NEXT-TO-INTERPOLATORY APPROXIMATION ON SETS WITH MULTIPLICITIES

T. S. MOTZKIN AND A. SHARMA

Introduction. It is known that given a set X of m $(\geqslant n)$ distinct real numbers and a real-valued function f defined on X, there exists a unique polynomial $p_{n-1,f,X}$ of degree $n - 1$ or less which approximates best to $f(x)$ on X, that is, which minimizes the deviation $\delta = \delta(f, p)$ defined by the αth-power metric $(\alpha > 1)$ with positive weights, or by the positively weighted maximum of $|f - p|$ on X; these deviations shall be denoted by δ_α and δ_∞. The polynomial $p_{n-1,f,X}$ has the property that $f - p_{n-1,f,X}$ has at least n strong sign changes; in other words, there are at least $n + 1$ points in X where the difference takes alternatingly positive and negative values.

When $m = n$ the polynomial of best approximation $p_{n-1,f,X}$ becomes a polynomial of interpolation. This is trivially so whenever δ is a non-negative function of the values of f and p that only vanishes when $f = p$ for all points of X. The polynomials of best approximation in the next case, $m = n + 1$, shall be called *next-to-interpolatory* polynomials.

The object of this paper is to establish a relation between interpolatory and next-to-interpolatory polynomials, and to generalize it to sets X whose points have a fixed multiplicity and to some other sets X not all of whose points are simple. This generalization requires an appropriate definition (similar to **(3**, pp. 225–226**)** where a special case is considered) of polynomials of best approximation for such sets.

We start (§1) by introducing in a general field, for a linear n-parameter family defined on $n + 1$ points, the concept of unisolvence relative to given functionals, and (§2) by exhibiting the members of a unisolvent family as weighted means of interpolators. In §§3 and 4 we find, for the complex field, the weighted mean representation of next-to-interpolators (members which minimize certain generalizations of δ_α and δ_∞) and determine the value of the minimum. We also show that for given α, $1 < \alpha \leqslant \infty$, every positively weighted mean of interpolators is a next-to-interpolator; viz., that it minimizes δ_α for a suitable choice of weights. These positively weighted means are the natural generalization of polynomials p such that $f - p$ has n sign changes.

In §5, the case of functions unisolvent on sets with fixed multiplicities is reduced to the case of simple sets. Applying this reduction we obtain the results of §6 for next-to-interpolatory polynomials that minimize δ_∞ on sets with multiplicities. In §7 we use, following and extending **(2**, p. 81**)**, the

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results of §6 to determine, in the case of real points, an upper bound for the deviation of the least maximum approximator in terms of bounds for the distances between consecutive points of X and for a derivative of $f(x)$. Lastly, in §8 we obtain results for trigonometric polynomials, similar to those of §6.

It would be of interest to establish the corresponding results for δ_α and other deviations, as well as for different sets of functionals.

1. Relative unisolvence. Let A be an arbitrary field and let $Z = \{z_0, \ldots, z_n\}$ be a set of $n + 1$ distinct elements of A. We consider functions f from Z into A. The expression

$$L(f) = \sum b_k f(z_k), \qquad b_k \in A,$$

will be called a *functional*; particular cases are the values

$$f(z_0), \ldots, f(z_n) \tag{1}$$

and, e.g., the derivative $f'(z_k)$ at z_k of the polynomial of degree $\leqslant n$ with coefficients in A whose values on Z coincide with those of f.

Given $n + 1$ linearly independent functionals $L_0, \ldots, L_n$, there exists a unique f for which these functionals take on given values. If $a_0, \ldots, a_n \in A$ are not all zero, then the set of all f for which

$$\sum_0^n a_k L_k(f) = 0 \tag{2}$$

will be a *linear family* F (of dimension n); we write $F = F(a_0, \ldots, a_n)$and call (2) the equation of F with respect to the functionals L_k. The family F consists of the linear combinations $\sum c_i f_i$ of a suitable basis $f_1, \ldots, f_n$.

For instance, the family P_{n-1} of polynomials p in z of degree $\leqslant n - 1$ has, with respect to the functionals (1), the equation

$$\sum_0^n \gamma_k^{-1} L_k(p) = 0 \tag{3}$$

where

$$\gamma_k = \omega'(z_k) = \omega_{(k)}(z_k) \neq 0, \qquad \omega(z) = \prod_0^n (z - z_k), \qquad \omega_{(k)}(z) = \omega(z)/(z - z_k).$$

This follows easily from the Lagrange interpolation formula

$$p(z) = \sum_0^n p(z_k)\omega_{(k)}(z)/\gamma_k. \tag{4}$$

If in (2) no a_k is zero, then any n among the $L_k(f)$ determine the remaining one and we say that the family F is *unisolvent* or interpolational relative to $L_0, \ldots, L_n$.

Example 1. The family P_{n-1} is unisolvent relative to the functionals (1), since in this case, according to (3), the a_k of formula (2) are given by $a_k = \gamma_k^{-1} \neq 0$.

Example 2. The family P_{n-1} is not unisolvent relative to the functionals

$$L_k(f) = f^{(k)}(z_k), \qquad k = 0, 1, \ldots, n. \tag{5}$$

Here $L_n(p) = 0$, i.e., $a_0 = \ldots = a_{n-1} = 0$. A basis of P_{n-1} related to the functionals (5) is formed by the Abel–Gontcharoff polynomials (**1**, pp. 46–47)

$$\begin{cases} q_k(z) = \displaystyle\int_{z_0}^{z} dz' \int_{z_1}^{z'} dz'' \ldots \int_{z_{k-1}}^{z^{(k-1)}} dz^{(k)}, \qquad k = 1, \ldots, n-1, \\ q_0(z) = 1, \end{cases} \tag{6}$$

with $L_j(q_k) = q_k^{(j)}(z_j) = \delta_{jk}$ (Kronecker delta).

