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Extended affine Weyl groups and Frobenius manifolds

BORIS DUBROVIN[1] and YOUJIN ZHANG[2]
[1] *SISSA, Via Beirut 2–4, 34014 Trieste, Italy, and Steklov Mathematics Institute, Moscow, e-mail: dubrovin@sissa.it*
[2]*SISSA, Via Beirut 2–4, 34014 Trieste, Italy, and Department of Mathematics, University of Science and Technology of China, Hefei, e-mail: youjin@sissa.it*

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Abstract. We define certain extensions of affine Weyl groups (distinct from these considered by K. Saito [S1] in the theory of extended affine root systems), prove an analogue of Chevalley Theorem for their invariants, and construct a Frobenius structure on their orbit spaces. This produces solutions $F(t_1, \dots, t_n)$ of WDVV equations of associativity polynomial in $t_1, \dots, t_{n-1}, \exp t_n$.

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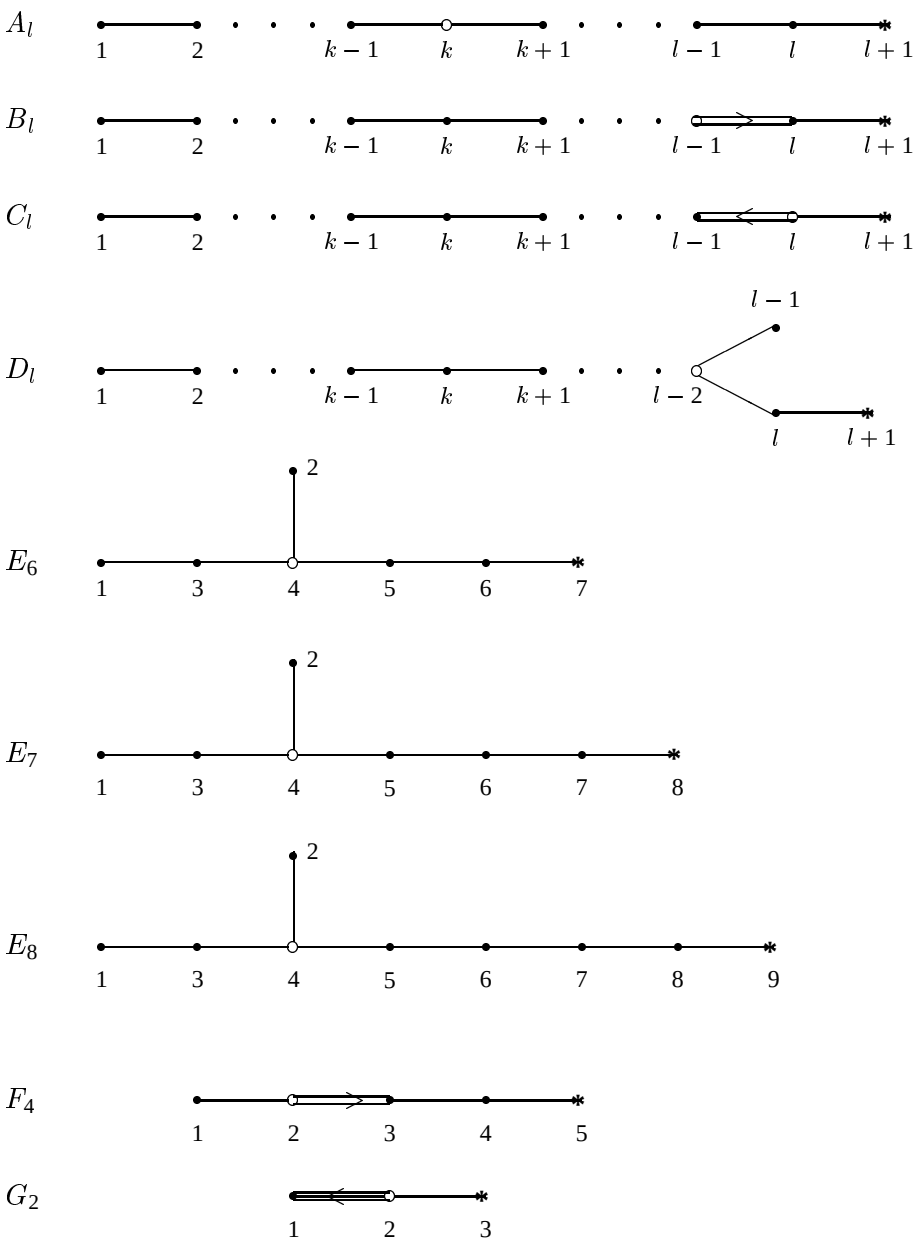
0. Introduction

Frobenius manifold is a geometric object (see precise definition in Section 2 below) designed as a coordinate-free formulation of *equations of associativity*, or *WDVV equations* (they were invented in the beginning of '90s by Witten, Dijkgraaf, E. and H. Verlinde in the setting of two dimensional topological field theory; see [D] and references therein). In [D] for an arbitrary n-dimensional Frobenius manifold a *monodromy group* was defined. It acts in n-dimensional linear space and it is an extension of a group generated by reflections. Looking at simple examples it might be conjectured that for a Frobenius manifold with good analytic properties (in the sense of [D], Appendix A) the monodromy group acts discretely in some domain of the space. The Frobenius manifold itself can be identified with the orbit space of the group in the sense to be specified for each class of monodromy groups.

In the present paper we introduce a new class of discrete groups that can be realized as monodromy groups of Frobenius manifolds (it was shown previously that any finite Coxeter group can serve as a monodromy group of a polynomial Frobenius manifold, see [D]). We define certain extensions of affine Weyl groups, and construct a Frobenius structure on their orbit spaces. Our groups coincide with the monodromy groups of the Frobenius manifolds. They are labelled by pairs (R, k) where R is an irreducible reduced root system, and k is a certain simple root (shown in white on Table I, next page). Our construction of Frobenius

Table I



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structure includes, particularly, a construction of the flat coordinates $t_1, \dots, t_n$ in the appropriate ring of invariants of the extended affine Weyl groups (flat coordinates in the ring of polynomial invariants of finite Coxeter groups were discovered by Saito, Yano, Sekiguchi [SYS, S]). The correspondent solutions of equations of associativity are weighted homogeneous (up to a quadratic function) polynomials in $t_1, \dots, t_{n-1}$, e^{t_n} with all positive weights of the variables. Here $n-1$ is equal to the rank of the root system R. It can be shown (see [D], Appendix A) that for $n \leqslant 3$ our construction exhausts all such solutions.

The paper is organized in the following way. In Section 1 we define extended affine Weyl groups and prove an analogue of Chevalley Theorem [B] for them. In Section 2 we construct Frobenius structure on the orbit spaces of our groups and compute explicitly all low-dimensional examples of these Frobenius manifolds. In Section 3 we show that in the case of the root system of A-type, our extended affine Weyl groups describe monodromy of roots of trigonometric polynomials of a given bidegree. We discuss topology of the complement to bifurcation variety of such trigonometric polynomials in terms of the correspondent Frobenius manifolds.

1. Extended affine Weyl groups and their invariants

Let R be an irreducible reduced root system in l-dimensional Euclidean space V with Euclidean inner product $(\ ,\ )$. We fix a basis $\alpha_1, \alpha_2, \dots, \alpha_l$ of simple roots. Let

$$\alpha_j^\vee = \frac{2\alpha_j}{(\alpha_j, \alpha_j)}, \quad j = 1, 2, \dots, l$$

be the correspondent coroots. All the numbers $A_{ij} := (\alpha_i, \alpha_j^\vee)$ are integers (these are the entries of the Cartan matrix $A = (A_{ij})$, $(\alpha_i, \alpha_i^\vee) = 2$, $(\alpha_i, \alpha_j^\vee) \leqslant 0$ for $i \neq j$). The Weyl group $W = W(R)$ is a finite group generated by the reflections $\sigma_1, \sigma_2, \dots, \sigma_l$

$$\sigma_j(\mathbf{x}) = \mathbf{x} - (\alpha_j^\vee, \mathbf{x})\alpha_j, \quad \mathbf{x} \in V. \tag{1.1}$$

We recall that the root system is one of the type A_l, B_l, C_l, D_l, E_6, E_7, E_8, F_4, G_2 (see [B]).

The affine Weyl group $W_a(R)$ acts in the space V by affine transformations

$$\mathbf{x} \mapsto w(\mathbf{x}) + \sum_{j=1}^{l} m_j \alpha_j^\vee, \quad w \in W,\ m_j \in \mathbf{Z}.$$

So it is isomorphic to the semidirect product of W by the lattice of coroots.

Let us introduce coordinates $x_1, x_2, \dots, x_l$ in V using the basis of coroots

$$\mathbf{x} = x_1\alpha_1^\vee + x_2\alpha_2^\vee + \cdots + x_l\alpha_l^\vee. \tag{1.2}$$

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We define *Fourier polynomial* as the following functions on V

$$f(\mathbf{x}) = \sum_{m_1,\dots,m_l \in \mathbf{Z}} a_{m_1,\dots,m_l}\, \mathrm{e}^{2\pi i(m_1x_1+\cdots+m_lx_l)},$$

the coefficients are arbitrary complex numbers and only finite number of them could be nonzero. Alternatively, introducing the fundamental weights $\omega_1, \dots, \omega_l \in V$

$$(\omega_i, \alpha_j^\vee) = \delta_{ij},$$

we can represent the Fourier polynomial as a sum over the weight lattice

$$f(\mathbf{x}) = \sum_{m_1,\dots,m_l \in \mathbf{Z}} a_{m_1,\dots,m_l}\, \mathrm{e}^{2\pi i(m_1\omega_1+\cdots+m_l\omega_l,\mathbf{X})}.$$

Thus the ring of our Fourier polynomials is identified with the group algebra of the weight lattice [B]. We define the operation of averaging of a Fourier polynomial

$$f(\mathbf{x}) \mapsto \bar{f}(\mathbf{x}) = S_W(f(\mathbf{x})) := n_f^{-1} \sum_{w\in W} f(w(\mathbf{x})), \tag{1.3}$$

where $n_f = \#\{w \in W \mid f(w(\mathbf{x})) = f(\mathbf{x})\}$. For any $f(\mathbf{x})$ the Fourier polynomial $\bar{f}(\mathbf{x}) = S_W(f(\mathbf{x}))$ is a function on V invariant with respect to the action of the affine Weyl group

$$\bar{f}\left(w(\mathbf{x}) + \sum_{j=1}^{l} m_j\alpha_j^\vee\right) = \bar{f}(\mathbf{x}).$$

Equivalently, this is a W-invariant Fourier polynomial.

THEOREM [B]. *The ring of W-invariant Fourier polynomials is isomorphic to the polynomial ring $\mathbf{C}[y_1, \dots, y_l]$, where $y_1 = y_1(\mathbf{x}), \dots, y_l = y_l(\mathbf{x})$ are the basic W-invariant Fourier polynomials defined by*

$$y_j = S_W(\mathrm{e}^{2\pi i(\omega_j,\mathbf{X})}), \quad j = 1, \dots, l. \tag{1.4}$$

EXAMPLE 1.1. The Weyl group $W(A_l)$ acts by permutations of the coordinates $z_1, \dots, z_{l+1}$ on the hyperplane

$$z_1 + \cdots + z_{l+1} = 0.$$

We choose the standard basis of simple roots $\alpha_j = \alpha_j^\vee$ as in [B, Planches I]. Then the coordinates $x_1, \dots, x_l$ are defined by

$$z_1 = x_1, \quad z_i = x_i - x_{i-1}, \quad i = 2, \dots, l, \quad z_{l+1} = -x_l. \tag{1.5}$$

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The basic W-invariant Fourier polynomials coincide with the elementary symmetric functions

$$y_j = s_j(\mathrm{e}^{2\pi i z_1}, \dots, \mathrm{e}^{2\pi i z_{l+1}}), \quad j = 1, \dots, l. \tag{1.6}$$

We are going now to define certain extensions of the affine Weyl group acting in the $(l+1)$-dimensional space with indefinite metric.

For any irreducible reduced root system R we fix a root α_k indicated in Table I. The Dynkin graphs of $A - B - C - D - E - F - G$ type are shown in the Table 1 with one more vertex added (this is indicated by asterisk). We will use this additional vertex later on. The white vertex of the Dynkin graph corresponds to the chosen root α_k. Observe that the Dynkin graph of $R_k := \{\alpha_1, \dots, \hat{\alpha}_k, \dots, \alpha_l\}$ (α_k is omitted) consists of 1, 2 or 3 branches of A_r type for some r. Another observation is that the number

$$\tfrac{1}{2}(\alpha_k, \alpha_k)$$

is an integer for our choice of k.

We construct a group

$$\widetilde{W} = \widetilde{W}^{(k)}(R)$$

acting in

$$\widetilde{V} = V \oplus \mathbf{R}$$

generated by the transformations

$$x = (\mathbf{x}, x_{l+1}) \mapsto \left(w(\mathbf{x}) + \sum_{j=1}^{l} m_j \alpha_j^\vee,\ x_{l+1} \right), \quad w \in W,\ m_j \in \mathbf{Z}, \tag{1.7a}$$

and

$$x = (\mathbf{x}, x_{l+1}) \mapsto (\mathbf{x} + \omega_k,\ x_{l+1} - 1). \tag{1.7b}$$

DEFINITION. $\mathcal{A} = \mathcal{A}^{(k)}(R)$ is the ring of all $\widetilde{W}$-invariant Fourier polynomials of $x_1, \dots, x_l, (1/f)x_{l+1}$ that are bounded in the limit

$$\mathbf{x} = \mathbf{x}^0 - i\,\omega_k \tau, \quad x_{l+1} = x_{l+1}^0 + i\,\tau, \quad \tau \to +\infty, \tag{1.8}$$

for any $x^0 = (\mathbf{x}^0, x_{l+1}^0)$, here f is the determinant of the Cartan matrix A of the root system R, see Table 2.

We put

$$d_j = (\omega_j, \omega_k), \quad j = 1, \dots, l, \tag{1.8}$$

these are certain positive rational numbers that can be found in Table II. All the numbers $f \cdot d_j$ are integers. Indeed, they are the elements of the kth column of the matrix A^{-1} times $\frac{1}{2}(\alpha_k, \alpha_k)$.

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Table II.

