τ solves the quadratic problem with parameter $\mathbb{E}[\mathrm{e}^{X_{\tau^*}}]$ and $\mathbb{E}[\mathrm{e}^{X_\tau}] = \mathbb{E}[\mathrm{e}^{X_{\tau^*}}]$. By Lemma 2 we obtain $\tau \geq \tau_0^{++}$ and, hence, $\mathbb{E}[\mathrm{e}^{X_\tau}] \leq \mathbb{E}[\exp(X_{\tau_0^{++}})]$, as e^{X_t} is a supermartingale. However, from the proof of Theorem 5, it follows that $\mathbb{E}[\exp(X_{\tau_0^{++}})] < \mathbb{E}[\mathrm{e}^{X_{\tau^*}}]$. Therefore, $\mathbb{E}[\mathrm{e}^{X_\tau}] < \mathbb{E}[\mathrm{e}^{X_{\tau^*}}]$ and we cannot have $\mathbb{V}[\mathrm{e}^{X_\tau}] = \mathbb{V}[\mathrm{e}^{X_{\tau^*}}]$.

(a) Let α_t^q be the q-fractile of X_t. That is, $\alpha_t^q = \inf\{\alpha \in \mathbb{R} : \mathbb{P}(X_t \leq \alpha) > q\}$. It follows from [8, Theorem 27.4] that the distribution of X_t is continuous and, hence, for every $t > 0$ and $q \in (0, 1)$, we have $\mathbb{P}(X_t \leq \alpha_t^q) = \mathbb{P}(X_t < \alpha_t^q) = q$.

Next, we choose a sequence of stopping times, $(\tau_n)_{n\in\mathbb{N}} \subset \mathcal{T}$, in the following way:

$$\tau_n = \inf\left\{t > \frac{1}{n} \,\middle|\, X_t > 0\right\} \mathbf{1}_{(X_{1/n}\in(\alpha_{1/n}^{(1-p)/2},\, \alpha_{1/n}^{(1+p)/2}))} + \frac{1}{n}\,\mathbf{1}_{(X_{1/n}\notin(\alpha_{1/n}^{(1-p)/2},\, \alpha_{1/n}^{(1+p)/2}))}\,.$$

That is, if the process at time $1/n$ is not between the fractiles $\alpha_{1/n}^{(1-p)/2}$ and $\alpha_{1/n}^{(1+p)/2}$, then the process is stopped. On the other hand, if the process at time $1/n$ is between the two fractiles then the process is stopped at the first time after $1/n$ when it gets above 0. We show that the variance at the stopping time τ_n approximates the variance at the randomized stopping time given in Theorem 4.

It holds that

$$\mathbb{E}[\mathrm{e}^{X_{\tau_n}}\,\mathbf{1}_{(X_{1/n}\notin(\alpha_{1/n}^{(1-p)/2},\alpha_{1/n}^{(1+p)/2}))}] = \mathbb{E}[\mathrm{e}^{X_{1/n}}] - \mathbb{E}[\mathrm{e}^{X_{1/n}}\,\mathbf{1}_{(X_{1/n}\in(\alpha_{1/n}^{(1-p)/2},\alpha_{1/n}^{(1+p)/2}))}]$$

is bounded from below by

$$\mathbb{E}[\mathrm{e}^{X_{1/n}}] - \mathrm{e}^{\alpha_{1/n}^{(1+p)/2}}\,\mathbb{P}(X_{1/n} \in (\alpha_{1/n}^{(1-p)/2}, \alpha_{1/n}^{(1+p)/2})) = \mathbb{E}[\mathrm{e}^{X_{1/n}}] - \mathrm{e}^{\alpha_{1/n}^{(1+p)/2}}\, p,$$

and from above by

$$\mathbb{E}[\mathrm{e}^{X_{1/n}}] - \mathrm{e}^{\alpha_{1/n}^{(1-p)/2}}\,\mathbb{P}(X_{1/n} \in (\alpha_{1/n}^{(1-p)/2}, \alpha_{1/n}^{(1+p)/2})) = \mathbb{E}[\mathrm{e}^{X_{1/n}}] - \mathrm{e}^{\alpha_{1/n}^{(1-p)/2}}\, p.$$