Example 3. The family $P_{n,0}$ of all polynomials of degree $\leqslant n$ with constant term zero is for infinite A in general unisolvent relative to the functionals (5). Indeed, we have for all $p \in P_n$, and the q_k in (6),

$$p(z) = \sum_0^n p^{(k)}(z_k) q_k(z); \tag{7}$$

hence, for $p \in P_{n,0}$,

$$\sum_0^n q_k(0) p^{(k)}(z_k) = 0. \tag{8}$$

Thus in (2), we can set $a_k = q_k(0)$. In general $q_k(0) \neq 0$, since

$$q_k(z) = z^k + \ldots \in P_k.$$

In particular $a_0 = 1$, $a_1 = -z_0$, $a_2 = z_0(2z_1 - z_0)$.

Example 4. In view of Example 2, it is interesting to observe that the family P_{n-1} is in general unisolvent relative to the functionals

$$L_0(f) = f(z_0), \qquad L_k(f) = f^{(k-1)}(z_k), \qquad k = 1, \ldots, n. \tag{9}$$

We introduce polynomials $r_k(z) \in P_n (k = 0, 1, \ldots, n)$ such that

$$\begin{cases} r_0(z_0) = 1, & r_0^{(j-1)}(z_j) = 0, \\ r_k(z_0) = 0, & r_k^{(j-1)}(z_j) = \delta_{jk}, \qquad j, k = 1, \ldots, n. \end{cases} \tag{10}$$

By an easy computation we can verify the following explicit formulae for the $r_k(z)$:

$$r_0(z) = h_n^{-1} \int_{z_1}^{z} dz' \int_{z_2}^{z'} dz'' \ldots \int_{z_n}^{z^{(n-1)}} dz^{(n)}, \tag{11}$$

$$r_1(z) = -h_n^{-1} \int_{z_0}^{z} dz' \int_{z_2}^{z'} dz'' \ldots \int_{z_n}^{z^{(n-1)}} dz^{(n)} = 1 - r_0(z), \tag{12}$$

and for $2 \leqslant k \leqslant n$,

$$r_k(z) = -h_{k-1} r_0(z) + \int_{z_1}^{z} dz' \int_{z_2}^{z'} dz'' \ldots \int_{z_{k-1}}^{z^{(k-2)}} dz^{(k-1)}, \tag{13}$$

where

$$h_k = \int_{z_1}^{z_0} dz' \int_{z_2}^{z'} dz'' \ldots \int_{z_k}^{z^{(k-1)}} dz^{(k)}, \qquad k = 1, \ldots, n. \tag{14}$$

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Thus for every polynomial $q(z)$ of degree $\leqslant n$, we have

$$q(z) = q(z_0)r_0(z) + \sum_1^n q^{(k-1)}(z_k)r_k(z)$$

whence for $p \in P_{n-1}$

$$a_0\, p(z_0) + \sum_1^n a_k\, p^{(k-1)}(z_k) = 0 \tag{15}$$

with $a_0 = -a_1 = h_n^{-1}$, $a_k = -h_{k-1}\, h_n^{-1}$ $(k = 2, \ldots, n)$.

When $z_k = k$ $(k = 0, 1, \ldots, n)$, the polynomials $r_k(z)$ have the following simple forms:

$$r_0(z) = (-1)^n(z-1)(z-n-1)^{n-1}/(n+1)^{n-1}, \qquad r_1(z) = 1 - r_0(z),$$

$$r_k(z) = (-1)^{n+k}(z-1)(z-n-1)^{n-1}k^{k-2}/((k-1)!\,(n+1)^{n-1}) + (z-1)(z-k)^{k-2}/(k-1)!,$$

with $h_k = (-1)^k(k+1)^{k-1}/k!$ $(k = 1, \ldots, n)$.

2. The interpolational basis. For $\phi \notin F, f \in F$, let $\delta_k(f) = L_k(\phi - f)$ and let

$$\delta = \sum_0^n a_k\, \delta_k(f) = \sum_0^n a_k\, L_k(\phi) \neq 0.$$

The *k-interpolator* $f_k \in F$ of ϕ is defined by

$$L_j(f_k) = L_j(\phi), \quad \text{i.e., } \delta_j(f_k) = 0, \qquad j = 0, \ldots, n; j \neq k. \tag{16}$$

Setting $\iota_k = \delta_k(f_k) = L_k(\phi - f_k)$, we have $\delta = a_k\, \iota_k$, since

$$\delta = \sum_0^n a_j\, L_j(\phi) = \sum_{j=0}^n a_j\, L_j(\phi - f_k).$$

Hence $\iota_k \neq 0$.

For $c_0, c_1, \ldots, c_n \in A$ with $\sum_0^n c_j = 0$, we have

$$\sum_{j=0}^n c_j\, L_k(f_j) = -c_k\, \iota_k + L_k(\phi) \sum_0^n c_j = -c_k\, \iota_k.$$

Hence, if

$$\sum_0^n c_j = 0, \qquad \sum_{j=0}^n c_j\, L_k(f_j) = 0, \qquad k = 0, \ldots, n,$$

then $c_k = 0$, $k = 0, \ldots, n$. Thus, the n rows

$$L_0(f_k) - L_0(f_0), \ldots, L_n(f_k) - L_n(f_0) \qquad (k = 1, \ldots, n)$$

are linearly independent. Consequently, for every $f \in F$ there exist unique values $\lambda_0, \ldots, \lambda_n \in A$ such that

$$\sum_0^n \lambda_j = 1, \qquad \sum_{j=0}^n \lambda_j\, L_k(f_j) = L_k(f), \qquad k = 0, \ldots, n.$$

In view of this representation of the functionals as weighted means, we call $f_0, \ldots, f_n$ the *interpolational basis* of F with respect to ϕ.