R	$d_1, \dots, d_l$	f	d_k
A_l	$d_i = \begin{cases} \frac{i(l-k+1)}{l+1}, & 1 \leqslant i \leqslant k \\ \frac{k(l-i+1)}{l+1}, & k+1 \leqslant i \leqslant l \end{cases}$	$l+1$	$\frac{k(l-k+1)}{l+1}$
B_l	$d_i = \begin{cases} i, & 1 \leqslant i \leqslant l-1 \\ \frac{l-1}{2}, & i = l \end{cases}$	2	$l-1$
C_l	$d_i = i$	2	l
D_l	$d_i = \begin{cases} i, & 1 \leqslant i \leqslant l-2 \\ \frac{l-2}{2}, & i = l-1, l \end{cases}$	4	$l-2$
E_6	2, 3, 4, 6, 4, 2	3	6
E_7	4, 6, 8, 12, 9, 6, 3	2	12
E_8	10, 15, 20, 30, 24, 18, 12, 6	1	30
F_4	3, 6, 4, 2	1	6
G_2	3, 6	1	6

LEMMA 1.1. *The Fourier polynomials*

$$\begin{aligned} \tilde{y}_j(x) &= \mathrm{e}^{2\pi i d_j x_{l+1}} y_j(\mathbf{x}), \quad j = 1, \dots, l, \\ \tilde{y}_{l+1}(x) &= e^{2\pi i x_{l+1}}, \end{aligned} \tag{1.10}$$

are $\widetilde{W}$-invariant.

Proof. We show that all $\tilde{y}_1, \dots, \tilde{y}_l$ are $\widetilde{W}$-invariant (invariance of $\tilde{y}_{l+1}$ is obvious). It suffices to prove that

$$y_j(\mathbf{x} + \omega_k) = \mathrm{e}^{2\pi i d_j}\, y_j(\mathbf{x}). \tag{1.11}$$

We can represent $y_j(\mathbf{x})$ as

$$y_j(\mathbf{x}) = n_j^{-1} \sum_{w \in W} \mathrm{e}^{2\pi i (w(\omega_j), \mathbf{x})}, \tag{1.12}$$

where $n_j = n_f$ for $f = e^{2\pi i (\omega_j, \mathbf{x})}$. According to [B, VI, Sect. 1.6, Prop. 18] for any $w \in W$

$$w(\omega_j) = \omega_j - \sum_{i=1}^{l} m_i \alpha_i, \tag{1.13}$$

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for some non-negative integers $m_1, \dots, m_l$. So

$$(w(\omega_j), \omega_k) = (\omega_j, \omega_k) - \sum_{i=1}^{l} m_i(\alpha_i, \omega_k) = d_j - m_w,$$

for an integer

$$m_w = \tfrac{1}{2} m_k\, (\alpha_k, \alpha_k), \tag{1.14}$$

this leads to (1.11), and we proved the lemma. □

Let us prove now boundedness of the functions $\tilde{y}_1, \dots, \tilde{y}_l$ in the limit (1.8).

LEMMA 1.2. *In the limit*

$$\mathbf{x} = \mathbf{x}^0 - i\omega_k \tau, \quad \tau \to +\infty, \tag{1.15}$$

the functions $y_1(\mathbf{x}), \dots, y_l(\mathbf{x})$ *have the expansion*

$$y_j(\mathbf{x}) = \mathrm{e}^{2\pi d_j \tau}[y_j^0(\mathbf{x}^0) + \mathcal{O}(\mathrm{e}^{-2\pi\tau})], \quad j = 1, \dots, l, \tag{1.16}$$

where

$$y_j^0(\mathbf{x}^0) = n_j^{-1} \sum_{\substack{w \in W \\ (w(\omega_j) - \omega_j, \omega_k) = 0}} \mathrm{e}^{2\pi i (w(\omega_j), \mathbf{x}^0)}. \tag{1.17}$$

Proof. From the representation (1.13) we see that the exponential $\exp[2\pi i(w(\omega_j), \mathbf{x})]$ in the limit (1.15) behaves as

$$\mathrm{e}^{2\pi(d_j - m_w)\tau}\, \mathrm{e}^{2\pi i (w(\omega_j), \mathbf{x}^0)},$$

where the non-negative integer m_w is defined in (1.14). Thus the leading contribution in the asymptotic behaviour of the sum (1.12) comes from those $w \in W$ satisfying $m_k = 0$. Lemma is proved. □

COROLLARY 1.1. *The functions* $\tilde{y}_1(x), \dots, \tilde{y}_{l+1}(x)$ *belong to* $\mathcal{A}$.

Proof. From (1.16) it follows that

$$\tilde{y}_j(x) \to \tilde{y}_j^0(x^0) = \mathrm{e}^{2\pi i d_j x^0_{l+1}} y_j^0(\mathbf{x}^0), \quad j = 1, \dots, l,$$

$$\tilde{y}_{l+1} \to 0$$

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in the limit (1.8), where the functions $y_j^0(\mathbf{x}^0)$ are defined in (1.17). Corollary is proved. □

The main result of this section is

THEOREM 1.1. *The ring $\mathcal{A}$ is isomorphic to the ring of polynomials of $\tilde{y}_1, \dots, \tilde{y}_{l+1}$.*

Proof. We will show that any element $f(x)$ of the ring $\mathcal{A}$ can be represented as a polynomial of $\tilde{y}_1, \dots, \tilde{y}_{l+1}$. This will be enough since the functions $\tilde{y}_1, \dots, \tilde{y}_{l+1}$ are algebraically independent.

From the invariance w.r.t. $\widetilde{W}$ it easily follows (using Theorem [B]) that any $f(x)$ can be represented as a polynomial of $\tilde{y}_1(x), \dots, \tilde{y}_l(x), \tilde{y}_{l+1}(x), \tilde{y}_{l+1}^{-1}$. We need to show that in $f(x)$ there are no negative powers of

$$\tilde{y}_{l+1}(x) = \mathrm{e}^{2\pi i x_{l+1}}.$$

Let's assume

$$f(x) = \sum_{n \geqslant -N} \tilde{y}_{l+1}^n \, P_n(\tilde{y}_1(x), \dots, \tilde{y}_l(x))$$

and the polynomial $P_{-N}(\tilde{y}_1(x), \dots, \tilde{y}_l(x))$ does not vanish identically for certain positive integer N. From Corollary 1.1 we obtain that in the limit (1.8) the function $f(x)$ behaves as

$$f(x) = \mathrm{e}^{2\pi N\tau - 2\pi i N x_{l+1}^0}[P_{-N}(\tilde{y}_1^0(x^0), \dots, \tilde{y}_l^0(x^0)) + \mathcal{O}(\mathrm{e}^{-2\pi\tau})],$$

where

$$\tilde{y}_j^0(x^0) = \mathrm{e}^{2\pi i d_j x_{l+1}^0} \, y_j^0(\mathbf{x}^0), \quad j = 1, \dots, l$$

and the functions $y_j^0(\mathbf{x}^0)$ are defined in (1.17).

To obtain a function bounded for $\tau \to +\infty$ it is necessary to have

$$P_{-N}(\tilde{y}_1^0(x^0), \dots, \tilde{y}_l^0(x^0)) \equiv 0,$$

for any $x^0 = (\mathbf{x}^0, x_{l+1}^0)$. We show now that this is impossible due to algebraic independence of the functions $\tilde{y}_1^0, \dots, \tilde{y}_l^0$. It is sufficient to prove algebraic independence of the functions $y_1^0(\mathbf{x}), \dots, y_l^0(\mathbf{x})$.

MAIN LEMMA. *The Fourier polynomials $y_1^0(\mathbf{x}), \dots, y_l^0(\mathbf{x})$ are algebraically independent.*

We will prove that these functions are functionally independent, i.e., that the Jacobian

$$\det\left(\frac{\partial y_j^0(\mathbf{x})}{\partial x_i}\right)$$

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does not vanish identically. At this end we derive explicit formulae for these functions and then prove nonvanishing of the Jacobian.

Consider

$$R_k = R^{(1)} \cup R^{(2)} \cup \dots,$$

here any subsystem $R^{(1)}, \dots$ is a root system of the type A_r for some r (see Table I).

Let ω_i be a fundamental weight orthogonal to $R \backslash R^{(1)}$.

LEMMA 1.3. *Let for some* $w \in W(R)$

$$w(\omega_i) = \omega_i - \sum_{m=1}^{l} c_m \alpha_m,$$

such that $c_k = 0$, *then* $c_m \neq 0$ *only if* $\alpha_m \in R^{(1)}$.

Proof. In $W(R_k \backslash R^{(1)})$ there exists w_0 such that it maps all positive roots of $R_k \backslash R^{(1)}$ into negative roots. Clearly, w_0 preserves all $\alpha_m \in R^{(1)}$, and $w_0(\omega_i) = \omega_i$. We represent

$$w(\omega_i) = \omega_i - \alpha^{(1)} - \alpha^{(2)} - \dots,$$

where $\alpha^{(i)}$'s are sum of some positive roots of $R^{(i)}$. Then

$$w_0 w(\omega_i) = \omega_i - \alpha^{(1)} - w_0(\alpha^{(2)} + \cdots) = \omega_i - \alpha^{(1)} + \sum_{\alpha_m \in R_k \backslash R^{(1)}} \tilde{c}_m \alpha_m,$$

for some nonnegative integers $\tilde{c}_m$, and not all of these integers vanish if there exists certain $\alpha_m \notin R^{(1)}$ such that $c_m \neq 0$. This contradicts to negativity of $w_0 w(\omega_i) - \omega_i$. Lemma is proved. □

A similar statement holds true for other components $R^{(2)}, \dots$ (if any) of R_k.

LEMMA 1.4. *If for some* $w \in W(R)$

$$w(\omega_k) = \omega_k - \sum_{m \neq k} c_m \alpha_m,$$

then all $c_m = 0$.

Proof. There exists $w_0 \in W(R_k)$ such that it maps any positive roots of R_k into negative ones, and preserves ω_k. So

$$w_0 w(\omega_k) = \omega_k + \text{sum of some positive roots},$$

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which leads to the result of the lemma. □

LEMMA 1.5. *Under the assumption of Lemma* 1.3 *there exists* $\tilde{w} \in W(R^{(1)})$ *such that*

$$\tilde{w}(\omega_i) = w(\omega_i) = \omega_i - \sum_{\alpha_m \in R^{(1)}} c_m \alpha_m.$$

Proof. We will use induction on the length of w. If the length of w equals one, then the lemma holds true obviously. We now assume that the lemma holds true when the length of w is less than p. Let w has the reduced expression $\sigma_{i_1} \dots \sigma_{i_p}$, then it follows from Lemma 1.3 that

$$\sigma_{i_1} \dots \sigma_{i_p}(\omega_i) = \omega_i - \sum_{\alpha_m \in R^{(1)}} c_m \alpha_m. \tag{1.18}$$

If all $c_m = 0$, then we can put $\tilde{w} = 1$, otherwise we rewrite (1.18) in the form

$$\omega_i = \sigma_{i_p} \sigma_{i_{p-1}} \dots \sigma_{i_1}(\omega_i) - \sum_{\alpha_m \in R^{(1)}} c_m \sigma_{i_p} \sigma_{i_{p-1}} \dots \sigma_{i_1}(\alpha_m).$$

We put

$$\sigma_{i_p} \sigma_{i_{p-1}} \dots \sigma_{i_1}(\omega_i) = \omega_i - \sum_{m=1}^{l} b_m \alpha_m,$$

for some nonnegative integers $b_1, \dots, b_l$. We claim now that there exists a root $\alpha_{m_1} \in R^{(1)}$ such that $c_{m_1} \neq 0$ in (1.18) and

$$w^{-1}(\alpha_{m_1}) = \sigma_{i_p} \sigma_{i_{p-1}} \dots \sigma_{i_1}(\alpha_{m_1})$$

is a negative root. Indeed, otherwise the root

$$\sum_{m=1}^{l} b_m \alpha_m + \sum_{\alpha_m \in R^{(1)}} c_m \sigma_{i_p} \sigma_{i_{p-1}} \dots \sigma_{i_1} \alpha_m$$

could not be equal to zero since c_m's are nonnegative integers. We use now the following proposition.

PROPOSITION [H, page 50]. *Let* $\alpha_{j_1}, \dots, \alpha_{j_t}$ *be some simple roots of* R (*not necessarily distinct*). *If*

$$\sigma_{j_1} \cdots \sigma_{j_{t-1}}(\alpha_{j_t})$$

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is a negative root, then for some $1 \leqslant s \leqslant t-1$ *we have*

$$\sigma_{j_1} \dots \sigma_{j_{t-1}} = \sigma_{j_1} \dots \hat{\sigma}_{j_s} \dots \sigma_{j_{t-1}} \sigma_{j_t},$$

where the hat above σ_{j_s} *means that this factor is omitted in the product.*

According to this statement we can represent

$$w^{-1} = \sigma_{i_p} \dots \hat{\sigma}_{i_s} \dots \sigma_{i_1} \sigma_{m_1},$$

for some $1 \leqslant s \leqslant p$. We can now rewrite (1.18) as

$$w'(\omega_i) = \sigma_{m_1}(\omega_i) - \sum_{\alpha_m \in R^{(1)}} c_m \sigma_{m_1}(\alpha_m) = \omega_i - \sum_{\alpha_m \in R^{(1)}} c'_m \alpha_m,$$

for some new nonnegative integers c'_m, and for

$$w' = \sigma_{i_1} \dots \hat{\sigma}_{i_s} \dots \sigma_{i_p}.$$

The length of w' is less than the length p of w. Using the induction assumption we complete the proof of the lemma. □

COROLLARY 1.2. (1) *Let* $R^{(s)}$ *be any branch of* R_k*, then for any* $\alpha_j \in R^{(s)}$ *we have*

$$y_j^0(\mathbf{x}) = m_j^{-1} \sum_{w \in W(R^{(s)})} \mathrm{e}^{2\pi i (w(\omega_j), \mathbf{x})}; \tag{1.19a}$$

(2)

$$y_k^0(\mathbf{x}) = e^{2\pi i (\omega_k, \mathbf{x})} = e^{2\pi i x_k}, \tag{1.19b}$$

where $m_j = \#\{w \in W(R^{(s)}) \mid w(\omega_j) = \omega_j\}$.

Proof of the Main Lemma. We proceed now to the proof of algebraic independence of the functions $y_1^0(\mathbf{x}), \dots, y_l^0(\mathbf{x})$ by analyzing the formulae (1.19) for all the cases of root systems. Let's define $z_i, i = 1, \dots, l$ as in (1.5), and denote the jth order elementary symmetric polynomial of n variables $u_1, \dots, u_n$ by $s_j(u_1, \dots, u_n)$ with $s_0(u_1, \dots, u_n) = 1$.