As $\mathbb{E}[\mathrm{e}^{X_{1/n}}]$ converges to 1 as n tends to ∞ and as X_t converges to 0 in probability as t converges to 0, it follows that both $\alpha_{1/n}^{(1-p)/2}$ and $\alpha_{1/n}^{(1+p)/2}$ converge to 0 as n tends to ∞. Hence, $\mathbb{E}[\mathrm{e}^{X_{\tau_n}}\,\mathbf{1}_{(X_{1/n}\notin(\alpha_{1/n}^{(1-p)/2},\alpha_{1/n}^{(1+p)/2}))}]$ converges to $1 - p$ as n tends to ∞.

Next, let $H(x) = \mathbb{E}_x[\mathrm{e}^{X_{\tau_0^{++}}}]$. Then

$$\mathbb{E}[\mathrm{e}^{X_{\tau_n}}\,\mathbf{1}_{(X_{1/n}\in(\alpha_{1/n}^{(1-p)/2},\alpha_{1/n}^{(1+p)/2}))}] = \mathbb{E}[H(X_{1/n})\,\mathbf{1}_{(X_{1/n}\in(\alpha_{1/n}^{(1-p)/2},\alpha_{1/n}^{(1+p)/2}))}]. \tag{16}$$

We claim that, for every $q \in (0, 1)$, there exists some $t^q > 0$ such that, for $t < t^q$, we have $\alpha_t^q \leq 0$. We prove the statement by contradiction. Thus, assume that there exists a $q \in (0, 1)$

and a sequence $t_n \to 0$ such that, for every n, $\alpha_{t_n}^q > 0$. Then

$$\mathbb{P}(\limsup_{n\to\infty} X_{t_n} > 0) = \lim_{n\to\infty} \mathbb{P}\bigg(\bigcup_{k=n}^{\infty}\{X_{t_k} > 0\}\bigg) \geq \lim_{n\to\infty}(1-q) = 1-q.$$

However, this is in contradiction to 0 being irregular for $(0,\infty)$. As H is nondecreasing, we conclude that (16) is bounded from below by

$$\mathbb{E}[H(\alpha_{1/n}^{(1-p)/2})\,\mathbf{1}_{(X_{1/n}\in(\alpha_{1/n}^{(1-p)/2},\alpha_{1/n}^{(1+p)/2}))}] = H(\alpha_{1/n}^{(1-p)/2})p,$$

and from above by

$$\mathbb{E}[H(\alpha_{1/n}^{(1+p)/2})\,\mathbf{1}_{(X_{1/n}\in(\alpha_{1/n}^{(1-p)/2},\alpha_{1/n}^{(1+p)/2}))}] = H(\alpha_{1/n}^{(1+p)/2})p.$$

Both terms converge to $H(0)p$ as n tends to ∞. Since it follows from (4) that H is continuous and increasing on $(-\infty,0]$, and since there exist some $t^q > 0$ such that, for $t < t^q$, we have $\alpha_t^q \leq 0$. Therefore, $\mathbb{E}[\mathrm{e}^{X_{\tau_n}}\,\mathbf{1}_{(X_{1/n}\in(\alpha_{1/n}^{(1-p)/2},\alpha_{1/n}^{(1+p)/2}))}]$ converges to $H(0)p$ as n tends to ∞. Taken together, these results imply that, when n tends to ∞,

$$\mathbb{E}[\mathrm{e}^{X_{\tau_n}}] \to (1-p) + H(0)p = (1-p)\mathbb{E}[\mathrm{e}^{X_{\tau_0^+}}] + p\mathbb{E}[\mathrm{e}^{X_{\tau_0^{++}}}] = \mathbb{E}[\mathrm{e}^{X_{\tau^*}}].$$

Similarly, it follows that $\mathbb{E}[\mathrm{e}^{2X_{\tau_n}}] \to \mathbb{E}[\mathrm{e}^{2X_{\tau^*}}]$, and hence, $\mathbb{V}[\mathrm{e}^{X_{\tau_n}}] \to \mathbb{V}[\mathrm{e}^{X_{\tau^*}}]$.