From

$$L_k(f) = \sum_{j=0}^{n} \lambda_j L_k(f_j) = \sum_{j=0}^{n} \lambda_j L_k(\phi) - \lambda_k \iota_k = L_k(\phi) - \lambda_k \iota_k$$

we see that

$$\delta_k(f) = \lambda_k \iota_k.$$

Hence

$$\sum_0^n \delta_k(f)/\iota_k = 1.$$

We formulate our result as a lemma:

LEMMA 1. *Given a linear family F of dimension n, unisolvent on $Z = \{z_0, \ldots, z_n\}$ relative to the functionals $L_0, \ldots, L_n$, and given a function $\phi \notin F$, there exists an interpolational basis $f_0, \ldots, f_n$ with respect to ϕ such that for every $f \in F$ we have*

$$L_k(f) = \sum_{j=0}^{n} \lambda_j L_k(f_j), \qquad \sum_0^n \lambda_j = 1, \qquad k = 0, \ldots, n, \tag{17}$$

with

$$\lambda_j = L_j(\phi - f)/L_j(\phi - f_j) = L_j(\phi - f)a_j/\delta, \qquad \delta = \sum_0^n a_k L_k(\phi), \qquad j = 0, \ldots, n. \tag{18}$$

Example 1′. For the family P_{n-1} and the functionals (1), the interpolational basis relative to $\phi \notin P_{n-1}$ consists of polynomials $p_k(z)$ $(k = 0, \ldots, n)$ such that

$$p_k(z_j) = \phi(z_j), \qquad j = 0, \ldots, k-1, k+1, \ldots, n,$$

and we have

$$p_k(z) = \sum \phi(z_j)\omega_{k,j}(z)/\omega_{(k)}'(z_j),$$

with

$$\omega_{(k)}(z) = \omega(z)/(z - z_k), \qquad \omega_{k,j}(z) = \omega_{(k)}(z)/(z - z_j),$$

where the sum is taken over $j = 0, \ldots, n, j \neq k$.

Example 3′. For the family $P_{n,0}$ of Example 3, the interpolational basis consists of polynomials $s_k(z) \in \mathrm{P}_{n,0}$ such that

$$s_k^{(j)}(z_j) = \phi^{(j)}(z_j), \qquad j = 0, \ldots, n, j \neq k.$$

From this an explicit formula for $s_k(z)$ as quotient of two determinants follows immediately.

Example 5. The family P_{n-1} is unisolvent relative to the functionals

$$\begin{cases} L_j(f) = f(z_j), & j = 0, \ldots, n-1, \\ L_n(f) = f^{(n-1)}(z_n). \end{cases} \tag{19}$$

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Indeed for any $p \in P_{n-1}$, we have by the Lagrange interpolation formula

$$(20) \qquad p(z) = \sum_{0}^{n-1} p(z_k) l_k(z)$$

where

$$(21) \qquad l_k(z) = \omega_{(n)}(z)/((z - z_k)\omega'_{(n)}(z_k)).$$

Differentiating both sides in (20) $n - 1$ times, we obtain

$$p^{(n-1)}(z_n) = p^{(n-1)}(z) = (n-1)! \sum_{0}^{n-1} p(z_k)/\omega'_{(n)}(z_k).$$

Thus the a_k of (2) are now given by

$$(22) \qquad \begin{cases} a_k = (n-1)!/\omega'_{(n)}(z_k), & k = 0, \dots, n-1, \\ a_n = -1. \end{cases}$$

The interpolational basis $\{p_0, p_1, \dots, p_n\}$ is easy to determine from the condition (16). Indeed, for any $\phi \notin P_{n-1}$ we have

$$p_n(z) = \sum_{0}^{n-1} \phi(z_k)\omega_{n,k}(z)/\omega_{(n)}'(z_k)$$

and for $k = 0, \dots, n - 1$,

$$p_k(z) = \sum \phi(z_j)\omega_{k,j}(z)/\omega'_{(k)}(z_j) + \omega_{k,n}(z)[\phi^{(n-1)}(z_n) - \sum \phi(z_j)/\omega_{(k)}'(z_j)]/(n-1)!,$$

both sums taken over $j = 0, \dots, n - 1; j \neq k$.

3. Least weighted αth power sum. Let A be the complex field. If the distance between ϕ and f is defined as

$$\sum_{0}^{n} b_k |L_k(\phi - f)|^\alpha$$

for given $\alpha > 1$, $b_k > 0$, $k = 0, \dots, n$, then the closest point $\sum_0^n \lambda_k f_k$, $\sum_0^n \lambda_k = 1$ to ϕ will minimize

$$(23) \qquad \sum_{0}^{n} b_k |\lambda_k \iota_k|^\alpha = \sum_{0}^{n} c_k |\lambda_k|^\alpha, \qquad c_k = b_k |\iota_k|^\alpha, \qquad \iota_k = L_k(\phi - f_k).$$

Now we have an elementary lemma:

LEMMA 2. *Whenever* $\sum_0^n \lambda_k = 1$ *and not all* $\lambda_k \geqslant 0$, *then there exist* $\mu_k \geqslant 0$, $\sum_0^n \mu_k = 1$ *with* $\mu_k = \theta|\lambda_k|$, $0 < \theta < 1$.

We have in fact $\theta = 1/\sum_0^n |\lambda_k|$.

From this lemma it follows that a sequence $\lambda_0, \dots, \lambda_n$ of coefficients which are not all $\geqslant 0$ cannot be a minimizing sequence. But neither can a minimizing sequence contain some $\lambda_j = 0$. For if $\lambda_j = 0$, $\lambda_k = \lambda > 0$, then replacement of

λ_j by ϵ and λ_k by $\lambda - \epsilon$ changes $c_j 0^\alpha + c_k \lambda_k^\alpha$ into $c_j \epsilon^\alpha + c_k(\lambda_k - \epsilon)^\alpha$, which for small $\epsilon > 0$ is smaller.