(1) For the root system of type A_l, from Example 1.1 and (1.19) we have

$$\begin{aligned}
&y_i^0(\mathbf{x}) = v_i + v_{i-1} v_k, \quad 1 \leqslant i \leqslant k-1, \\
&y_k^0(\mathbf{x}) = v_{k-1} v_k, \\
&y_{k+1}^0(\mathbf{x}) = v_{k+1} v_{k-1} v_k + \frac{1}{v_l}, \\
&y_{k+j}^0(\mathbf{x}) = v_{k+j} v_{k-1} v_k + \frac{v_{k+j-1}}{v_l}, \quad 2 \leqslant j \leqslant l-k,
\end{aligned} \tag{1.20}$$

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where

$$\begin{aligned} v_i &= s_i(\mathrm{e}^{2\pi i z_1}, \dots, \mathrm{e}^{2\pi i z_{k-1}}), \quad 1 \leqslant i \leqslant k-1, \\ v_k &= \mathrm{e}^{2\pi i z_k}, \\ v_{k+j} &= s_j(\mathrm{e}^{2\pi i z_{k+1}}, \dots, \mathrm{e}^{2\pi i z_l}), \quad 1 \leqslant j \leqslant l-k. \end{aligned} \tag{1.21}$$

Clearly the variables $v_1, \dots, v_l$ are algebraically independent. We have

$$\det\left(\frac{\partial y_i^0}{\partial v_j}\right) = (-1)^{k-1} v_{k-1}^{l-k} v_k^{l-1} + \text{lower order terms of } v_k. \tag{1.22}$$

So the Jacobian $\det(\partial y_i^0/\partial x_j)$ does not vanish identically.

(2) For the root system of type B_l and C_l, we have $k = l-1$ and $k = l$ respectively, all the formulae in (1.20)–(1.22) hold true if we replace k by $l-1$ and l respectively. So in these cases the Jacobian does not vanish identically neither.

(3) For the root system of type $D_l, k = l-2$, we have

$$\begin{aligned} y_i^0(\mathbf{x}) &= v_i + v_{i-1} v_{l-2}, \quad 1 \leqslant i \leqslant l-3, \\ y_{l-2}^0(\mathbf{x}) &= v_{l-3} v_{l-2}, \\ y_{l-1}^0(\mathbf{x}) &= v_{l-3} v_{l-2} v_{l-1} + \frac{1}{v_{l-1}}, \\ y_l^0(\mathbf{x}) &= v_{l-3} v_{l-2} v_{l-1} v_l + \frac{1}{v_{l-1} v_l}, \end{aligned} \tag{1.23}$$

where $v_i, 1 \leqslant i \leqslant l-2$ are defined in (1.21) with $k = l-2$, and $v_{l-1} = \mathrm{e}^{2\pi i z_{l-1}}, v_l = \mathrm{e}^{2\pi i z_l}$. We have

$$\det\left(\frac{\partial y_i^0}{\partial v_j}\right) = (-1)^{l+1} v_{l-1} v_{l-3}^2 v_{l-2}^{l-1} + \text{lower order terms of } v_{l-2}.$$

Since the functions $v_1, \dots, v_l$ are algebraically independent, the Jacobian $\det(\partial y_i^0/\partial x_j)$ does not vanish identically.

(4) For the root system of type $E_l, k = 4$, let's define

$$\beta_1 = x_l, \quad \beta_i = x_{l-i+1} - x_{l-i+2}, \quad 2 \leqslant i \leqslant l$$

and

$$\begin{aligned} v_i &= s_i(\mathrm{e}^{2\pi i \beta_1}, \dots, \mathrm{e}^{2\pi i \beta_{l-4}}), \quad 1 \leqslant i \leqslant l-4, \\ v_{l-3} &= \mathrm{e}^{2\pi i \beta_{l-3}}, \end{aligned}$$

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$$v_{l-2} = \mathrm{e}^{2\pi i(\beta_{l-2}+\beta_{l-1}+\beta_l)},$$

$$v_{l-1} = \mathrm{e}^{2\pi i(\beta_{l-1}+\beta_l)},$$

$$v_l = \mathrm{e}^{2\pi i\beta_l}.$$

Then we have

$$\begin{aligned}
y_1^0 &= v_{l-4}v_{l-3}v_{l-2} + \frac{1}{v_{l-1}} + \frac{v_{l-1}}{v_{l-2}},\\
y_2^0 &= v_{l-4}v_{l-3}\frac{v_{l-2}}{v_l} + \frac{v_l}{v_{l-2}},\\
y_3^0 &= v_{l-4}v_{l-3}\frac{v_{l-2}}{v_{l-1}} + v_{l-4}v_{l-3}v_{l-1} + \frac{1}{v_{l-2}},\\
y_4^0 &= v_{l-4}v_{l-3},\\
y_{l-i+1}^0 &= v_i + v_{i-1}v_{l-3}, \quad 1 \leqslant i \leqslant l-4
\end{aligned} \tag{1.24}$$

and

$$\det\left(\frac{\partial y_i^0}{\partial v_j}\right) = \varepsilon\left(1 - \frac{v_{l-2}}{v_{l-1}^2}\right)\frac{v_{l-2}(v_{l-4})^3}{v_l^2}v_{l-3}^{l-1} + \text{lower order terms of } v_{l-3},$$

where $\varepsilon = -1$ or 1, which shows that the Jacobian $\det(\partial y_i^0/\partial x_j)$ does not vanish identically.

(5) For the root system of type $F_4, k = 2$, define β_i as for the E_l case with $l = 4$, we have

$$\begin{aligned}
y_1^0 &= v_2v_3v_4 + \frac{1}{v_4},\\
y_2^0 &= v_2v_3,\\
y_3^0 &= v_2 + v_1v_3,\\
y_4^0 &= v_1 + v_3,
\end{aligned} \tag{1.25}$$

where $v_j = s_j(\mathrm{e}^{2\pi i\beta_1}, \mathrm{e}^{2\pi i\beta_2}), j = 1, 2, v_3 = \mathrm{e}^{2\pi i\beta_3}, v_4 = \mathrm{e}^{2\pi i\beta_4}$. Then

$$\det\left(\frac{\partial y_i^0}{\partial v_j}\right) = v_2v_3^3 + \text{lower order terms of } v_3.$$

Since the functions $v_1, \ldots, v_4$ are algebraically independent, the Jacobian $\det(\partial y_i^0/\partial x_j)$ does not vanish identically.

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(6) For the root system of type G_2, $k = 2$, we have

$$y_1^0 = \mathrm{e}^{2\pi i x_1} + \mathrm{e}^{2\pi i(x_1 - x_2)}, \quad y_2^0 = \mathrm{e}^{2\pi i x_2}, \tag{1.26}$$

so y_1^0 and y_2^0 are algebraically independent.

We thus proved the Main lemma and also Theorem 1.1. □

COROLLARY 1.3. *The function* deg *defined as*

$$\deg \tilde{y}_j = d_j, \quad j = 1, \dots, l,$$

$$\deg \tilde{y}_{l+1} = 1$$

determines on $\mathcal{A}$ *a structure of graded polynomial ring.*

We end this section with an important observation about the numbers $d_1, \dots, d_l$. Let us again consider the components of the Dynkin graph of $R_k = R \backslash \alpha_k$. By $R^{(1)}$ we denote the component that touches the added vertex (the asterisk) on Table I. We put

$$\hat{R}^{(1)} = R^{(1)} \cup \{\alpha_k\} \cup \{*\}.$$

This is again an A_r-type diagram. By $R^{(2)}, \dots$ we denote other components of R_k. We put also

$$d_{l+1} = 0.$$

On any of the diagram $\hat{R}^{(1)}, R^{(2)}, \dots$ there is an involution

$$\alpha_i \mapsto \alpha_{i^*}$$

corresponding to the reflection of the component w.r.t. the center.

LEMMA 1.6. *The numbers* $d_1, \dots, d_{l+1}$ *satisfy the duality relation*

$$d_i + d_{i^*} = d_k. \tag{1.27}$$

Proof. By obvious inspection of Table II. □

2. Differential geometry of the orbit spaces of extended affine Weyl groups

Let $\mathcal{M} = \mathcal{M}(R, k) = \operatorname{Spec} \mathcal{A}$. We call it *orbit space* of the extended affine Weyl group $\widetilde{W}$. According to Theorem 1.1 and Corollary 1.3 this is a graded affine algebraic variety of the dimension $n = l + 1$. The functions $\tilde{y}_1(x), \dots, \tilde{y}_{l+1}(x)$ serve as global coordinates on $\mathcal{M}$. We will however use the local coordinates $y^1 =$

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$\tilde{y}_1(x), \dots, y^l = \tilde{y}_l(x)$ and $y^{l+1} = \log \tilde{y}_{l+1}(x) = 2\pi i x_{l+1}$. The last coordinate is multivalued on $\mathcal{M}$. In other words, it lives on a covering $\widetilde{\mathcal{M}}$ of $\mathcal{M}\backslash\{\tilde{y}_{l+1} = 0\}$.

The projection map

$$P\colon \widetilde{V} \to \widetilde{\mathcal{M}}$$

is given by the formulae

$$(x_1, \dots, x_{l+1}) \mapsto (\tilde{y}_1(x), \dots, \tilde{y}_l(x), 2\pi i x_{l+1}) = (y^1, \dots, y^{l+1}). \tag{2.1}$$

For the Jacobian of the projection map we have a formula

$$\begin{aligned} \det\left(\frac{\partial y^j}{\partial x_p}\right) &= 2\pi i\, \mathrm{e}^{2\pi i(d_1+\cdots+d_l)x_{l+1}} \det\left(\frac{\partial y_j(\mathbf{x})}{\partial x_p}\right) \\ &= c \exp\left[2\pi i(d_1+\cdots+d_l)x_{l+1} - \sum_{\alpha\in\Phi^+} \pi i(\alpha,\mathbf{x})\right] \\ &\quad \times \prod_{\beta\in\Phi^+} (\mathrm{e}^{2\pi i(\beta,\mathbf{x})} - 1), \end{aligned} \tag{2.2}$$

where Φ^+ is the set of all positive roots and c is a nonzero constant [B, page 185, 228]. So the projection map is a local diffeomorphism outside the hyperplanes

$$\{(\mathbf{x}, x_{l+1}) | (\beta, \mathbf{x}) = m \in \mathbf{Z},\ x_{l+1} = \text{arbitrary}\}, \quad \beta \in \Phi^+. \tag{2.3}$$

Recall that the hyperplanes $\{\mathbf{x}|(\beta,\mathbf{x}) = m \in \mathbf{Z}\}$ are the mirrors of the affine Weyl group.

In this section, we will introduce a structure of Frobenius manifold on $\mathcal{M}$ (see definition below, and also in [D]). We define first an indefinite metric $(\ ,\ )\tilde{}$ on $\widetilde{V} = V \oplus \mathbf{R}$. The restriction of $(\ ,\ )\tilde{}$ onto V coincides with the W-invariant Euclidean metric $(\ ,\ )$ on V times $4\pi^2$. The coordinate x_{l+1} is orthogonal to V (so $V \oplus \mathbf{R}$ is orthogonal direct sum). Finally we put

$$(\mathrm{e}_{l+1}, \mathrm{e}_{l+1})\tilde{} = -4\pi^2(\omega_k, \omega_k) = -4\pi^2 d_k,$$

where e_{l+1} is the unit vector along the x_{l+1} axis.

We introduce now a symmetric bilinear form on $\mathrm{T}^*\mathcal{M}$ taking projection of $(\ ,\ )\tilde{}$. More explicitly (cf. [A1]) the bilinear form on $\mathrm{T}^*\mathcal{M}$ in the coordinates $y^1, \dots, y^{l+1}$ is given by a $(l+1)\times(l+1)$ matrix (g^{ij}) of the form

$$(dy^i, dy^j)\tilde{} \equiv g^{ij} := \sum_{a,b=1}^{l+1} \frac{\partial y^i}{\partial x^a}\frac{\partial y^j}{\partial x^b}(dx^a, dx^b)\tilde{}, \tag{2.4}$$

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here $x^a = x_a, 1 \leqslant a \leqslant l+1$, and this notation will also be used later.

LEMMA 2.1. *The matrix entries* g^{ij} *of* (2.4) *are weighted homogeneous polynomials in* $\tilde{y}_1, \ldots, \tilde{y}_{l+1}$ *of the degree*

$$\deg g^{ij} = \deg y^i + \deg y^j,$$

(*here* $\deg y^{l+1} = d_{l+1} = 0$). *The matrix* (g^{ij}) *does not degenerate outside the* P*-images of the hyperplanes* (2.3)

Proof. We have for $1 \leqslant i, j \leqslant l$

$$g^{ij} = \frac{d_i d_j}{d_k} \tilde{y}_i \tilde{y}_j + \frac{1}{4\pi^2} \sum_{p,q=1}^{l} \frac{\partial \tilde{y}_i}{\partial x_p} \frac{\partial \tilde{y}_j}{\partial x_q} (\omega_p, \omega_q). \tag{2.5a}$$

This is a Fourier polynomial invariant w.r.t. $\widetilde{W}$. Clearly it is bounded on the limit (1.8). According to Theorem 1.1 this is a polynomial in $\tilde{y}_1, \ldots, \tilde{y}_{l+1}$. The homogeneity is obvious.

For $g^{j,l+1}$ the computation is even simpler. For $1 \leqslant j \leqslant l$ we obtain

$$g^{j,l+1} = \frac{d_j}{d_k} y^j. \tag{2.5b}$$

Finally

$$g^{l+1,l+1} = \frac{1}{d_k}. \tag{2.5c}$$

Nondegeneracy of (g^{ij}) follows from (2.4) and from the formula (2.2) for the Jacobian. Lemma is proved. □

According to Lemma 2.1, the image of nonregular orbits (2.3) is an algebraic subvariety Σ in $\mathcal{M}$ given by the polynomial equation

$$\Sigma = \{y | \det(g^{ij}(y)) = 0\}.$$

We will call Σ the *discriminant* of the extended affine Weyl group. On $\mathcal{M} \setminus \Sigma$ the matrix (g^{ij}) is invertible; the inverse matrix

$$(g_{ij}) = (g^{ij})^{-1}$$

determines a metric[1] on $\mathcal{M} \setminus \Sigma$. Of course, this is a flat metric since it is obtained from a constant metric on $\widetilde{V}$ by a change of coordinates (see formula (2.4)).