(b) We prove the statement by contradiction. Assume that there exists a sequence, $(\tau_n)_{n\in\mathbb{N}} \subset \mathcal{T}$ such that $\lim_{n\to\infty} \mathbb{V}[\mathrm{e}^{X_{\tau_n}}] = \mathbb{V}[\mathrm{e}^{X_{\tau^*}}]$. The proof is quite lengthy and, therefore, is broken into parts to clarify the structure.

Part (i). It follows from (9) that $\lim_{n\to\infty} \mathbb{E}[\mathrm{e}^{X_{\tau_n}}] = \mathbb{E}[\mathrm{e}^{X_{\tau^*}}] = c$. Let $\tau \in \mathcal{T}$ and $y < 0$ be given. Then

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{X_\tau}] &= \mathbb{E}[\mathrm{e}^{X_\tau}\,\mathbf{1}_{(\tau<\tau_0^{++})}\,\mathbf{1}_{(X_\tau>y)}] + \mathbb{E}[\mathrm{e}^{X_\tau}\,\mathbf{1}_{(\tau<\tau_0^{++})}\,\mathbf{1}_{(X_\tau\leq y)}] + \mathbb{E}[\mathrm{e}^{X_\tau}\,\mathbf{1}_{(\tau\geq\tau_0^{++})}]\\ &\leq \mathbb{P}(\tau < \tau_0^{++},\ X_\tau > y) + \mathrm{e}^y + \mathbb{E}[\mathrm{e}^{X_\tau}\,\mathbf{1}_{(\tau\geq\tau_0^{++})}]\\ &\leq \mathbb{P}(\tau < \tau_0^{++},\ X_\tau > y) + \mathrm{e}^y + \mathbb{E}[\mathrm{e}^{X_{\tau_0^{++}}}].\end{aligned}$$

Thus, to obtain $\lim_{n\to\infty}\mathbb{E}[\mathrm{e}^{X_{\tau_n}}] = c$, then, for any $\varepsilon > 0$ and $y < 0$, we need, for large enough n, $\mathbb{P}(\tau_n < \tau_0^{++}, X_{\tau_n} > y) + \mathrm{e}^y - (c - \mathbb{E}[\exp(X_{\tau_0^{++}})]) \geq -\varepsilon$. As X is not a compound Poisson process, it follows from Theorem 4 that $\hat{y} = 0$. Thus, $\mathbb{E}[\exp(X_{\tau_0^{++}})] < c$, and we may choose ε and y small enough that $(c - \mathbb{E}[\exp(X_{\tau_0^{++}})]) - \varepsilon - \mathrm{e}^y > 0$. We conclude that there exists $p > 0$, $y < 0$, and $N \in \mathbb{N}$ such that for all $n > N$,

$$\mathbb{P}(\tau_n < \tau_0^{++},\ X_{\tau_n} > y) \geq p. \tag{17}$$

Part (ii). Let T denote the first jump time of the process. Assume that $\mathbb{P}(\tau < T) > 0$ for some $\tau \in \mathcal{T}$. With t^* as in Lemma 3(a) below, it follows that

$$\mathbb{E}[\mathrm{e}^{X_\tau}] = \mathbb{E}[\mathrm{e}^{X_\tau}\,\mathbf{1}_{(T\leq t^*)}] + \mathbb{E}[\mathrm{e}^{X_\tau}\,\mathbf{1}_{(T>t^*)}] = \mathbb{E}[\mathrm{e}^{X_\tau}\,\mathbf{1}_{(T\leq t^*)}] + \mathrm{e}^{-dt^*}\mathbb{P}(T > t^*) \geq \mathrm{e}^{-(d+\lambda)t^*}.$$