Hence, all $\lambda_k > 0$. To determine λ_k explicitly, let

$$l = \sum_0^n c_j \lambda_j^\alpha - \lambda\left(\sum_0^n \lambda_j - 1\right), \qquad c_j > 0, \alpha > 1.$$

Then

$$\partial l / \partial \lambda_k = \alpha c_k \lambda_k^{\alpha-1} - \lambda = 0;$$

by $\lambda_k > 0$ we have $\lambda > 0$ and

$$\lambda_k = (\lambda \alpha^{-1} c_k^{-1})^{1/(\alpha-1)}.$$

From

$$\sum_0^n \lambda_k = 1$$

we obtain

$$\lambda_k = \mu_k \Big/ \sum_0^n \mu_j, \qquad \mu_k = c_k^{-1/(\alpha-1)}, \qquad c_k = b_k|\iota_k|^\alpha \tag{24}$$

or

$$\mu_k = b_k^{-1/(\alpha-1)}|a_k|^{\alpha/(\alpha-1)}$$

since $\iota_k a_k = \delta$.

We have thus proved the following theorem on next-to-interpolators:

THEOREM 1. *Given a linear family F of dimension n, unisolvent on $Z = \{z_0, \ldots, z_n\}$ relative to the functionals $L_0, \ldots, L_n$, and given any $\phi \notin F$, the element $f \in F$ that minimizes*

$$\sum_0^n b_k|L_k(\phi) - L_k(f)|^\alpha, \qquad b_k > 0, \alpha > 1, \tag{25}$$

is unique and can be written

$$f = \sum_0^n \lambda_k f_k \tag{26}$$

where the λ_k are determined by (24) *and the f_k by* (16). *The value of the minimum is*

$$\rho_\alpha = \left(\sum_0^n b_k^{-1/(\alpha-1)}|\iota_k|^{-\alpha/(\alpha-1)}\right)^{-(\alpha-1)} = \left(\sum_0^n b_k^{-1/(\alpha-1)}|a_k|^{\alpha/(\alpha-1)}\right)^{-(\alpha-1)}|\delta^\alpha|, \tag{27}$$

$$\delta = \sum_0^n a_j L_j(\phi).$$

In particular, if $L_j(\phi) = \phi(z_j), j = 0, \ldots, n$, and if F is the family of polynomials P_{n-1}, then the minimum of

$$\sum_0^{n-1} b_k|\phi(z_k) - p(z_k)|^\alpha, \qquad p \in P_{n-1},$$

is attained for

$$p(z) = \sum_0^n \lambda_k p_k(z)$$

where λ_k is given by (24) with $\iota_j = \phi(z_j) - p_j(z_j)$ and $p_k(z)$ is the polynomial that interpolates $\phi(z)$ in all points of Z except z_k.

4. Least weighted maximum. We now establish the corresponding result for next-to-interpolators in the sense of δ_∞.

THEOREM 2. *Under the assumptions of Theorem* 1, *the element* $f \in F$ *that minimizes*

$$\max_k b_k|L_k(\phi - f)| \tag{28}$$

for given $b_k > 0$ $(k = 0, \ldots, n)$ *can be written*

$$f = \sum \mu_k f_k / \sum \mu_k, \qquad \mu_k = (b_k|\iota_k|)^{-1} \text{ or } \mu_k = b_k^{-1}|a_k|, \tag{29}$$

where

$$\iota_k = L_k(\phi - f_k) = \delta/a_k, \qquad \delta = \sum a_j L_j(\phi).$$

We have

$$L_k(\phi - f) = \mu_k \delta/(a_k \sum \mu_j).$$

This can be proved either by choosing weights b_k^α in place of b_k and letting $\alpha \to \infty$, or as follows.

If we define the distance between ϕ and f by (28), then the closest point $\sum_0^n \lambda_k f_k$, $\sum_0^n \lambda_k = 1$ of F to ϕ will minimize

$$l = \max b_k|\lambda_k \iota_k| = \max c_k|\lambda_k|, \qquad c_k = b_k|\iota_k| > 0.$$

By Lemma 2 we may assume that all $\lambda_k \geqslant 0$. However, if some $\lambda_j = 0$ and $\lambda_k = \lambda > 0$, then since $\max\{c_j 0, c_k \lambda\} > \max\{c_j \epsilon, c_k(\lambda - \epsilon)\}$ for sufficiently small $\epsilon > 0$, we can diminish l without changing the sum $\sum_0^n \lambda_k$; hence all $\lambda_k > 0$. Thus, except for a factor, the λ_k are determined by the requirement that all $b_k \lambda_k|\iota_k|$ be equal, while this factor is given by the normalization $\sum_0^n \lambda_k = 1$. We obtain (29), where λ is the weighted harmonic sum of $|\iota_k|$, i.e., $(\sum b_k^{-1})^{-1}$ times the weighted harmonic mean.

In particular for $F = P_{n-1}$, $\phi(z) = z^n$, $\phi - f$ becomes the "Tchebycheff" polynomial relative to the functionals $L_0, \ldots, L_n$.

5. Sets with multiplicities. Let $r \geqslant 2$ and $N = r(n + 1) - 2$. For a general field A, let Φ be a linear $(N + 1)$-dimensional family unisolvent on $Z = \{z_0, \ldots, z_n\}$, in the sense that its members are given by their values, their first, ..., $(r - 2)$nd derivatives on Z, and by their $(r - 1)$st derivatives at n points of Z. The subfamily consisting of those members of Φ that vanish together with their first, ..., $(r - 2)$nd derivatives on Z shall be called Φ_1; it is of dimension n. By F we denote the set of all functions on Z whose values are the $(r - 1)$st derivatives of some member of Φ_1; because of the unisolvence of Φ, this derivation does not diminish the dimension.