[1] The word 'metric' in this paper will denote a symmetric bilinear non-degenerate quadratic form on T$\mathcal{M}$. The metric is called flat if by a local change of coordinates it can be reduced to a constant form.

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Let us now compute the coefficients of the correspondent Levi–Civita connection ∇ for the metric $(\ ,\ )^{\sim}$ defined by (2.4). It is convenient to consider the 'contravariant components' of the connection

$$\Gamma_m^{ij} = (dy^i, \nabla_m dy^j)^{\sim} \tag{2.6}$$

They are related to the standard Christoffel coefficients by the formula

$$\Gamma_m^{ij} = -g^{is}\Gamma_{sm}^j.$$

For the contravariant components we have the formula

$$\Gamma_m^{ij} dy^m = \frac{\partial y^i}{\partial x^p}\frac{\partial^2 y^j}{\partial x^q\,\partial x^r}(dx^p, dx^q)^{\sim}\,\mathrm{d}x^r, \tag{2.7}$$

here and henceforth summation over the repeated indices is assumed.

LEMMA 2.2. *Γ_m^{ij}'s are weighted homogeneous polynomials in $\tilde{y}_1, \ldots, \tilde{y}_{l+1}$ of the degree* $\deg y^i + \deg y^j - \deg y^m$.

Proof. From (2.2) and (2.7) we can represent

$$\Gamma_m^{ij} = \mathrm{e}^{2\pi i(d_i+d_j-d_m)x_{l+1}}\frac{P_m^{ij}(\mathbf{x})}{J(\mathbf{x})},$$

where P_m^{ij} is certain Fourier polynomial in $x_1, \ldots, x_l$,

$$J(\mathbf{x}) = \mathrm{e}^{-\sum_{\alpha\in\Phi^+}\pi i(\alpha,\mathbf{x})}\prod_{\beta\in\Phi^+}(\mathrm{e}^{2\pi i(\beta,\mathbf{x})}-1)$$

is anti-invariant w.r.t the Weyl group W, it has a simple zero on any mirror of the Weyl group, and it changes sign w.r.t. the reflection in the mirrors. But Γ_m^{ij} must be invariant w.r.t. the Weyl group (it is invariant even w.r.t. the extended affine Weyl group $\widetilde{W}$). So P_m^{ij} must be anti-invariant. Hence it is divisible by $J(\mathbf{x})$ [B]. It follows that Γ_m^{ij} is a Fourier polynomial in $x_1, \ldots, x_l, (1/f)x_{l+1}$, where f is the determinant of the Cartan matrix of the root system (see Table II). Since it is invariant w.r.t. the extended group $\widetilde{W}$ and is bounded in the the limit (1.8), it belongs to $\mathcal{A}$, we conclude from Theorem 1.1 that it is a polynomial in $\tilde{y}_1, \ldots, \tilde{y}_{l+1}$. The homogeneity property is then obvious. Lemma is proved. □

COROLLARY 2.1. *The polynomials $g^{ij}(y)$ and $\Gamma_m^{ij}(y)$ are at most linear in y^k.*

Proof. This follows from weighted homogeneity and from the following important observation

$$d_k > d_j \quad \text{for any } j \neq k,$$

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(see Table II). We need however to prove linearity in y^k of the components g^{kk} and Γ^{kk}_{l+1}. According to (2.5a) we have

$$g^{kk} = d_k \tilde{y}_k^2 + \frac{1}{4\pi^2} \sum_{p,q=1}^{l} \frac{\partial \tilde{y}_k}{\partial x_p} \frac{\partial \tilde{y}_k}{\partial x_q} (\omega_p, \omega_q)$$

$$= \mathrm{e}^{4\pi i d_k x_{l+1}} \left(d_k y_k^2 + \frac{1}{4\pi^2} \sum_{p,q=1}^{l} \frac{\partial y_k(\mathbf{x})}{\partial x_p} \frac{\partial y_k(\mathbf{x})}{\partial x_q} (\omega_p, \omega_q) \right).$$

The second term in the bracket is a W-invariant Fourier polynomial of the form

$$g = \frac{1}{4\pi^2} \sum_{p,q=1}^{l} \frac{\partial y_k(\mathbf{x})}{\partial x_p} \frac{\partial y_k(\mathbf{x})}{\partial x_q} (\omega_p, \omega_q)$$

$$= -n_k^{-2} \sum_{w,w' \in W} (w(\omega_k), w'(\omega_k)) \, \mathrm{e}^{2\pi i (w(\omega_k) + w'(\omega_k), \mathbf{x})}.$$

We use now the standard partial ordering of the weights [H, page 69]

$$\omega \succ \omega'$$

iff

$$\omega - \omega' = \sum_{m=1}^{l} c_m \alpha_m$$

for some nonnegative integers $c_1, \ldots, c_l$. In this case we will also write

$$\mathrm{e}^{2\pi i(\omega, \mathbf{x})} \succ \mathrm{e}^{2\pi i(\omega', \mathbf{x})}.$$

The W-invariant Fourier polynomial g has unique maximal term

$$-(\omega_k, \omega_k) \, \mathrm{e}^{2\pi i (2\omega_k, \mathbf{x})} = -d_k \, \mathrm{e}^{2\pi i (2\omega_k, \mathbf{x})}.$$

Because of this all the terms in the W-invariant Fourier polynomial

$$d_k y_k^2 + g \tag{2.8}$$

are strictly less than $\mathrm{e}^{2\pi i (2\omega_k, \mathbf{x})}$. Hence the representation of (2.8) as a polynomial in $y_1(\mathbf{x}), \ldots, y_l(\mathbf{x})$ does not contain y_k^2. That means that g^{kk} is at most linear in y^k.

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For Γ^{kk}_{l+1} we have

$$\begin{aligned}\Gamma^{kk}_{l+1} &= \frac{\partial x^\gamma}{\partial y^{l+1}}\frac{\partial y^k}{\partial x^p}\frac{\partial^2 y^k}{\partial x^q\,\partial x^\gamma}(dx^p, dx^q)^\sim \\ &= \frac{\partial y^k}{\partial x^p}\frac{\partial}{\partial y^{l+1}}\left(\frac{\partial y^k}{\partial x^q}\right)(dx^p, dx^q)^\sim \\ &= \frac{1}{2}\frac{\partial g^{kk}}{\partial y^{l+1}}.\end{aligned}$$

so it also depends at most linearly on y^k. Corollary is proved. □

We define a new metric on $\mathrm{T}^*\mathcal{M}$ putting

$$\eta^{ij}(y) = \frac{\partial g^{ij}}{\partial y^k}. \tag{2.9}$$

Up to multiplication by a nonzero constant this metric does not depend on the choice of basic homogeneous $\tilde{W}$-invariant Fourier polynomials. Indeed, since

$$\deg y^k > \deg y^j \quad \text{for any } j \neq k$$

the ambiguity in the choice of the basis $y^1, \ldots, y^{l+1}$ is of the form

$$\begin{aligned}&y^k \mapsto cy^k + f^k(y^1, \ldots, \hat{y}^k, \ldots, y^l, \exp y^{l+1}),\\ &y^j \mapsto f^j(y^1, \ldots, \hat{y}^k, \ldots, y^l, \exp y^{l+1}), \quad j \neq k, l+1,\\ &y^{l+1} \mapsto y^{l+1},\end{aligned}$$

where c is a nonzero constant and the polynomials f^j are weighted homogeneous of the degree d_j resp. So the vector field $\partial/\partial y^k$ is invariant within a constant

$$\frac{\partial}{\partial y^k} \mapsto c\frac{\partial}{\partial y^k}.$$

The same formulae prove that the matrix $\eta^{ij}(y)$ behaves like a (2,0)-tensor (i.e., a symmetric bilinear form on the cotangent bundle) w.r.t. the changes of homogeneous coordinates on the orbit space.

MAIN LEMMA. *The determinant of the matrix* (η^{ij}) *is a nonzero constant.*

To prove this lemma, we need the following lemmas.

LEMMA 2.3. *Let* R *be a root system of type* $A_l - B_l - C_l - D_l$, *denote* $R_k \equiv R\backslash\alpha_k = R^{(1)} \cup R^{(2)} \cup \ldots$.

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(1) *If α_i and α_j belong to different components of R_k, then $\eta^{ij} = 0$.*
(2) *The block $\eta^{(t)} = (\eta^{ij})|_{\alpha_i,\alpha_j\in R^{(t)}}$ of the matrix $\eta = (\eta^{ij})$ corresponding to any branch $R^{(t)}$ has triangular form, i.e. it has all zero entries above or under the antidiagonal. The antidiagonal elements of $\eta^{(t)}$ consist of the constant numbers η^{ii^*} for $\alpha_i \in R^{(t)}$.*
(3) $\eta^{i(l+1)} = \eta^{(l+1)i} = (1/d_k)\delta_{ik}$.

Proof. As we did in the proof of Corollary 2.1, we use the standard partial ordering of the weights of R. When $1 \leqslant i, j \leqslant l$, we know from (2.5a) that $g^{ij} = \mathrm{e}^{2\pi i(d_i+d_j)x_{l+1}}\, h(\mathbf{x})$, where

$$h(\mathbf{x}) = \frac{d_i d_j}{d_k} y_i(\mathbf{x}) y_j(\mathbf{x}) - (n_i\, n_j)^{-1} \times \sum_{w,w'\in W} (w(\omega_i), w'(\omega_j))\, \mathrm{e}^{2\pi i(w(\omega_i)+w'(\omega_j),\mathbf{x})}.$$

All the terms in the Fourier polynomial $h(\mathbf{x})$ are strictly less than $\mathrm{e}^{2\pi i(\omega_i+\omega_j,\mathbf{x})}$ except the term $c\,\mathrm{e}^{2\pi i(\omega_i+\omega_j,\mathbf{x})}$, where c is certain constant. So if $h(\mathbf{x})$ as a polynomial in $y_1, \ldots, y_l$ contains a monomial $y_1^{p_1}, \ldots, y_l^{p_l}$ with $p_k = 1$ then

$$\omega_i + \omega_j = p_1\omega_1 + \cdots + p_l\omega_l + \sum_{s=1}^{l} q_s\alpha_s, \tag{2.10}$$

for some nonnegative integers $q_1, \ldots, q_l$.

Let's first consider the root system A_l, and assume $1 \leqslant i < k < j \leqslant l$. We multiply (2.10) by ω_1 to obtain

$$\frac{l-i+1+l-j+1}{l+1} = \frac{\sum p_s(l-s+1)}{l+1} + q_1. \tag{2.11}$$

If $q_1 \geqslant 1$ then we obtain inequality

$$l+1+k-i-j-\sum_{s\neq k} p_s(l-s+1) \geqslant l+1.$$

This is impossible since $k-i-j<0$. Hence $q_1 = 0$ and

$$l-i+1-(j-k) = \sum_{s\neq k} p_s(l-s+1).$$

From the last equation we conclude that $p_s = 0$ for $s \leqslant i$.

Next we multiply the Equation (2.10) by $\alpha_1, \ldots, \alpha_{i-1}$ to prove recursively that also $q_2 = \cdots = q_i = 0$. The last step is to multiply the same equation by α_i. We obtain

$$1 = -q_{i+1}.$$

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This contradicts nonnegativity of q's. So (2.10) is not possible for $i < k < j$.

For the root system $B_l, k = l - 1$, we assume (2.10) holds true for $1 \leqslant i < l - 1, j = l$. We multiply (2.10) by ω_1 to obtain

$$1 + \tfrac{1}{2} = p_1 + \cdots + p_{l-2} + 1 + \tfrac{1}{2}p_l + q_1,$$

which leads to $p_1 = \cdots = p_{l-2} = q_1 = 0, p_l = 1$. We now multiply (2.10) by ω_{i+1} to obtain

$$i + (\omega_l, \omega_{i+1}) = i + 1 + (\omega_l, \omega_{i+1}) + q_{i+1},$$

which is impossible since q_{i+1} is a nonnegative integer. So we proved that (2.10) is not valid for $1 \leqslant i < l - 1, j = l$.

For the root system $D_l, k = l - 2$, similar to the case of B_l we can see that when $1 \leqslant i < l - 2$ and $j = l - 1$, or $1 \leqslant i < l - 2$ and $j = l$ the relation (2.10) cannot be valid. When $i = l - 1, j = l$ we multiply (2.10) by ω_1 to obtain

$$\tfrac{1}{2} + \tfrac{1}{2} = p_1 + \cdots + p_{l-3} + 1 + \tfrac{1}{2}p_{l-1} + \tfrac{1}{2}p_l + q_1,$$

which leads to $p_1 = \cdots = p_{l-3} = p_{l-1} = p_l = q_1 = 0$. We now multiply (2.10) by ω_{l-1} to obtain

$$\tfrac{1}{4}l + \tfrac{1}{4}(l - 2) = \tfrac{1}{2}(l - 2) + q_{l-1},$$

which leads to $q_{l-1} = \frac{1}{2}$, this contradicts to the fact that q_{l-1} is an integer. So under our assumption on i, j, (2.10) is not valid.

The first statement of the lemma follows from the above arguments and from the fact that for the root system C_l, R_k has only one component. To prove the second statement of the lemma, we note that in any component of R_k, the numbers d_i are distinct and ordered monotonically (see Table II). Since $\eta^{ij}(y)$ is a polynomial of degree $d_i + d_j - d_k$, we have $\eta^{ij}(y) = 0$ when $d_i + d_j < d_k$, and $\eta^{ij} =$ constant when $d_i + d_j = d_k$, note that this happens if the labels i and j are dual to each other in the sense of Lemma 1.6. So we proved the second statement of the lemma. The third statement of the lemma follows from (2.5b) and (2.5c). Lemma is proved. □

COROLLARY 2.2. $\det(\eta^{ij})$ *is a constant, modulo a sign it equals to* $\prod_{i=1}^{l} \eta^{ii^*}$.

Proof. For the root system $A_l - B_l - C_l - D_l$ the above statement follows from Lemma 2.3.

For the root system E_8, F_4, G_2 we observe that all the numbers d_i are distinct. We can re-label the simple roots in such a way that the numbers d_i are ordered monotonically. Since $\eta^{ij}(y)$ is a polynomial of degree $d_i + d_j - d_k$, we have $\eta^{ij} = 0$ when $d_i + d_k < d_k$ and $\eta^{ij} =$ constant when $d_i + d_j = d_k$. The last equality holds true when $j = i^*$. We conclude that the matrix (η^{ij}) is triangular, and its anti-diagonal elements are the numbers η^{ii^*}. So the statement of the corollary holds true.