Now, for each n, where $\mathbb{P}(\tau_n < T) > 0$, let t_n^* be as in Lemma 3. If there are infinitely many such n then to have $\lim_{n\to\infty}\mathbb{E}[\mathrm{e}^{X_{\tau_n}}] = c$, it must hold that, for any $\varepsilon > 0$ then for

https://doi.org/10.1239/aap/1427814584 Published online by Cambridge University Press

large enough n, $\mathbb{E}[e^{X_{\tau_n}}] < c+\varepsilon$ and, thus, $e^{-(d+\lambda)t_n^*} < c+\varepsilon$. Hence, we need, from a certain step, $t_n^* \geq -\log(c+\varepsilon)/(d+\lambda)$. By Lemma 3(b), it follows that if $\mathbb{P}(\tau_n < T) > 0$ then $\mathbb{P}(\tau_n < T,\ \tau_n < t_n^*) = 0$. Therefore, for large enough n, it must hold that

$$\begin{aligned}\mathbb{P}\bigg(\tau_n < T,\ X_{\tau_n} > \frac{\log(c+\varepsilon)d}{d+\lambda}\bigg) &= \mathbb{P}\bigg(\tau_n < T,\ \tau_n < \frac{\log(c+\varepsilon)}{d+\lambda}\bigg)\\ &\leq \mathbb{P}(\tau_n < T,\ \tau_n < t_n^*)\\ &= 0.\end{aligned}$$

Let $u_1 = \log(c+\varepsilon)d/(d+\lambda)$. Note that $d<0$, as 0 would be regular for $(0,\infty)$ otherwise. Hence, there exist $u_1 < 0$ and $N \in \mathbb{N}$, such that for all $n > N$,

$$\mathbb{P}(\tau_n < T,\ X_{\tau_n} > u_1) = 0. \tag{18}$$

Part (iii). Let $Y = \sup\{X_t \mid t \in [T, \tau_0^{++})\}\,\mathbf{1}_{(T<\tau_0^{++})}$. We want to show that $Y < 0$ a.s. Let $T_n = \inf\{t > T \colon X_t > -1/n\}$. Note that this is an increasing sequence of stopping times bounded by τ_0^{++}. Hence, the sequence of stopping times will a.s. converge to some random time $\tilde{T}$, with $\tilde{T} \leq \tau_0^{++}$ a.s. From the quasi-left-continuity we obtain $X_{\tilde{T}} = \lim_{n\to\infty} X_{T_n} \geq \lim_{n\to\infty}(-1/n) = 0$ and, hence, $\tilde{T} = \sigma_0^+$. It then follows that

$$\begin{aligned}\mathbb{P}(Y=0) &= \mathbb{P}(T_n < \tau_0^{++} \text{ for all } n \in \mathbb{N})\\ &= \mathbb{P}(X_{T_n} < 0 \text{ for all } n)\\ &\leq \mathbb{P}(X_{\tilde{T}} = 0)\\ &= \mathbb{P}(X_{\sigma_0^+} = 0)\\ &= 0.\end{aligned}$$

Hence, there exist some $u_2 < 0$ such that $\mathbb{P}(Y > u_2) < \frac{1}{2}p$, and this implies that, for all $p > 0$, there exist $u_2 < 0$ and $N \in \mathbb{N}$ such that for all $n > N$,

$$\mathbb{P}(\tau_n \in [T, \tau_0^{++}),\ X_{\tau_n} > u_2) < \tfrac{1}{2}p. \tag{19}$$