If we now have to find a member of Φ with given values and derivatives

$$v_k^{(j)}, \qquad k = 0, 1, \ldots, n; j = 0, 1, \ldots, r - 2,$$

such that its $(r - 1)$st derivatives on Z approximate given values

$$v_0^{(r-1)}, \ldots, v_n^{(r-1)},$$

we first choose an $f \in \Phi$ with

$$f^{(j)}(z_k) = v_k^{(j)}, \qquad k = 0, \ldots, n; j = 0, \ldots, r-2.$$

We are then searching only among all $f + f_1$, $f_1 \in \Phi_1$, and trying to approximate the $v_k^{(r-1)}$ by $f^{(r-1)}(z_k) + f_1^{(r-1)}(z_k)$, i.e., $v_k^{(r-1)} - f^{(r-1)}(z_k)$ by the $f_1^{(r-1)}(z_k)$ where $f_1^{(r-1)} \in F$. Now the theory of §§3 and 4 applies.

6. Polynomials on sets with multiplicities. Specializing again to polynomials so that Φ is the set of all polynomials of degree N or less, we see that Φ_1 is then the set of all polynomials of the form $\omega^{r-1}u$, where u is of degree $n - 1$ or less, and

$$\omega(z) = \prod_0^n (z - z_j).$$

Hence the members of F are of the form $(D = d/dz)$

$$D^{r-1}(\omega^{r-1}u). \tag{30}$$

Setting

$$(D^{r-1}\omega^{r-1})_{z_j} = \omega_j, \qquad j = 0, \ldots, n,$$

the polynomial u is completely determined by assigning the values of (30) on $Z - \{z_k\}$. Indeed, if these values are $v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n$, then

$$u(z) = \sum v_j\, \omega_j^{-1}\omega_{(j)}(z)/\gamma_j,$$

where the sum is taken over $j = 0, \ldots, n, j \neq k$.

Now we can apply Theorem 2 to obtain

THEOREM 3. *The polynomial $p_N(z)$ of degree N that minimizes*

$$\max_{z_k \in Z} b_k|p_N^{(r-1)}(z_k) - f^{(r-1)}(z_k)| \tag{31}$$

for given $b_k > 0$ among all polynomials $p(z)$ in the family P_N for which

$$p^{(i)}(z_k) = f^{(i)}(z_k), \qquad i = 0, 1, \ldots, r-2; k = 0, 1, \ldots, n, \tag{32}$$

is the weighted arithmetic mean of the polynomials $q_{N,k}(z)$ with weights $b_k^{-1}|\gamma_k|^{-r}$, where $q_{N,k}(z)$ is uniquely determined by the conditions

$$q_{N,k}^{(i)}(z_j) = f^{(i)}(z_j), \qquad i = 0, 1, \ldots, r-1; j = 0, 1, \ldots, n, \tag{33}$$

except $i = r - 1$ and $j = k$.

(For $b_k = |\gamma_k|^{1-r}$ the weights are $|\gamma_k|^{-1}$.)

In the particular case when $r = 2$ and all $b_k = 1$, the polynomial $q_{2n,k}(z)$ is given by

$$\begin{aligned}(34)\qquad q_{2n,k}(z) &= f(z_k)\pi_k^2(z) \\ &+ \sum f(z_j)(z_j - z_k)(z - z_k)^{-1}\pi_j^2(z)\{1 + c_j(z - z_j)\} \\ &+ \sum f'(z_j)(z_j - z_k)(z - z_k)^{-1}(z - z_j)\pi_j^2(z)\end{aligned}$$

where the two sums are taken over $j = 0, \ldots, n, j \neq k$, and

$$(35)\qquad \begin{cases} c_j = (z_j - z_k)^{-1} - \omega''(z_j)\gamma_j^{-1}, \\ \pi_k(z) = \omega(z)(z - z_k)^{-1}\gamma_k^{-1}. \end{cases}$$

In this case we have

$$(36)\qquad \rho = \left|\sum_0^n f(z_k)\omega''(z_k)\gamma_j^{-3} - \sum_0^n f'(z_k)\gamma_k^{-2}\right| \Big/ \sum_0^n |\gamma_k|^{-1}$$

which for real $z_0, \ldots, z_n$ fits the explicit form of $p_{2n}(z)$:

$$\begin{aligned}(37)\qquad p_{2n}(z) &= \sum_0^n f(z_k)\{1 - (z - z_k)\omega''(z_k)\gamma_k^{-1}\}\pi_k^2(z) \\ &+ \sum_0^n \{f'(z_k) + (-1)^{k-1}\rho\gamma_k\}(z - z_k)\pi_0^2(z)\end{aligned}$$

obtained from the Hermite–Fejér formula of interpolation (**4**, p. 328).

We formulate a theorem slightly more general than Theorem 3:

THEOREM 4. *The polynomial $p_{N+m}(z)$ which among all polynomials of degree $N + m$ that interpolate $f(z)$ in $Z \cup Y_m$ ($Y_m = \{y_1, \ldots, y_m\}$) and fulfill* (32) *minimizes*

$$\max_{z_k \in Z} b_k|p_{N+m}^{(r-1)}(z_k) - f^{(r-1)}(z_k)|/|\psi(z_k)|, \qquad \psi(z) = \prod_1^m (z - y_i),$$

is the arithmetic mean, with weights $b_k^{-1}|\gamma_k|^{-r}$, of the polynomials $q_{N+m,k}(z)$, $k = 0, \ldots, n$, determined by

$$q_{N+m,k}^{(i)}(z_j) = f^{(i)}(z_j), \qquad i = 0, 1, \ldots, r-1; j = 0, 1, \ldots, n,$$
$$\textit{except } i = r - 1 \textit{ and } j = k,$$
$$q_{N+m,k}(y_h) = f(y_h), \qquad h = 1, \ldots, m.$$

It is of some interest that for real $z_0, \ldots, z_n$ a proof of Theorems 3 and 4 can be based on the following lemma and on the equioscillation of the members of F, instead of on Theorem 2.