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For the root system E_6, we have

$$d_1 = 2, \quad d_2 = 3, \quad d_3 = 4, \quad d_4 = 6, \quad d_5 = 4, \quad d_6 = 2, \quad d_7 = 0.$$

We change the labels as follows

$$4 \mapsto 1, \quad 3 \mapsto 2, \quad 5 \mapsto 3, \quad 2 \mapsto 4, \quad 6 \mapsto 5, \quad 1 \mapsto 6, \quad 7 \mapsto 7.$$

Under the new labels the numbers d_i are ordered as follows

$$\tilde{d}_1 = 6, \quad \tilde{d}_2 = 4, \quad \tilde{d}_3 = 4, \quad \tilde{d}_4 = 3, \quad \tilde{d}_5 = 2, \quad \tilde{d}_6 = 2, \quad \tilde{d}_7 = 0$$

and the matrix (η^{ij}) becomes $(\tilde{\eta}^{ij})$. We claim that the matrix $(\tilde{\eta}^{ij})$ is triangular, and its anti-diagonal elements consist of the constant numbers η^{ii^*}. To see this, it suffices to show that $\tilde{\eta}^{36} = \eta^{15} = 0$. If $\eta^{15} \neq 0$, then by using a similar argument as we gave in the proof of Lemma 2.3 we have

$$\omega_1 + \omega_5 = p_1\omega_1 + \cdots + p_6\omega_6 + \sum_{s=1}^{6} q_s\alpha_s,$$

with $p_4 = 1$ and p_i, q_j are some nonnegative integers. This is impossible due to

$$\omega_1 + \omega_5 - \omega_4 = \tfrac{2}{3}\alpha_1 + \tfrac{1}{3}\alpha_3 + \tfrac{2}{3}\alpha_5 + \tfrac{1}{3}\alpha_6$$

and the fact that $(\omega_i, \omega_j) > 0$ for $1 \leqslant i, j \leqslant 6$. So we proved that the corollary holds true for the root system E_6.

For the root system E_7, we have

$$d_1 = 4, \quad d_2 = 6, \quad d_3 = 8, \quad d_4 = 12,$$

$$d_5 = 9, \quad d_6 = 6, \quad d_7 = 3, \quad d_8 = 0.$$

Similar to the E_6 case, to prove the statement of the corollary, we only need to prove that $\eta^{26} = 0$, this easily follows from

$$\omega_2 + \omega_6 - \omega_4 = -\alpha_1 - \tfrac{5}{4}\alpha_2 - 2\alpha_3 - 3\alpha_4 - \tfrac{7}{4}\alpha_5 - \tfrac{1}{2}\alpha_6 - \tfrac{1}{4}\alpha_7.$$

Corollary is proved. □

LEMMA 2.4. *There exists* $\mathbf{x}^0$ *such that* $y_j^0(\mathbf{x}^0) = 0$ *for* $j \neq k$ *and* $y_k^0(\mathbf{x}^0) \neq 0$.

Proof. We give the required $\mathbf{x}^0$ explicitly for all cases of the root systems. As in the proof of the Main Lemma in Section 1, we denote $z_1^0 = x_1^0$, $z_i^0 = x_i^0 - x_{i-1}^0$ and $\beta_1^0 = x_l^0$, $\beta_i^0 = x_{l-i+1}^0 - x_{l-i+2}^0$ for $2 \leqslant i \leqslant l$.

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For A_l, we take

$$(z_1^0, \ldots, z_l^0) = \left(0, \frac{1}{k}, \frac{2}{k}, \ldots, \frac{k-1}{k}, c, c - \frac{1}{l-k+1}, \ldots, c - \frac{l-k-1}{l-k+1}\right),$$

where $c = (l - 2k + 1)/2(l - k + 1)$.

For B_l, we take

$$(z_1^0, \ldots, z_l^0) = \left(0, \frac{1}{l-1}, \ldots, \frac{l-2}{l-1}, \frac{3-l}{4}\right).$$

For C_l, we take

$$(z_1^0, \ldots, z_l^0) = \left(0, \frac{1}{l}, \ldots, \frac{l-1}{l}\right).$$

For D_l we take

$$(z_1^0, \ldots, z_l^0) = \left(0, \frac{1}{l-2}, \ldots, \frac{l-3}{l-2}, \frac{4-l}{4}, 0\right).$$

For E_6 we take

$$(\beta_1^0, \ldots, \beta_6^0) = (\tfrac{2}{3}, \tfrac{1}{3}, 0, -\tfrac{2}{3}, -\tfrac{1}{12}, -\tfrac{1}{4}).$$

For E_7 we take

$$(\beta_1^0, \ldots, \beta_7^0) = (\tfrac{3}{4}, \tfrac{2}{4}, \tfrac{1}{4}, 0, -\tfrac{5}{6}, -\tfrac{1}{6}, -\tfrac{1}{3}).$$

For E_8 we take

$$(\beta_1^0, \ldots, \beta_8^0) = (\tfrac{4}{5}, \tfrac{3}{5}, \tfrac{2}{5}, \tfrac{1}{5}, 0, -1, -\tfrac{1}{4}, -\tfrac{5}{12}).$$

For F_4 we take

$$(x_1^0, x_2^0, x_3^0, x_4^0) = (0, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{1}{2}).$$

For G_2 we take

$$(x_1^0, x_2^0) = (\tfrac{1}{2}, \tfrac{3}{2}).$$

It is now easy to see from the formulae in (1.20), (1.23)–(1.26) that these $\mathbf{x}^0 = (x_1^0, \ldots, x_l^0)$ satisfy the requirement of the lemma. Lemma is proved. □

Remark. The $\mathbf{x}^0$ given in Lemma 2.4 satisfies the following relation

$$\sigma_1\sigma_2 \ldots \sigma_{k-1}\sigma_{k+1} \ldots \sigma_l(\mathbf{x}^0) = -\frac{1}{d_k}\omega_k + \sum_{i=1}^{k} \alpha_i^\vee + \mathbf{x}^0, \tag{2.12}$$

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where σ's are defined in (1.1). From this relation and (1.17) we obtain

$$\begin{aligned} & y_j^0(\sigma_1 \dots \sigma_{k-1}\sigma_{k+1} \dots \sigma_l(\mathbf{x}^0)) \\ & = n_j^{-1} \sum_{\substack{w \in W \\ (w(\omega_j)-\omega_j,\omega_k)=0}} \mathrm{e}^{2\pi i(w(\omega_j),-(1/d_k)\omega_k+\sum_{i=1}^k \alpha_i^\vee+\mathbf{x}^0)} \\ & = n_j^{-1} \sum_{\substack{w \in W \\ (w(\omega_j)-\omega_j,\omega_k)=0}} \mathrm{e}^{-(2\pi i/d_k)(\omega_j,\omega_k)+2\pi i(w(\omega_j),\mathbf{x}^0)} \\ & = \mathrm{e}^{-(2\pi i d_j/d_k)} y_j^0(\mathbf{x}^0). \end{aligned} \tag{2.13}$$

On the other hand from (1.17) we have

$$\begin{aligned} & y_j^0(\sigma_1 \dots \sigma_{k-1}\sigma_{k+1} \dots \sigma_l(\mathbf{x}^0)) \\ & = n_j^{-1} \sum_{\substack{w \in W \\ (w(\omega_j)-\omega_j,\omega_k)=0}} \mathrm{e}^{2\pi i(\sigma_l \dots \sigma_{k+1}\sigma_{k-1}\dots\sigma_1 w(\omega_j),\mathbf{x}^0)} = y_j^0(\mathbf{x}^0). \end{aligned} \tag{2.14}$$

So from (2.13), (2.14) and the fact that $0 < d_j/d_k < 1$ when $j \neq k$ it follows that $y_j^0(\mathbf{x}^0) = 0$ when $j \neq k$.

Proof of the Main Lemma. By using Corollary 2.2 we only need to prove that $\det(\eta^{ij})$ does not vanish.

Let's denote

$$\Psi^+ = \{\alpha \in \Phi^+ | (\alpha, \omega_k) = 0\}, \tag{2.15}$$

where Φ^+ is the set of all positive roots of R. Let's take $x = (\mathbf{x}, x_{l+1}) = (\mathbf{x}^0 - i\tau\omega_k, i\tau)$, where $\mathbf{x}^0$ is given by Lemma 2.4, then from (2.2) we have

$$\begin{aligned} & \det\left(\frac{\partial y^i(x)}{\partial x_j}\right) \\ & = c\ \exp\left(2\pi i(d_1 + \dots + d_l)x_{l+1} - \sum_{\alpha \in \Phi^+} \pi i(\alpha, \mathbf{x})\right) \\ & \quad \times \prod_{\beta \in \Phi^+} (\mathrm{e}^{2\pi i(\beta,\mathbf{x})} - 1) \\ & = c' \prod_{\beta \in \Psi^+} (\mathrm{e}^{2\pi i(\beta,\mathbf{x}^0)} - 1) \prod_{\beta \in \Phi^+ \setminus \Psi^+} (\mathrm{e}^{2\pi i(\beta,\mathbf{x}^0)+2\pi(\beta,\omega_k)\tau} - 1), \end{aligned} \tag{2.16}$$

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here

$$c' = c\,\mathrm{e}^{-\sum_{\alpha\in\Phi^+}\pi i(\alpha,\mathbf{x}^0)-\sum_{\alpha\in\Phi^+}2\pi(\alpha,\omega_k)\tau}$$

$$= c\,\mathrm{e}^{-\sum_{\alpha\in\Phi^+}\pi i(\alpha,\mathbf{x}^0)-\sum_{\alpha\in\Phi^+\setminus\Psi^+}2\pi(\alpha,\omega_k)\tau}$$

and we have used the identity [B, H]

$$\frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha=\sum_{i=1}^{l}\omega_i.$$

Now let's take the limit $\tau\to+\infty$, by using (2.16) we obtain

$$\chi := \lim_{\tau\to+\infty}\det\left(\frac{\partial y^i(x)}{\partial x_j}\right)$$

$$= c\,\mathrm{e}^{-\sum_{\alpha\in\Phi^+}\pi i(\alpha,\mathbf{x}^0)}\prod_{\beta\in\Psi^+}(\mathrm{e}^{2\pi i(\beta,\mathbf{x}^0)}-1)\prod_{\beta\in\Phi^+\setminus\Psi^+}\mathrm{e}^{2\pi i(\beta,\mathbf{x}^0)}.$$

From the explicit form of $\mathbf{x}^0$ given in Lemma 2.4 and the above formula we know that $\chi\neq 0$.

Finally, by using Lemma 1.2, Lemma 2.1, Lemma 2.3–2.4 and Corollary 2.1 we have

$$\det(\eta^{ij}) = \frac{1}{(y_k^0(\mathbf{x}^0)^{l+1}}\lim_{\tau\to+\infty}\det(g^{ij}(x))$$

$$= c''\lim_{\tau\to+\infty}\left(\det\left(\frac{\partial y^i(x)}{\partial x_j}\right)\right)^2 = c''\chi^2\neq 0,$$

where c'' is a nonzero constant. The main lemma is proved. □

COROLLARY 2.3. *The function η^{ij} is equal to a nonzero constant if and only if j is dual to i, i.e., $j=i^*$.*

Proof. For the root system of type $A_l-B_l-C_l-D_l-E_8-F_4-G_2$, the corollary follows from the above Main Lemma, Lemma 2.3, Corollary 2.2, Table II and the weighted homogeneity of η^{ij}.

For the root system of type E_6-E_7, the corollary follows from the above Main Lemma, Corollary 2.2, Table II, the weighted homogeneity of η^{ij} and the fact that $\eta^{15}=\eta^{36}=0$ for E_6 and $\eta^{26}=0$ for E_7. In the proof of Corollary 2.2 we showed that $\eta^{15}=0$ for E_6 and $\eta^{26}=0$ for E_7, we can show in a same way that $\eta^{36}=0$ for E_6. Corollary is proved. □

COROLLARY 2.4. *The orbit space $\mathcal{M}$ carries a flat pencil of metrics*

$$g^{ij}(y) \quad \textit{and} \quad \eta^{ij}(y)=\frac{\partial g^{ij}(y)}{\partial y^k}$$

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and the correspondent contravariant Levi–Civita connections

$$\Gamma^{ij}_m(y) \quad \textit{and} \quad \gamma^{ij}_m(y) = \frac{\partial \Gamma^{ij}_m(y)}{\partial y^k}.$$

Particularly, the metric $(\eta^{ij}(y))$ is flat.

Proof. This follows from the linearity of $g^{ij}(y)$ and Γ^{ij}_m in y^k and from nondegeneracy of $\eta^{ij}(y)$ (cf. [D], Appendix D). Corollary is proved. □

We recall [D] that this means that the Levi–Civita connection for a linear combination of the metrics

$$ag^{ij}(y) + b\eta^{ij}(y) \tag{2.17}$$

must have the form

$$a\Gamma^{ij}_m(y) + b\gamma^{ij}_m(y), \tag{2.18}$$

for arbitrary values of the constants a, b, and the metric (2.17) must be flat for any a, b.

Note that $g^{ij}(y), \Gamma^{ij}_m(y)$, and also $\eta^{ij}(y), \gamma^{ij}_m(y)$ are weighted homogeneous polynomials in $y^1, \dots, y^l, \mathrm{e}^{y^{l+1}}$, where

$$\deg y^j = d_j, \quad 1 \leqslant j \leqslant l; \qquad \deg \mathrm{e}^{y^{l+1}} = 1.$$

COROLLARY 2.5. *There exist weighted homogeneous polynomials*

$$t^\alpha = t^\alpha(y^1, \dots, y^l, \mathrm{e}^{y^{l+1}}), \quad \alpha = 1, \dots, l$$

of the degrees d_α such that the metric $\eta^{ij}(y)$ becomes constant in the coordinates $t^1, \dots, t^l, t^{l+1} = y^{l+1}$, and the linear part of t^α is equal to y^α.