Part (iv). Combining (18) and (19) we obtain, with $u = \max\{u_1, u_2\}$,

$$\mathbb{P}(\tau_n < \tau_0^{++},\ X_{\tau_n} > u) = \mathbb{P}(\tau_n \in [T, \tau_0^{++}),\ X_{\tau_n} > u) + \mathbb{P}(\tau_n < T,\ X_{\tau_n} > u)) < \tfrac{1}{2}p.$$

Combining this with (17) it follow that, for all $p > 0$, there exist $y, u < 0$ and $N \in \mathbb{N}$ such that for all $n > N$,

$$\mathbb{P}(\tau_n < \tau_0^{++},\ X_{\tau_n} \in (y, u]) > \tfrac{1}{2}p. \tag{20}$$

Part (v). First, we recall from the proof of Theorem 1 that

$$G(x) = v^*(x) - \frac{e^{2x}}{\mathbb{E}[e^{2\bar{X}_\infty}]}(e^{-x}\mathbb{E}[e^{\bar{X}_\infty}\mathbf{1}_{(\bar{X}_\infty < -x)}] - \mathbb{E}[e^{2\bar{X}_\infty}\mathbf{1}_{(\bar{X}_\infty < -x)}]).$$

Define

$$D(x_1, x_2) = \frac{e^{2x_1}}{\mathbb{E}[e^{2\bar{X}_\infty}]}(e^{-x_2}\mathbb{E}[e^{\bar{X}_\infty}\mathbf{1}_{(\bar{X}_\infty < -x_2)}] - \mathbb{E}[e^{2\bar{X}_\infty}\mathbf{1}_{(\bar{X}_\infty < -x_2)}]),$$

https://doi.org/10.1239/aap/1427814584 Published online by Cambridge University Press

and note that when $x_1, x_2 \in (y, u]$,

$$D(x_1, x_2) \ge D(y, u) \ge \frac{\mathrm{e}^y}{\mathbb{E}[\mathrm{e}^{2\bar{X}_\infty}]}(\mathrm{e}^{-u} - 1)\mathbb{P}(\bar{X}_\infty = 0) > 0. \tag{21}$$

Therefore, it follows that, for every $\tau \in \mathcal{T}$,

$$\begin{aligned}
&\mathbb{E}[(\mathrm{e}^{X_\tau} - c)^2 \mathbf{1}_{(\tau<\tau_0^{++}, X_\tau \in (y,u])}] \\
&\quad = \mathbb{E}[G(X_\tau)\mathbf{1}_{(\tau<\tau_0^{++}, X_\tau \in (y,u])}] \\
&\quad = \mathbb{E}[(v^*(X_\tau) - D(X_\tau, X_\tau))\mathbf{1}_{(\tau<\tau_0^{++}, X_\tau \in (y,u])}] \\
&\quad \le \mathbb{E}[(v^*(X_\tau) - D(y, u))\mathbf{1}_{(\tau<\tau_0^{++}, X_\tau \in (y,u])}] \\
&\quad = \mathbb{E}[v^*(X_\tau)\mathbf{1}_{(\tau<\tau_0^{++}, X_\tau \in (y,u])}] - D(y, u)\mathbb{P}(\tau < \tau_0^{++},\ X_\tau \in (y, u]) \\
&\quad \le \mathbb{E}[G(X_{\tau_0^{++}})\mathbf{1}_{(\tau<\tau_0^{++}, X_\tau \in (y,u])}] - D(y, u)\tfrac{1}{2}p.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\mathbb{V}[\mathrm{e}^{X_\tau}] &= \mathbb{E}[(\mathrm{e}^{X_\tau} - c)^2] - (c - \mathbb{E}[\mathrm{e}^{X_\tau}])^2 \\
&= \mathbb{E}[(\mathrm{e}^{X_\tau} - c)^2 \mathbf{1}_{(\tau<\tau_0^{++}, X_\tau \in (y,u])}] \\
&\quad + \mathbb{E}[(\mathrm{e}^{X_\tau} - c)^2 \mathbf{1}_{(\{\tau \ge \tau_0^{++}\}\cup\{X_\tau \notin (y,u]\})}] - (c - \mathbb{E}[\mathrm{e}^{X_\tau}])^2 \\
&\le \mathbb{E}[(\mathrm{e}^{X_{\tau_0^{++}}} - c)^2 \mathbf{1}_{(\tau<\tau_0^{++}, X_\tau \in (y,u])}] - D(y, u)\tfrac{1}{2}p \\
&\quad + \mathbb{E}[(\mathrm{e}^{X_\tau} - c)^2 \mathbf{1}_{(\{\tau \ge \tau_0^{++}\}\cup\{X_\tau \notin (y,u]\})}] - (c - \mathbb{E}[\mathrm{e}^{X_\tau}])^2 \\
&= \mathbb{E}[(\mathrm{e}^{X_{\hat{\tau}}} - c)^2)] - D(y, u)\tfrac{1}{2}p - (c - \mathbb{E}[\mathrm{e}^{X_\tau}])^2 \\
&\le \mathbb{V}[\mathrm{e}^{X_{\tau_0^{++}}}] - D(y, u)\tfrac{1}{2}p,
\end{aligned}$$