LEMMA 3. *If*

$$\omega(z) = \prod_0^n (z - z_k), \qquad \gamma_k = \omega'(z_k), \qquad \psi(z) = \prod_1^m (z - y_i), \qquad \psi_k = \psi(z_k),$$

then $\delta_k(f)\cdot\psi_k^{-1}\gamma_k^{-r}$ *is independent of* k *for* $k = 0, 1, \ldots, n$, *where*

$$\delta_k(f) = q_{N+m,k}^{(r-1)}(z_k) - f^{(r-1)}(z_k).$$

To prove the lemma we observe that the polynomial

$$q_{N+m,i}(z) - q_{N+m,j}(z)$$

vanishes at $z_0, \ldots, z_n$ together with its first $r - 2$ derivatives, and its $(r - 1)$st derivative vanishes at all these points except at z_i and z_j. Hence

$$q_{N+m,i}(z) - q_{N+m,j}(z) = \lambda_{ij}(\omega(z))^r\psi(z)(z - z_i)^{-1}(z - z_j)^{-1}.$$

Dividing both sides by $(\omega(z))^{r-1}$, we at once have for $z = z_i$, by l'Hôpital's rule,

$$\{q_{N+m}^{(r-1)}(z_i) - q_{N+m,j}^{(r-1)}(z_j)\}(\omega'(z_i))^{1-r}/(r - 1)! = \lambda_{ij}\,\psi_i\,\omega'(z_i)(z_i - z_j)^{-1},$$

and for $z = z_j$,

$$\{q_{N+m,i}^{(r-1)}(z_j) - q_{N+m,j}^{(r-1)}(z_j)\}(\omega'(z_j))^{1-r}/(r - 1)! = \lambda_{ij}\,\psi_j\,\omega'(z_j)(z_j - z_i)^{-1}.$$

It then follows from the definition of the polynomials $q_{N+m,i}(z)$ that

$$\begin{aligned}-\delta_j(f) &= f^{(r-1)}(z_j) - q_{N+m,j}^{(r-1)}(z_j) - (f^{(r-1)}(z_j) - q_{N+m,i}^{(r-1)}(z_j))\\ &= q_{N+m,i}^{(r-1)}(z_j) - q_{N+m,j}^{(r-1)}(z_j) = (r - 1)!\,\lambda_{ij}\,\gamma_j^{\,r}\,\psi_j(z_j - z_i)^{-1}.\end{aligned}$$

Similarly,

$$-\delta_i(f) = q_{N+m,j}^{(r-1)}(z_i) - q_{N+m,i}^{(r-1)}(z_i) = (r - 1)!\,\lambda_{ij}\,\gamma_i^{\,r}\,\psi_i(z_j - z_i)^{-1}.$$

This completes the proof.

7. An upper bound for the deviation. We shall now give an application of Theorem 3 for $r = 2$. For a set X of $n + 1$ real points $x_0, \ldots, x_n$ (not necessarily in increasing order), let ρ be given as the minimum of (31) with $z_k = x_k$. We shall show that for boundedly smooth f the points of X cannot be too close. For given points this provides an upper bound for ρ. In fact, we shall prove

THEOREM 5. *If* $f(x)$ *has in the interval* $[a, b]$ *a continuous derivative of order* $2n + 1$ *with* $|f^{(2n+1)}(x)| \leqslant \mu$ *and if the deviation of the next-to-interpolatory polynomial* $p_{2n}(x)$ *of Theorem* 3 *(with* $b_k = |\gamma_k|^{-1}$, $r = 2$*) is* ρ, *then the shortest distance* δ *between two points of* X *satisfies the condition*

$$\delta \geqslant (2\rho/(5\mu))(2n + 1)!\,(b - a)^{-(n-1)}. \tag{38}$$

To prove Theorem 5 we note first that the best approximation ρ as given by (36) remains unchanged when f is replaced by

$$g = f - f(x_0) - \omega(x)f'(x_0)/\gamma_0.$$

Indeed, the first and second terms of the numerator in (36) will be increased by multiples of $\sum_0^n \omega''(x_k)\gamma_k^{-3}$ and $\sum_0^n \gamma_k^{-1}$, respectively; but the latter is seen to be zero from the Lagrange formula of interpolation and the former vanishes by virtue of (36), $f \equiv 1$, $\rho = 0$.

Since $g(x_0) = 0$, $g'(x_0) = 0$, we have

$$\rho < \left(\sum_1^n g(x_k)\omega''(x_k)(\omega'(x_k))^{-3} - \sum_1^n g'(x_k)(\omega'(x_k))^{-2}\right) \Big/ \sum_1^n |\omega'(x_k)|^{-1} \tag{39}$$

after increasing the expression on the right by omitting the term $|\omega'(x_0)|^{-1}$ in the denominator.