Proof. (cf. [D, page 272]) Local existence of the coordinates t^α follows from vanishing of the curvature of η^{ij} (see Corollary 2.4). The flat coordinates $t = t(y)$ are to be found from the system of linear differential equations

$$\begin{aligned} &\eta^{is}\frac{\partial \xi_j}{\partial y^s} + \gamma^{is}_j \xi_s = 0, \\ &\frac{\partial t}{\partial y^s} = \xi_s. \end{aligned} \tag{2.19}$$

From the Main Lemma we know that the inverse matrix $(\eta_{ij}) = (\eta^{ij})^{-1}$ is also polynomial in $y^1, \dots, y^l, \mathrm{e}^{y^{l+1}}$. By using the formulae (2.7) we have

$$\gamma^{i,l+1}_j = \frac{\partial \Gamma^{i,l+1}_j}{\partial y^k} = 0,$$

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it follows that

$$t^{l+1} = y^{l+1}$$

is one of the solutions of the system (2.19). We choose remaining solutions $t^\alpha(y^1, \dots, y^l, \mathrm{e}^{y^{l+1}})$ in such a way that

$$\frac{\partial t^\alpha}{\partial y^j}(0, \dots, 0, 0) = \delta^\alpha_j, \quad \alpha, j = 1, \dots, l.$$

The solutions $t^\alpha(y)$ are power series in $y^1, \dots, y^l, \mathrm{e}^{y^{l+1}}$. The system (2.19) is invariant w.r.t. the transformations

$$y^j \mapsto c^{d_j} y^j, \quad j = 1, \dots, l, \quad y^{l+1} \mapsto y^{l+1} + \log c,$$

for any nonzero constant c. So the functions $t^\alpha(y), 1 \leqslant \alpha \leqslant l$ are weighted homogeneous of the same degrees $d_\alpha > 0$. Hence the power series $t^\alpha(y)$ must be polynomials. Corollary is proved. □

COROLLARY 2.6. *In the flat coordinates* $t^1, \dots, t^{l+1}$ *we have*

(1)
$$\eta^{ij} = \frac{\partial g^{ij}(t^1, \dots, t^{l+1})}{\partial t^k}.$$

(2) η^{ij} *is equal to a nonzero constant if and only if* $j = i^*$, *and*

$$\eta^{ii^*}(t^1, \dots, t^{l+1}) = \eta^{ii^*}(y^1, \dots, y^{l+1}).$$

Proof. From Corollary 2.5 we have

$$\frac{\partial}{\partial t^k} = \frac{\partial}{\partial y^k},$$

which leads to the first statement of the corollary.

The second statement of the corollary follows from the fact that the linear part of t^α is y^α. Corollary is proved. □

It follows from our normalization of the flat coordinates that

$$\eta^{l+1,\alpha} = \delta^\alpha_k.$$

Remark. For the orbit spaces of finite reflection groups flat coordinates were constructed by Saito, Yano and Sekiguchi in [SYS] (see also [S]).

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We recall now the definition of Frobenius manifold.

DEFINITION. (Smooth, polynomial etc.) *Frobenius structure* on a n-dimensional manifold M consists of:

(1) A structure of commutative Frobenius algebra with a unity e on the tangent plane $\mathrm{T}_t M$ that depends smoothly, polynomially etc. on $t \in M$. (We recall that a commutative associative algebra A is called Frobenius algebra if it is equipped with a symmetric nondegenerated bilinear form $\langle\ ,\ \rangle$ satisfying the invariance condition

$$\langle a\ b, c\rangle = \langle a, bc\rangle,$$

for any $a, b, c \in A$.)

(2) A vector field E is fixed on M. We will call it *Euler vector field*. These objects must satisfy the following properties:

(i) The curvature of the invariant metric $\langle\ ,\ \rangle$ on M is equal to zero;

(ii) denoting ∇ the Levi–Civita connection for $\langle\ ,\ \rangle$, we require that

$$\nabla \mathrm{e} = 0; \tag{2.20}$$

(iii) the four-tensor

$$d(a_1, \ldots, a_4) := (\nabla_{a_4} c)(a_1, a_2, a_3),$$

where

$$c(a_1, a_2, a_3) = \langle a_1 a_2, a_3\rangle,$$

must be symmetric w.r.t. any permutation of the vectors $a_1, \ldots, a_4$.

(iv) The vector field E must be linear w.r.t. ∇

$$\nabla\nabla E = 0. \tag{2.21}$$

The eigenvalues $q_1, \ldots, q_n$ of the linear operator $\mathrm{id} - \nabla E$ are called *charges* of the Frobenius structure.

(v) The Lie derivative $\mathcal{L}_E$ along the vector field E must act by rescalings

$$\begin{aligned} &\mathcal{L}_E e = -e, \\ &\mathcal{L}_E(ab) - (\mathcal{L}_E a)b - a(\mathcal{L}_E b) = ab, \\ &\mathcal{L}_E\langle a, b\rangle - \langle \mathcal{L}_E a, b\rangle - \langle a, \mathcal{L}_E b\rangle = (2-d)\langle a, b\rangle, \end{aligned} \tag{2.22}$$

for arbitrary vector fields a, b and for certain constant d. (Observe that, due to (2.20) and (2.22), zero and d are among the charges $q_1, \ldots, q_n$).

A manifold M with a Frobenius structure on it is called a *Frobenius manifold*.

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If we choose locally flat coordinates $t^1, \dots, t^n$ for the invariant metric, then the condition (iii) provides local existence of a function $F(t^1, \dots, t^n)$ such that

$$\langle a\, b, c\rangle = a^\alpha b^\beta c^\gamma \frac{\partial^3 F}{\partial t^\alpha\, \partial t^\beta\, \partial t^\gamma}, \tag{2.23}$$

for any three vectors $a = a^\alpha(\partial/\partial t^\alpha), b = b^\beta(\partial/\partial t^\beta), c = c^\gamma(\partial/\partial t^\gamma)$. Choosing the coordinate t^1 along the unity vector field e we obtain

$$\frac{\partial^3 F(t^1, \dots, t^n)}{\partial t^1\, \partial t^\alpha\, \partial t^\beta} = \eta_{\alpha\beta}, \tag{2.24a}$$

for a constant symmetric nondegenerate matrix $(\eta_{\alpha\beta})$ coinciding with the metric $\langle\ ,\ \rangle$ in the chosen coordinates. Associativity of the algebras implies an overdetermined system of equations for the function F

$$\frac{\partial^3 F}{\partial t^\alpha\, \partial t^\beta\, \partial t^\lambda}\eta^{\lambda\mu}\frac{\partial^3 F}{\partial t^\mu\, \partial t^\gamma\, \partial t^\delta} = \frac{\partial^3 F}{\partial t^\gamma\, \partial t^\beta\, \partial t^\lambda}\eta^{\lambda\mu}\frac{\partial^3 F}{\partial t^\mu\, \partial t^\alpha\, \partial t^\delta}, \tag{2.24b}$$

for arbitrary $\alpha, \beta, \gamma, \delta$ from 1 to n. The components of the Euler vector field E in the basis $\partial/\partial t^\alpha$ are linear functions of $t^1, \dots, t^n$. They enter into the following scaling condition for the function F

$$\mathcal{L}_E F = (3 - d)F + \text{quadratic polynomial in } t. \tag{2.24c}$$

The system (2.24(a)–(c)) is just the *WDVV equations of associativity* being equivalent to our definition of Frobenius manifold in the chosen system of local coordinates.

We recall also an important construction of *intersection form* of a Frobenius manifold. This is a symmetric bilinear form $(\ ,\ )^*$ on T^*M defined by the formula

$$(w_1, w_2)^* = i_E(w_1 \cdot w_2),$$

here the product of two 1-forms w_1, w_2 at a point $t \in M$ is defined by using the algebra structure on T_tM and the isomorphism

$$\mathrm{T}_tM \to \mathrm{T}_t^*M$$

established by the invariant metric $\langle\ ,\ \rangle$. Choosing the flat coordinates $t^1, \dots, t^n$ for the invariant metric, we can rewrite the definition of the intersection form as

$$(dt^\alpha, dt^\beta)^* = \mathcal{L}_E F^{\alpha\beta}, \tag{2.25}$$

where

$$F^{\alpha\beta} = \eta^{\alpha\alpha'}\eta^{\beta\beta'}\frac{\partial^2 F(t)}{\partial t^{\alpha'}\, \partial t^{\beta'}}$$

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and the function $F(t)$ is defined in (2.23). According to the general theory of Frobenius manifolds, $(\ ,\ )^*$ defines a new flat metric on the open subset of M where $\det(\ ,\ )^* \neq 0$. The *discriminant* $\Sigma = \{t \mid \det(\ ,\ )_t^* = 0\}$ is a proper analytic subvariety in an analytic Frobenius manifold M. The holonomy of the local Euclidean structure defined on $M\backslash\Sigma$ by the intersection form $(\ ,\ )^*$ gives a representation

$$\pi_1(M\backslash\Sigma) \to \text{Isometries } V,$$

where V is the standard complex Euclidean space. The image of this representation is called *monodromy group* of the Frobenius manifold.

THEOREM 2.1. *There exists a unique Frobenius structure on the orbit space* $\mathcal{M} = \mathcal{M}(R,k)$ *polynomial in* $t^1, \dots, t^l, \mathrm{e}^{t^{l+1}}$ *such that*

(1) *the unity vector field coincides with* $\partial/\partial y^k = \partial/\partial t^k$;
(2) *the Euler vector field has the form*

$$E = \frac{1}{2\pi i d_k}\frac{\partial}{\partial x_{l+1}} = \sum_{\alpha=1}^{l}\frac{d_\alpha}{d_k}\,t^\alpha\frac{\partial}{\partial t^\alpha} + \frac{1}{d_k}\frac{\partial}{\partial t^{l+1}}. \tag{2.26}$$

(3) *The intersection form of the Frobenius structure coincides with the metric* $(\ ,\ )^{\sim}$ *on* $\mathcal{M}$.

Observe that the charges $q_1, \dots, q_l$ are

$$q_j = \frac{(\omega_k - \omega_j, \omega_k)}{(\omega_k, \omega_k)}, \quad j = 1, \dots, l, \quad q_{l+1} = d = 1.$$

COROLLARY 2.7. *The monodromy group of* $\mathcal{M}(R,k)$ *is isomorphic to the group* $\widetilde{W}^{(k)}(R)$.

The proof of the theorem will be based on the following lemmas (cf. [D, pp. 273–275]).

LEMMA 2.5. *In the coordinates* $t^1, \dots, t^{l+1}$

$$g^{\alpha,l+1} = \frac{d_\alpha}{d_k}t^\alpha, \quad \alpha = 1, \dots, l, \quad g^{l+1,l+1} = \frac{1}{d_k},$$

$$\Gamma_\beta^{l+1,\alpha} = \frac{d_\alpha}{d_k}\delta_\beta^\alpha, \quad 1 \leqslant \alpha, \beta \leqslant l+1.$$

The proof of this lemma is straightforward using (2.5b), (2.5c), (2.7), $t^{l+1} = y^{l+1}$ and the quasi-homogeneity of $t^1, \dots, t^l$.

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LEMMA 2.6. *There exists a unique weighted homogeneous polynomial* $G = G(t^1, \dots, t^{k-1}, t^{k+1}, \dots, t^l, \mathrm{e}^{t^{l+1}})$ *of the degree* $2d_k$ *such that the function*

$$F = \tfrac{1}{2}(t^k)^2 t^{l+1} + \tfrac{1}{2} t^k \sum_{\alpha,\beta \neq k} \eta_{\alpha\beta} t^\alpha t^\beta + G$$

satisfies the equations

$$(dt^\alpha, dt^\beta)^\sim = \mathcal{L}_E F^{\alpha\beta}. \tag{2.27}$$

Proof. Let $\Gamma_\gamma^{\alpha\beta}(t)$ be the coefficients of the Levi–Civita connection for the metric $(\ ,\ )^\sim$ in the coordinates $t^1, \dots, t^{l+1}$. We use now the theory of flat pencils of metrics (see [D, Appendix D]). According to Proposition D.1 of [D] we can represent these functions as

$$\Gamma_\gamma^{\alpha\beta}(t) = \eta^{\alpha\varepsilon} \partial_\varepsilon \, \partial_\gamma f^\beta(t), \tag{2.28}$$

for some functions $f^\beta(t)$. From the weighted homogeneity of $\Gamma_\gamma^{\alpha\beta}(t)$ and Corollary 2.6 we obtain that

$$\partial_\alpha \partial_\gamma \left(\mathcal{L}_E f^\beta - \frac{d_k + d_\beta}{d_k} f^\beta \right) = 0,$$

for any α, β. So

$$\mathcal{L}_E f^\beta(t) = \frac{(d_\beta + d_k)}{d_k} f^\beta(t) + A_\sigma^\beta t^\sigma + B^\beta, \tag{2.29}$$

for some constants A_σ^β, B^β. Doing a transformation

$$f^\beta(t) \mapsto \tilde{f}^\beta(t) = f^\beta(t) + R_\lambda^\beta t^\lambda + Q^\beta,$$

we can kill all the coefficients A_σ^β, B^β in (2.29) except A_k^{l+1}. Indeed, after the transformation we obtain

$$\begin{aligned} \mathcal{L}_E \tilde{f}^\beta(t) \ = \ & \frac{(d_\beta + d_k)}{d_k} \tilde{f}^\beta(t) + \sum_{\gamma=1}^{l} \left[R_\gamma^\beta \frac{d_\gamma - d_k - d_\beta}{d_k} + A_\gamma^\beta \right] t^\gamma \\ & + \frac{1}{d_k} R_{l+1}^\beta + B^\beta - \frac{d_k + d_\beta}{d_k} Q^\beta \\ & + \left[A_{l+1}^\beta - \frac{d_k + d_\beta}{d_k} R_{l+1}^\beta \right] t^{l+1}. \end{aligned}$$

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The function $\tilde{f}^\beta(t)$ does still satisfy (2.28). Choosing

$$R^\beta_{l+1} = \frac{d_k}{d_k + d_\beta} A^\beta_{l+1},$$

$$Q^\beta = \frac{d_k}{d_k + d_\beta} \left[B^\beta + \frac{1}{d_k} R^\beta_{l+1} \right],$$

we kill the constant term in the right-hand side of (2.29) and the term linear in t^{l+1}.

To kill other linear terms we are to put

$$R^\beta_\gamma = \frac{d_k}{d_\beta + d_{\gamma^*}} A^\beta_\gamma,$$

where γ^* is the index dual to γ in the sense of duality (1.27). We can do this unless $d_\beta = d_{\gamma^*} = 0$. The last equation holds only for $\beta = \gamma^* = l + 1$, i.e. for $\beta = l + 1$, $\gamma = k$. So, we can kill all the linear terms but A^{l+1}_k in (2.29).

Thus for $\beta \neq l + 1$ the polynomials $f^\beta(t)$ can be assumed to be homogeneous of the degree $d_\beta + d_k$.