where $\hat{\tau} = \tau\,\mathbf{1}_{(\tau<\tau_0^{++}, X_\tau \in (y,u])} + \tau_0^{++}\,\mathbf{1}_{(\{\tau \ge \tau_0^{++}\}\cup\{X_\tau \notin (y,u]\})}$. We conclude that

$$\sup_{\tau \in \mathcal{T}^*} \mathbb{V}[\mathrm{e}^{X_\tau}] \le \mathbb{V}[\mathrm{e}^{X_{\tau_0^{++}}}] - D(y, u)\tfrac{1}{2}p,$$

where y and u are as given in (17) and (20). From (21) we have $D(y, u) > 0$; thus, the statement of the theorem follows. This completes the proof of Theorem 7.

The following lemma, used in the proof of Theorem 7, is intuitive. Before the first jump of a compound Poison process, the process has not created any other information than the fact that there have been no jumps. Hence, if a stopping time with respect to $\mathcal{F}$ has a positive probability of stopping before the first jump then, given that the stopping time occurs before the first jump, it is deterministic.

The lemma is the key reason why there is a gap between the variances at the stopping times of the solutions to the variance problem with and without randomized stopping, and that is the reason we have included a proof of it.

Lemma 3. *Let X be a compound Poisson process with negative drift. Let T denote the first jump time of the process, let τ be a stopping time with respect to $\mathcal{F}$, and assume that $\mathbb{P}(\tau < T) > 0$.*

https://doi.org/10.1239/aap/1427814584 Published online by Cambridge University Press