If we set $\omega_k(x) = \prod_{j=k}^n (x - x_j)$, we have

$$\left.\begin{aligned} \omega'(x_k) &= (x_k - x_0)\omega_1'(x_k) \\ \omega''(x_k) &= (x_k - x_0)\omega_1''(x_k) + 2\omega_1'(x_k) \end{aligned}\right\} \qquad k = 1, \ldots, n,$$

so that

$$\begin{aligned} \omega''(x_k)(\omega'(x_k))^{-3} = (x_k - x_0)^{-2}\omega_1''(x_k)(\omega_1'(x_k))^{-3} \\ + 2(x_k - x_0)^{-3}(\omega_1'(x_k))^{-3}. \end{aligned}$$

Hence the numerator on the right side of (39) becomes

$$\begin{aligned} \sum_1^n g(x_k)(x_k - x_0)^{-2}\omega_1''(x_k)(\omega_1'(x_k))^{-3} \\ - \sum_1^n \{g'(x_k)(x_k - x_0)^{-2} - 2g(x_k)(x_k - x_0)^{-3}\}(\omega_1'(x_k))^{-2} \end{aligned} \tag{40}$$

and the denominator is greater than

$$(b - a)^{-1}\sum_1^n |\omega_1'(x_k)|^{-1}. \tag{41}$$

Furthermore, letting

$$h_1(x) = g(x)(x - x_0)^{-2}$$

so that

$$h_1'(x_k) = g'(x_k)(x_k - x_0)^{-2} - 2g(x_k)(x_k - x_0)^{-3},$$

we have from (39), (40), and (41)

$$\left\{\begin{aligned} &\rho \leqslant (b - a)\rho_1, \\ &\rho_1 = \left(\sum_1^n h_1(x_k)\omega_1''(x_k)(\omega_1'(x_k))^{-3} \right.\\ &\qquad\qquad \left. - \sum_1^n h_1'(x_k)(\omega_1'(x_k))^{-2}\right) \Big/ \sum_1^n |\omega_1'(x_k)|^{-1}, \end{aligned}\right. \tag{42}$$

where ρ_1 is the best approximation to $h_1(x)$ by polynomials of degree $2n - 2$ with weights $|\omega_1'(x_k)|^{-1}$ on $x_1, \ldots, x_n$ in the sense of (31), $r = 2$.

Similarly, let

$$h_{k+1}(x) = [h_k(x) - h_k(x_k) - \omega_k(x)h_k'(x_k)(\omega_1'(x_k))^{-1}]/(x - x_k)^2$$

for $k = 0, 1, \ldots, n - 2$, where we take $h_0 = f$. If we denote by ρ_{k+1} the number ρ of Theorem 3 when f is replaced by h_{k+1} and ω by $\omega_{k+1}(x)$, it is easy to see from (42) that

$$\rho \leqslant (b - a)^{n-1}\rho_{n-1}. \tag{43}$$

Now ρ_{n-1} is the best approximation on x_{n-1}, x_n to the function $h_{n-1}(x)$ by polynomials of second degree which interpolate $h_{n-1}(x)$ in x_{n-1}, x_n and whose derivative approximates $h_{n-1}'(x)$ in the weighted maximum sense with weight $|2x - x_{n-1} - x_n|^{-1}$ on x_{n-1}, x_n. Then from (35), we have

$$\begin{aligned}\rho_{n-1} = \tfrac{1}{2}|x_n - x_{n-1}|\cdot|(h_{n-1}'(x_{n-1}) + h_{n-1}'(x_n))|(x_n - x_{n-1})^2 \\ - 2\cdot(h_{n-1}(x_n) - h_{n-1}(x_{n-1}))|(x_n - x_{n-1})^3|,\end{aligned}$$

whence, using the relations

$$\begin{aligned}h_{n-1}(x_n) &= h_{n-1}(x_{n-1}) + (x_n - x_{n-1})h_{n-1}'(x_{n-1}) + (x_n - x_{n-1})^2 h_{n-1}''(x_{n-1})/2! \\ &\qquad + (x_n - x_{n-1})^3 h_{n-1}'''(\xi)/3!, \\ h_{n-1}'(x_n) &= h_{n-1}'(x_{n-1}) + (x_n - x_{n-1})h_{n-1}''(x_{n-1}) \\ &\qquad + (x_n - x_{n-1})^2 h_{n-1}'''(\eta)/2!, \quad x_{n-1} \leqslant \xi, \eta \leqslant x_n,\end{aligned}$$

we have on simplifying

$$\rho_{n-1} \leqslant (5/12)|x_n - x_{n-1}|\cdot\mu_{n-1} \tag{44}$$

where

$$\mu_{n-1} = \max_{x\in[a,b]} |h_{n-1}'''(x)|.$$

From the definition of $h_{k+1}(x)$ it is easy to see that for $k = 0, \ldots, n - 2$,

$$\begin{aligned}h_{k+1}(x) = \Bigg[\omega'_k(x_k) \int_0^1 t\,dt \int_0^1 h_k''(x_k + tu(x - x_k))\,du \\ -h_k'(x_k) \int_0^1 t dt \int_0^1 \omega_k''(x_k + tu(x - x_k))\,du\Bigg]\Bigg/ \omega_k'(x_k).\end{aligned}$$

We recall that $\omega_k(x)$ is a polynomial of degree $n - k + 1$ and that in particular $\omega_{n-2}''(x)$ is a first-degree polynomial so that when we evaluate $h_{n-1}'''(x)$, this term vanishes. Hence we have for $x_{n-1} \leqslant x \leqslant x_n$,

$$h_{n-1}'''(x) = \int_0^1 t\,dt \int_0^1 h_{n-2}^{(5)}(x_{n-2} + tu(x - x_{n-2}))(tu)^3\,du.$$

Similarly for $x_{n-1} \leqslant x \leqslant x_n$,

$$h_{k+1}^{(2n-2k-1)}(x) = \int_0^1 t\,dt \int_0^1 h_k^{(2n-2k+1)}(x_k + tu(x - x_k))(tu)^{2n-2k-1}\,du$$

$$(k = 0, \ldots, n - 2).$$

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If

$$\max_{x\in[a,b]} |h_k^{(2n-2k+1)}(x)| = \mu_k,$$

we obtain

$$\mu_{k+1} \leqslant \mu_k/(2n-2k+1)(2n-2k), \qquad k = 0, \ldots, n-2,$$

so that multiplying these inequalities for $k = 0, \ldots, n-2$,

(45) $$\mu_{n-1} \leqslant 6\mu/(2n+1)!$$

where

$$\mu = \mu_0 = \max_{x\in[a,b]} |f^{(2n+1)}(x)|.$$

Thus from (43), (44), and (45), we have

$$\rho < (5\mu/2)(b-a)^{n-1}(2n+1)!\,|x_n - x_{n-1}|.$$

Since the order of $x_0, \ldots, x_n$ was arbitrary, we can assume that $|x_n - x_{n-1}| = \delta$; then (38) follows immediately.