We show now that for $1 \leqslant \beta \leqslant l$ the functions f^β are polynomials in $t^1, \dots, t^l$, $\exp t^{l+1}$. We already know that this is true for the Christoffel coefficients $\Gamma^{\alpha\beta}_\gamma$. Let us denote

$$\eta_{\alpha\varepsilon} \Gamma^{\varepsilon\beta}_{l+1}(t) = \sum_{m=0}^{N} C^\beta_{\alpha,m} \exp mt^{l+1} \equiv \partial_\alpha \, \partial_{l+1} f^\beta(t),$$

where the coefficients $C^\beta_{\alpha,m}$ are polynomials in $t^1, \dots, t^l$ and N is certain positive integer. From compatibility

$$\partial_{l+1}(\partial_\alpha \, \partial_{l+1} f^\beta) = \partial_\alpha(\partial^2_{l+1} f^\beta),$$

we obtain that

$$\partial_\alpha C^\beta_{l+1,0} = 0, \quad \alpha = 1, \dots, l.$$

So $C^\beta_{l+1,0}$ is a constant. But $\partial^2_{l+1} f^\beta$ must be a weighted homogeneous polynomial of the positive degree $d_k + d_\beta$. Hence $C^\beta_{l+1,0} = 0$ and we obtain

$$f^\beta = \sum_{m=1}^{N} \frac{1}{m^2} C^\beta_{l+1,m} \exp mt^{l+1} + t^{l+1} D^\beta + H^\beta,$$

for some new polynomials $D^\beta = D^\beta(t^1, \dots, t^l)$ and $H^\beta = H^\beta(t^1, \dots, t^l)$. Since the derivatives $\partial_\alpha \, \partial_\gamma f^\beta$ must not contain terms linear in t^{l+1}, the polynomial D^β is at most linear in $t^1, \dots, t^l$. Using homogeneity of f^β we conclude that $D^\beta = 0$.

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The coefficient $\Gamma_\gamma^{\alpha\beta}(t)$ must also satisfy the conditions [D]

$$g^{\alpha\sigma}\Gamma_\sigma^{\beta\gamma} = g^{\beta\sigma}\Gamma_\sigma^{\alpha\gamma}. \tag{2.30}$$

For $\alpha = l+1$ because of (2.28), (2.30) and Lemma 2.5 we obtain

$$\mathcal{L}_E(\eta^{\beta\varepsilon}\partial_\varepsilon f^\gamma) = \frac{d_\gamma}{d_k} g^{\beta\gamma}.$$

Because of $\deg f^\gamma = d_\gamma + d_k$ we have $\deg(\eta^{\beta\varepsilon}\partial_\varepsilon f^\gamma) = d_\beta + d_\gamma$ for $\gamma \neq l+1$, so

$$(d_\gamma + d_\beta)\eta^{\beta\varepsilon}\partial_\varepsilon f^\gamma = d_\gamma g^{\beta\gamma}, \quad \gamma \neq l+1. \tag{2.31}$$

We introduce functions F^γ for $\gamma \neq l+1$ putting

$$F^\gamma = \frac{d_k}{d_\gamma} f^\gamma.$$

From (2.31) we obtain the equation

$$\eta^{\beta\varepsilon}\partial_\varepsilon F^\gamma = \eta^{\gamma\varepsilon}\partial_\varepsilon F^\beta, \quad 1 \leqslant \gamma, \beta \leqslant l. \tag{2.32}$$

From (2.32) it follows that a function $F(t)$ exists such that

$$F^\gamma = \eta^{\gamma\varepsilon}\partial_\epsilon F, \quad 1 \leqslant \gamma \leqslant l. \tag{2.33}$$

The dependence of F on t^k is not determined from (2.33). However, putting $\beta = l+1$ in (2.31) we obtain

$$\partial_k F^\gamma = t^\gamma, \quad 1 \leqslant \gamma \leqslant l, \tag{2.34}$$

from (2.33), (2.34) and Corollary 2.6 we obtain

$$\partial_{l+1}(\partial_k F) = t^k,$$

$$\partial_\gamma(\partial_k F) = \sum_{\alpha=1}^{l} \eta_{\gamma\alpha} t^\alpha, \quad \gamma \neq k, l+1.$$

Hence we have

$$\partial_k F = t^k t^{l+1} + \frac{1}{2} \sum_{\alpha,\beta \neq k, l+1} \eta_{\alpha\beta} t^\alpha t^\beta + g(t^k),$$

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for some function g. Shifting $F \mapsto F + \int g(t^k)\,\mathrm{d}t^k$ we can kill this function, and the equations in (2.33) still hold true due to $\eta^{ik} = \delta_{i,l+1}$. We obtain the representation

$$F = \frac{1}{2}(t^k)^2 t^{l+1} + \tfrac{1}{2} t^k \sum_{\alpha,\beta\neq k,l+1} \eta_{\alpha\beta} t^\alpha t^\beta + G, \tag{2.35}$$

with some G independent on t^k.

From the definition (2.33) of F and the weighted homogeneity of $f^\gamma, \gamma \neq l+1$ we obtain that

$$\mathcal{L}_E F(t) = 2F(t) + a(t^k),$$

for some unknown function a. Using the duality condition (1.27) and Corollary 2.6 we obtain

$$\mathcal{L}_E F(t) = \frac{1}{2d_k}(t^k)^2 + (t^k)^2 t^{l+1} + t^k \sum_{\alpha,\beta\neq k,l+1} \eta_{\alpha\beta} t^\alpha t^\beta + \mathcal{L}_E G(t),$$

or, equivalently

$$\mathcal{L}_E G(t) = 2G(t) + a(t^k) - \frac{1}{2d_k}(t^k)^2.$$

But $\mathcal{L}_E G(t)$ does not depend on t^k. We obtain

$$a(t^k) = \frac{1}{2d_k}(t^k)^2 + c,$$

for some constant c. Killing the constant by a shift, we obtain that $G(t^1, \dots, \hat{t}^k, \dots, t^{l+1})$ is a weighted homogeneous function of the degree $2d_k$. The above conditions determine this function uniquely. Clearly G is a polynomial in $t^1, \dots, \hat{t}^k, \dots, t^l$, $\exp t^{l+1}$ (it was obtained by integrating polynomials).

Substituting $F(t)$ into (2.31) we obtain (2.27) for $1 \leqslant \alpha \leqslant l$. Finally, for $\alpha = \beta = l+1$ the Equation (2.27) reads

$$\frac{1}{d_k} = \mathcal{L}_E \frac{\partial^2 F}{\partial t^k\, \partial t^k}.$$

This follows immediately from the explicit form of F. Lemma is proved. □

LEMMA 2.7. *The functions*

$$c_\gamma^{\alpha\beta}(t) = \eta^{\alpha\alpha'}\eta^{\beta\beta'} \frac{\partial^3 F(t)}{\partial t^{\alpha'}\, \partial t^{\beta'}\, \partial t^\gamma}$$

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are weighted homogeneous polynomials in $t^1, \dots, t^l, \mathrm{e}^{t^{l+1}}$ *of the degrees* $d_\alpha + d_\beta - d_\gamma$. *They satisfy the associativity equations*

$$c_\sigma^{\alpha\beta} c_\delta^{\sigma\gamma} = c_\delta^{\alpha\sigma} c_\sigma^{\beta\gamma}.$$

Proof. Weighted homogeneity of $c_\gamma^{\alpha\beta}$ for α or $\beta \neq l+1$ follows from those of the functions $\Gamma_\gamma^{\alpha\beta}$ since

$$\Gamma_\gamma^{\alpha\beta} = \frac{d_\beta}{d_k} c_\gamma^{\alpha\beta}, \tag{2.36}$$

which follows from (2.28). Due to (2.35) we also have

$$c_\gamma^{\alpha,l+1} = \delta_\gamma^\alpha. \tag{2.37}$$

This is also a weighted homogeneous function.

To prove associativity we use again the theory of linear pencils of the flat metrics $g^{\alpha\beta}, \eta^{\alpha\beta}$. Using [D, Eqn. (D.2)] we obtain

$$\Gamma_\sigma^{\alpha\beta} \Gamma_\delta^{\sigma\gamma} = \Gamma_\sigma^{\alpha\gamma} \Gamma_\delta^{\sigma\beta}. \tag{2.38}$$

Substitute (2.36), (2.37) into (2.38) we obtain

$$c_\sigma^{\alpha\beta} c_\delta^{\sigma\gamma} = c_\sigma^{\alpha\gamma} c_\delta^{\sigma\beta}.$$

Due to commutativity of the multiplication we obtain needed associativity. Lemma is proved. □

Proof of Theorem 2.1. From Lemma 2.5–2.7 we know that we only need to verify that $e = \partial/\partial t^k$ is the unity of the algebra and that

$$\mathcal{L}_E e = -e.$$

This is very simple. Theorem is proved. □

Remark. Any orthogonal map $T: V \to V$ preserving the set of simple roots defines an isomorphism of Frobenius manifolds

$$\mathcal{M}(R, k) \to \mathcal{M}(R, k'),$$

where

$$T(\alpha_k) = \alpha_{k'}.$$

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Particularly, for the root system of the type A_l we obtain an isomorphism

$$\mathcal{M}(A_l, k) \simeq \mathcal{M}(A_l, l-k+1)$$

corresponding to the reflection of the Dynkin graph w.r.t. the center.

We now give some examples to illustrate our above construction. For convenience, instead of $t^1, \dots, t^{l+1}$ we will denote the flat coordinates of the metric (η^{ij}) by $t_1, \dots, t_{l+1}$, and we will also denote $\partial_i = \partial/\partial t_i$ in these examples.

EXAMPLE 2.1. For the root system of the type A_1 the affine Weyl group acts on x_1-line by transformations

$$x_1 \mapsto \pm x_1 + m,$$

for an integer m. Our extension $\tilde{W}(A_1)$ consists of transformations of (x_1, x_2)-plane of the form

$$(x_1, x_2) \mapsto (\pm x_1 + m + \tfrac{1}{2}n, x_2 - n),$$

for arbitrary integers m, n. Basic invariants of this group bounded along the lines

$$(x_1, x_2) = (x_1^0 - \tfrac{1}{2}i\tau, x_2^0 + i\tau), \quad \tau \to +\infty$$

are

$$t_1 = 2\,\mathrm{e}^{\pi i x_2} \cos 2\pi x_1 \quad \text{and} \quad \mathrm{e}^{2\pi i x_2}.$$

The extended invariant metric on the dual space has the matrix

$$((dx_i, dx_j)^{\sim}) = \frac{1}{8\pi^2} \begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix}.$$

In the coordinates t_1, $t_2 = 2\pi i x_2$ this metric has the matrix

$$(g^{\alpha\beta}) = \begin{pmatrix} 2\,\mathrm{e}^{t_2} & t_1 \\ t_1 & 2 \end{pmatrix}.$$

So the Frobenius structure is determined by the function

$$F = \tfrac{1}{2}t_1^2 t_2 + \mathrm{e}^{t_2}.$$

Up to normalization this is the free energy of $\mathbf{CP}^1$ topological sigma-model [D].

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EXAMPLE 2.2. Let R be the root system A_2, we take $k = 1$, then $d_1 = \frac{2}{3}, d_2 = \frac{1}{3}$, and

$$y^1 = \mathrm{e}^{(4/3)\pi i x_3}(\mathrm{e}^{2\pi i x_1} + \mathrm{e}^{-2\pi i x_2} + \mathrm{e}^{2\pi i(x_2 - x_1)});$$

$$y^2 = \mathrm{e}^{(2/3)\pi i x_3}(\mathrm{e}^{2\pi i x_2} + \mathrm{e}^{-2\pi i x_1} + \mathrm{e}^{2\pi i(x_1 - x_2)});$$

$$y^3 = 2\pi i x_3.$$

The metric $(\ ,\ )^{\sim}$ has the form

$$((dx_i, dx_j)^{\sim}) = \frac{1}{12\pi^2}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -\frac{9}{2} \end{pmatrix}.$$

The flat coordinates $t_1 = y^1, t_2 = y^2, t_3 = y^3$, the intersection form is given by

$$(g^{ij}) = \begin{pmatrix} 2t_2\,\mathrm{e}^{t_3} & 3\,\mathrm{e}^{t_3} & t_1 \\ 3\,\mathrm{e}^{t_3} & 2t_1 - \frac{1}{2}t_2^2 & \frac{1}{2}t_2 \\ t_1 & \frac{1}{2}t_2 & \frac{3}{2} \end{pmatrix}.$$

The free energy

$$F = \tfrac{1}{2}t_1^2 t_3 + \tfrac{1}{4}t_1 t_2^2 + t_2\,\mathrm{e}^{t_3} - \tfrac{1}{96}t_2^4$$

and the Euler vector field reads

$$E = t_1\partial_1 + \tfrac{1}{2}t_2\partial_2 + \tfrac{3}{2}\partial_3.$$

EXAMPLE 2.3. Let R be the root system C_2, then $k = 2, d_1 = 1, d_2 = 2$, and

$$y^1 = \mathrm{e}^{2\pi i x_3}(\mathrm{e}^{2\pi i x_1} + \mathrm{e}^{-2\pi i x_1} + \mathrm{e}^{2\pi i(x_2 - x_1)} + \mathrm{e}^{-2\pi i(x_2 - x_1)}).$$

$$y^2 = \mathrm{e}^{4\pi i x_3}(\mathrm{e}^{2\pi i x_2} + \mathrm{e}^{-2\pi i x_2} + \mathrm{e}^{2\pi i(2x_1 - x_2)} + \mathrm{e}^{-2\pi i(2x_1 - x_2)}),$$

$$y^3 = 2\pi i x_3.$$

The metric $(\ ,\ )^{\sim}$ has the form

$$((dx_i, dx_j)^{\sim}) = \frac{1}{4\pi^2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}.$$

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The flat coordinates $t_1 = y^1, t_2 = y^2 + 2\,\mathrm{e}^{2y^3}, t_3 = y^3$, the intersection form is given by

$$(g^{ij}) = \begin{pmatrix} 2t_2 - \frac{1}{2}t_1^2 + 4\,\mathrm{e}^{2t_3} & 6t_1\,\mathrm{e}^{2t_3} & \frac{1}{2}t_1 \\ 6t_1\,\mathrm{e}^{2t_3} & 8\,\mathrm{e}^{4t_3} + 4t_1^2\,\mathrm{e}^{2t_3} & t_2 \\ \frac{1}{2}t_1 & t_2 & \frac{1}{2}. \end{pmatrix}.$$

The free energy

$$F = \tfrac{1}{4}t_1^2t_2 + \tfrac{1}{2}t_2^2t_3 - \tfrac{1}{96}t_1^4 + \tfrac{1}{2}t_1^2\,\mathrm{e}^{2t_3} + \tfrac{1}{4}\,\mathrm{e}^{4t_3}$$

and the Euler vector field reads

$$E = \tfrac{1}{2}t_1\partial_1 + t_2\partial_2 + \tfrac{1}{2}\partial_3.$$

Remark. The root system B_2 gives a Frobenius structure which is isomorphic to the one given by the root system C_2.