Then there exists some $t^* \geq 0$ *such that*

(a) $\{T > t^*\} \cap \{\tau = t^*\} = \{T > t^*\}$ *a.s.,*

(b) $\{T > \tau\} \cap \{\tau = t^*\} = \{T > \tau\}$ *a.s.*

Proof. (a) Let $\mathcal{F}_t^* = \sigma(X_s \mid s \in [0, t])$. Then all sets in $\mathcal{F}_t^*$ will either contain $\{T > t\}$ or be contained in $\{T > t\}^c$. As τ is a stopping time with respect to $\mathcal{F}$, then, for every $t \geq 0$, $\{\tau \leq t\}$ is a.s equal to some set which is measurable with respect to $\mathcal{F}_t^*$. Thus, for all $t \geq 0$, it follows that $\{T > t\} \cap \{\tau \leq t\} = \{T > t\}$ a.s. or $\{T > t\} \cap \{\tau \leq t\} = \varnothing$ a.s. As $\mathbb{P}(\tau < T) > 0$, we cannot have $\{T > t\} \cap \{\tau \leq t\} = \varnothing$ a.s. for all $t \geq 0$. Hence, we define $t^* = \inf\{t \geq 0 \colon \{T > t\} \cap \{\tau \leq t\} = \{T > t\}\ a.s.\}$ and there exists a sequence $t_n \downarrow t^*$ such that $\{T > t_n\} \cap \{\tau \leq t_n\} = \{T > t_n\}$ a.s. for every n. Hence, it follows that $\{T > t^*\} \cap \{\tau \leq t^*\} \supseteq \bigcap_{k=1}^{\infty}(\{\tau \leq t_k\} \cap \{T > t_k\}) = \{T > t^*\}$ a.s Thus, we obtain $\{T > t^*\} \cap \{\tau \leq t^*\} = \{T > t^*\}$ a.s. Besides, $\{T > t^*\} \cap \{\tau < t^*\} \subseteq \bigcup_{n=1}^{\infty}(\{T > t^*\} \cap \{\tau \leq t^* - 1/n\}) = \varnothing$ a.s. and part (a) follows.

(b) Define a new process $Y_t = X_{\tau+t} - X_\tau$, and let T^Y denote the time of the first jump of Y. Thus, if $T > \tau$ then $T = T^Y + \tau$. Note that $\mathbb{P}(T^Y > t^*) > 0$ because Y has the same distribution as X. As T^Y is independent of $\mathcal{F}_\tau$ and using part (a), we find that

$$\begin{aligned}
\mathbb{P}(\tau < T,\ \tau \neq t^*)\mathbb{P}(T^Y > t^*) &= \mathbb{P}(\tau < T,\ \tau \neq t^*, T^Y > t^*) \\
&= \mathbb{P}(\tau < T,\ \tau \neq t^*,\ T > t^* + \tau) \\
&= \mathbb{P}(\tau < T,\ \tau \neq t^*,\ T > t^* + \tau,\ \tau = t^*) \\
&= 0.
\end{aligned}$$

It must therefore hold that $\mathbb{P}(\tau < T,\ \tau \neq t^*) = 0$ and, thus, part (b) follows.

Example 7. In this example we show that there exist compound Poisson processes with negative drift and $\psi(2) < 0$ for which the embedding equations have no solution. Let X be the compound Poisson process of Example 4 with $p = \frac{1}{11}$ and $\alpha = \log(2)$. For this process, we have $\psi(2) < 0$ and the embedding equations have no solution. We define compound Poisson processes with negative drift by $Y_t = X_t - dt$, where $d > 0$ is a constant. Then Y also has $\psi(2) < 0$. When d converges to 0, then Y converges a.s. to X dominated by X. Therefore, by choosing sufficiently small d, the embedding equations for Y also have no solution.

References

[1] Alili, L. and Kyprianou, A. E. (2005). Some remarks on first passage of Lévy processes, the American put and pasting principles. *Ann. Appl. Prob.* **15,** 2062–2080.

[2] Kyprianou, A. E. (2006). *Introductory Lectures on Fluctuations of Lévy Processes with Applications*. Springer, Berlin.

[3] Markowitz, H. (1952). Portfolio selection. *J. Finance* **7,** 77–91.

[4] Mikosch, T. (2004). *Non-Life Insurance Mathematics*. Springer, Berlin.

[5] Mordecki, E. (2002). Optimal stopping and perpetual options for Lévy processes. *Finance Stoch.* **6,** 473–493.

[6] Pedersen, J. L. (2011). Explicit solutions to some optimal variance stopping problems. *Stochastics* **83,** 505–518.

[7] Peskir, G. and Shiryaev, A. (2006). *Optimal Stopping and Free-Boundary Problems*. Birkhäuser, Basel.

[8] Sato, K.-I. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press.

[9] Shiryaev, A. N. (1978). *Optimal Stopping Rules*. Springer, Berlin.