8. Trigonometric polynomials. Let

$$z_k = e^{i\theta_k}, \qquad k = 0, \ldots, n; \theta_k \text{ real},$$

be the set Z of $n+1$ points and let $p \in P_{2n}$. Then for $z = e^{i\theta}$, $t(\theta) = z^{-n}p(z)$ is a trigonometric polynomial of order n or less. The set T_n of all $t(\theta)$ is a $(2n+1)$-dimensional family unisolvent on Z in the sense that its members can be determined when their values are prescribed on Z and their first derivatives on any n points of Z.

If $f(\theta)$ is a 2π-periodic function $\notin T_n$ and has a continuous derivative, then the problem of minimizing

(46) $$\max_k b_k|t'(\theta_k) - f'(\theta_k)|$$

where $t(\theta) \in T_n$ and $t(\theta_k) = f(\theta_k)$, $k = 0, \ldots, n$, can be reduced to the polynomial case, since if $f(\theta) = g(z)$, $z = e^{i\theta}$, then (46) becomes

$$\max_k b_k|p'(z_k) - h'(z_k)|, \qquad h(z) = z^n g(z).$$

By Theorem 2, $p(z)$ is the weighted arithmetic mean of the polynomials $q_{2n,k}(z)$ of (34) after replacing $f(z)$ by $h(z)$, with weights

$$b_k^{-1}|\iota_k|^{-1}, \qquad \iota_k = h'(z_k) - q_{2n,k}'(z_k).$$

If $b_k = |\gamma_k|^{-1}$, we obtain from (36)

$$\rho = \left| \sum_0^n h(z_k)\omega''(z_k)\gamma_k^{-3} - \sum_0^n h'(z_k)\gamma_k^{-2} \right| \Big/ \left(\sum_0^n |\gamma_k|^{-1} \right).$$

Since

$$\omega(z) = c\tilde{\omega}(\theta)z^{(n+1)/2}, \qquad \tilde{\omega}(\theta) = \prod_0^n \sin \tfrac{1}{2}(\theta - \theta_k), \qquad |c| = 2^{n+1},$$

we see that the minimum ρ of

$$\max_k |t'(\theta_k) - f'(\theta_k)||\tilde{\omega}(\theta_k)|^{-1}$$

is given by

$$(47) \quad \rho = \left| \sum_0^n f(\theta_k)\tilde{\omega}''(\theta_k)\tilde{\gamma}_k^{-3} - \sum_0^n f'(\theta_k)\tilde{\gamma}_k^{-2} \right| \Big/ \left(\sum_0^n |\tilde{\gamma}_k|^{-1} \right), \quad \tilde{\gamma}_k = \tilde{\omega}'(\theta_k).$$

We have thus proved

THEOREM 6. *If $f(\theta)$ is 2π-periodic and has a continuous derivative, and $0 \leqslant \theta_0 < \theta_1 < \ldots < \theta_n < 2\pi$ is a set E of $n + 1$ points, then the trigonometric polynomial $t(\theta)$ of order n or less that interpolates $f(\theta)$ in E and that minimizes*

$$\max_k |t'(\theta_k) - f'(\theta_k)||\tilde{\gamma}_k|^{-1}, \qquad \tilde{\gamma}_k = \tilde{\omega}'(\theta_k),$$

is the weighted arithmetic mean with weights $|\tilde{\gamma}_j|^{-1}$ of the interpolatory trigonometric polynomials $t_k(\theta)$ of order n or less for which

$$t_k(\theta_j) = f(\theta_j) \text{ for all } j, \qquad t'_k(\theta_j) = f'(\theta_j), \qquad j \neq k.$$

The minimum is given by (47).

In particular, if $\theta_k = 2k\pi/(n+1)$, $k = 0, 1, \ldots, n$, then

$$\tilde{\omega}(\theta) = \sin \tfrac{1}{2}(n+1)\theta, \qquad \tilde{\omega}'(\theta_k) = \tfrac{1}{2}(n+1)(-1)^k, \qquad \tilde{\omega}''(\theta_k) = 0,$$

and from (47) we obtain

$$(48) \qquad \rho = 2(n+1)^{-2} \left| \sum_0^n f'(2k\pi/(n+1)) \right|.$$

If

$$f(\theta) = \frac{1}{2} \sum_{-\infty}^{\infty} a_m e^{im\theta},$$

then a simple formula is obtained for ρ in terms of the Fourier coefficients. In fact, if n is even,

$$\sum_{k=0}^n e^{im.2k\pi/(n+1)} = \begin{cases} 0, & \text{if } m \not\equiv 0 \pmod{(n+1)}, \\ n+1, & \text{if } m \equiv 0 \pmod{(n+1)}. \end{cases}$$

Hence from (48)

$$(49) \qquad \rho = \left| \sum_{\lambda=-\infty}^{\infty} \lambda a_{(n+1)\lambda} \right|.$$

If n is odd, it is easy to verify that the same formula for ρ is valid.

It would be interesting to give the trigonometric analogue of Theorem 5 by the method of (**2**, p. 99).

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University of California
and University of Alberta