EXAMPLE 2.4. Let R be the root system G_2, then $k = 2, d_1 = 3, d_2 = 6$, and

$$\begin{aligned} y^1 &= \mathrm{e}^{6\pi i x_3}(\mathrm{e}^{2\pi i x_1} + \mathrm{e}^{-2\pi i x_1} + \mathrm{e}^{2\pi i(2x_1 - x_2)} + \mathrm{e}^{-2\pi i(2x_1 - x_2)} \\ &\quad + \mathrm{e}^{2\pi i(x_1 - x_2)} + \mathrm{e}^{-2\pi i(x_1 - x_2)}). \\ y^2 &= \mathrm{e}^{12\pi i x_3}(\mathrm{e}^{2\pi i x_2} + \mathrm{e}^{-2\pi i x_2} + \mathrm{e}^{2\pi i(3x_1 - 2x_2)} + \mathrm{e}^{-2\pi i(3x_1 - 2x_2)} \\ &\quad + \mathrm{e}^{2\pi i(x_2 - 3x_1)} + \mathrm{e}^{-2\pi i(x_2 - 3x_1)}), \\ y^3 &= 2\pi i x_3. \end{aligned}$$

The metric $(\ ,\ )\tilde{}$ has the form

$$((dx_i, dx_j)\tilde{}) = \frac{1}{4\pi^2}\begin{pmatrix} 2 & 3 & 0 \\ 3 & 6 & 0 \\ 0 & 0 & -\frac{1}{6} \end{pmatrix}.$$

The flat coordinates $t_1 = y^1 + 2\,\mathrm{e}^{3y^3}, t_2 = y^2 + 3y^1\,\mathrm{e}^{3y^3} + 6\,\mathrm{e}^{6y^3}, t_3 = y^3$, the intersection form is given by

$$(g^{ij}) = \begin{pmatrix} 2t_2 - \frac{1}{2}t_1^2 + 8t_1\,\mathrm{e}^{3t_3} + 4\,\mathrm{e}^{6t_3} & 9(2t_1\,\mathrm{e}^{6t_3} + t_1^2\,\mathrm{e}^{3t_3}) & \frac{1}{2}t_1 \\ 9(2t_1\,\mathrm{e}^{6t_3} + t_1^2\,\mathrm{e}^{3t_3}) & 6(6t_1^2\,\mathrm{e}^{6t_3} + t_1^3\,\mathrm{e}^{3t_3} + 4\,\mathrm{e}^{12t_3}) & t_2 \\ \frac{1}{2}t_1 & t_2 & \frac{1}{6}. \end{pmatrix}.$$

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The free energy

$$F = \tfrac{1}{4}t_1^2t_2 + \tfrac{1}{2}t_2^2t_3 - \tfrac{1}{96}t_1^4 + \tfrac{1}{3}t_1^3\,\mathrm{e}^{3t_3} + \tfrac{1}{2}t_1^2\,\mathrm{e}^{6t_3} + \tfrac{1}{12}\,\mathrm{e}^{12t_3}$$

and the Euler vector field

$$E = \tfrac{1}{2}t_1\partial_1 + t_2\partial_2 + \tfrac{1}{6}\partial_3.$$

Remark. All the above examples was found in [D] (although the relation of Example 2.4 to an extension of the affine Weyl group of the type G_2 was not proved in [D]). It is important to notice that these are all n-dimensional Frobenius manifolds with $n \leqslant 3$ being polynomial in $t_1, \dots, t_{n-1}, \exp t_n$ with $\deg \exp t_n > 0$ (see [D], Appendix A). It would be natural to conjecture that our construction gives all such Frobenius manifolds (with semisimplicity condition [D] added) for any n.

We proceed now to the list of all four-dimensional Frobenius manifolds given by our construction.

EXAMPLE 2.5. Let R be the root system A_3, take $k = 1$, then $d_1 = \frac{3}{4}, d_2 = \frac{1}{2}, d_3 = \frac{1}{4}$, and

$$\begin{aligned}
y^1 &= \mathrm{e}^{(3/2)\pi i x_4}\big(\mathrm{e}^{2\pi i x_1} + \mathrm{e}^{2\pi i(x_2-x_1)} + \mathrm{e}^{2\pi i(x_3-x_2)} + \mathrm{e}^{-2\pi i x_3}\big);\\
y^2 &= \mathrm{e}^{\pi i x_4}\big(\mathrm{e}^{2\pi i x_2} + \mathrm{e}^{-2\pi i x_2} + \mathrm{e}^{2\pi i(x_1-x_3)} + \mathrm{e}^{-2\pi i(x_1-x_3)}\\
&\quad + \mathrm{e}^{2\pi i(x_1+x_3-x_2)} + \mathrm{e}^{-2\pi i(x_1+x_3-x_2)}\big);\\
y^3 &= \mathrm{e}^{(1/2)\pi i x_4}\big(\mathrm{e}^{2\pi i x_3} + \mathrm{e}^{2\pi i(x_1-x_2)} + \mathrm{e}^{2\pi i(x_2-x_3)} + \mathrm{e}^{-2\pi i x_1}\big);\\
y^4 &= 2\pi i x_4.
\end{aligned}$$

The metric $(\ ,\ )\tilde{}$ has the form

$$((dx_i, dx_j)\tilde{}) = \frac{1}{16\pi^2}\begin{pmatrix} 3 & 2 & 1 & 0\\ 2 & 4 & 2 & 0\\ 1 & 2 & 3 & 0\\ 0 & 0 & 0 & -\frac{16}{3}\end{pmatrix}.$$

The flat coordinates $t_1 = y^1, t_2 = y^2 - \frac{1}{6}(y^3)^2, t_3 = y^3, t_4 = y^4$, the intersection form is given by

$$(g^{ij}) = \begin{pmatrix} 2(t_2 + \frac{1}{6}t_3^2)\,\mathrm{e}^{t_4} & \frac{5}{3}t_3\,\mathrm{e}^{t_4} & 4\,\mathrm{e}^{t_4} & t_1\\ \frac{5}{3}t_3\,\mathrm{e}^{t_4} & \frac{2}{9}t_2t_3^2 - \frac{2}{3}t_2^2 - \frac{1}{54}t_3^4 + 4\,\mathrm{e}^{t_4} & \frac{1}{18}t_3^3 - t_2t_3 + 3t_1 & \frac{2}{3}t_2\\ 4\,\mathrm{e}^{t_4} & \frac{1}{18}t_3^3 - t_2t_3 + 3t_1 & 2t_2 - \frac{1}{3}t_3^2 & \frac{1}{3}t_3\\ t_1 & \frac{2}{3}t_2 & \frac{1}{3}t_3 & \frac{4}{3}\end{pmatrix}.$$

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The free energy

$$F = \tfrac{1}{2}t_1^2t_4 + \tfrac{1}{3}t_1t_2t_3 + \tfrac{1}{18}t_2^3 - \tfrac{1}{36}t_2^2t_3^2 + \tfrac{1}{648}t_2t_3^4 - \tfrac{1}{19440}t_3^6 + (t_2 + \tfrac{1}{6}t_3^2)\,\mathrm{e}^{t_4}$$

and the Euler vector field reads

$$E = t_1\partial_1 + \tfrac{2}{3}t_2\partial_2 + \tfrac{1}{3}t_3\partial_3 + \tfrac{4}{3}\partial_4.$$

EXAMPLE 2.6. Let R be the root system A_3, take $k = 2$, then $d_1 = \frac{1}{2}$, $d_2 = 1, d_3 = \frac{1}{2}$, and

$$\begin{aligned}
y^1 &= \mathrm{e}^{\pi i x_4}(\mathrm{e}^{2\pi i x_1} + \mathrm{e}^{2\pi i(x_2-x_1)} + \mathrm{e}^{2\pi i(x_3-x_2)} + \mathrm{e}^{-2\pi i x_3});\\
y^2 &= \mathrm{e}^{2\pi i x_4}(\mathrm{e}^{2\pi i x_2} + \mathrm{e}^{-2\pi i x_2} + \mathrm{e}^{2\pi i(x_1-x_3)} + \mathrm{e}^{-2\pi i(x_1-x_3)}\\
&\quad + \mathrm{e}^{2\pi i(x_1+x_3-x_2)} + \mathrm{e}^{-2\pi i(x_1+x_3-x_2)});\\
y^3 &= \mathrm{e}^{\pi i x_4}(\mathrm{e}^{2\pi i x_3} + \mathrm{e}^{2\pi i(x_1-x_2)} + \mathrm{e}^{2\pi i(x_2-x_3)} + \mathrm{e}^{-2\pi i x_1});\\
y^4 &= 2\pi i x_4.
\end{aligned}$$

The metric $(\ ,\ )^{\sim}$ has the form

$$((dx_i, dx_j)^{\sim}) = \frac{1}{16\pi^2}\begin{pmatrix} 3 & 2 & 1 & 0\\ 2 & 4 & 2 & 0\\ 1 & 2 & 3 & 0\\ 0 & 0 & 0 & -4 \end{pmatrix}.$$

The flat coordinates $t_1 = y^1, t_2 = y^2, t_3 = y^3, t_4 = y^4$, the intersection form is given by

$$(g^{ij}) = \begin{pmatrix} 2t_2 - \frac{1}{2}t_1^2 & 3t_3e^{t_4} & 4e^{t_4} & \frac{1}{2}t_1\\ 3t_3\,\mathrm{e}^{t_4} & 2t_1t_3\,\mathrm{e}^{t_4} + 4\,\mathrm{e}^{2t_4} & 3t_1\,\mathrm{e}^{t_4} & t_2\\ 4\,\mathrm{e}^{t_4} & 3t_1\,\mathrm{e}^{t_4} & 2t_2 - \frac{1}{2}t_3^2 & \frac{1}{2}t_3\\ \frac{1}{2}t_1 & t_2 & \frac{1}{2}t_3 & 1 \end{pmatrix}.$$

The free energy

$$F = \tfrac{1}{4}t_1^2t_2 + \tfrac{1}{2}t_2^2t_4 + \tfrac{1}{4}t_2t_3^2 - \tfrac{1}{96}t_1^4 - \tfrac{1}{96}t_3^4 + t_1t_3\,\mathrm{e}^{t_4} + \tfrac{1}{2}\,\mathrm{e}^{2t_4}$$

and the Euler vector field is given by

$$E = \tfrac{1}{2}t_1\partial_1 + t_2\partial_2 + \tfrac{1}{2}t_3\partial_3 + \partial_4.$$

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EXAMPLE 2.7. Let R be the root system B_3, then $k = 2, d_1 = 1, d_2 = 2, d_3 = 1$, and

$$\begin{aligned}
y^1 &= 2\,\mathrm{e}^{2\pi i x_4}(\cos(x_1) + \cos(x_2 - x_1) + \cos(2x_3 - x_2));\\
y^2 &= 4e^{4\pi i x_4}(\cos(x_1)\cos(x_2 - x_1) + \cos(x_1)\cos(2x_3 - x_2)\\
&\quad + \cos(x_2 - x_1)\cos(2x_3 - x_2));\\
y^3 &= 8\,\mathrm{e}^{2\pi i x_4}\left(\cos\left(\frac{x_1}{2}\right)\cos\left(\frac{x_2 - x_1}{2}\right)\cos\left(\frac{2x_3 - x_2}{2}\right)\right);\\
y^4 &= 2\pi i x_4.
\end{aligned}$$

The metric $(\ ,\ )\tilde{}$ has the form

$$((dx_i, dx_j)\tilde{}) = \frac{1}{4\pi^2}\begin{pmatrix} 1 & 1 & \frac{1}{2} & 0\\ 1 & 2 & 1 & 0\\ \frac{1}{2} & 1 & \frac{3}{4} & 0\\ 0 & 0 & 0 & -\frac{1}{2}\end{pmatrix}.$$

The flat coordinates $t_1 = y^1 + 2\,\mathrm{e}^{y^4}, t_2 = y^2 + 2y_1\,\mathrm{e}^{y^4} + 6\,\mathrm{e}^{2y^4}, t_3 = y^3, t_4 = y^4$, the elements of the intersection form are given by

$$\begin{aligned}
g^{11} &= 2t_2 - \tfrac{1}{2}t_1^2 + 4\,\mathrm{e}^{2t_4},\\
g^{12} &= 3t_3^2\,\mathrm{e}^{t_4} + 6t_1\,\mathrm{e}^{2t_4},\\
g^{13} &= 4t_3\,\mathrm{e}^{t_4},\\
g^{14} &= \tfrac{1}{2}t_1,\\
g^{22} &= 2t_1t_3^2\,\mathrm{e}^{t_4} + 4t_1^2\,\mathrm{e}^{2t_4} + 8t_3^2\,\mathrm{e}^{2t_4} + 8\,\mathrm{e}^{4t_4},\\
g^{23} &= 3t_1t_3\,\mathrm{e}^{t_4} + 6t_3\,\mathrm{e}^{2t_4},\\
g^{24} &= t_2,\\
g^{33} &= t_2 - \tfrac{1}{4}t_3^2 + 2t_1\,\mathrm{e}^{t_4} + 2\,\mathrm{e}^{2t_4},\\
g^{34} &= \tfrac{1}{2}t_3,\\
g^{44} &= \tfrac{1}{2}.
\end{aligned}$$

The free energy

$$\begin{aligned}
F &= \tfrac{1}{4}t_2t_1^2 + \tfrac{1}{2}t_2t_3^2 + \tfrac{1}{2}t_2^2t_4 - \tfrac{1}{96}t_1^4 - \tfrac{1}{48}t_3^4\\
&\quad + t_1t_3^2\,\mathrm{e}^{t_4} + \tfrac{1}{2}t_1^2\,\mathrm{e}^{2t_4} + t_3^2\,\mathrm{e}^{2t_4} + \tfrac{1}{4}\,\mathrm{e}^{4t_4}
\end{aligned}$$

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and the Euler vector field is given by

$$E = \tfrac{1}{2}t_1\partial_1 + t_2\partial_2 + \tfrac{1}{2}t_3\partial_3 + \tfrac{1}{2}\partial_4.$$

EXAMPLE 2.8. Let R be the root system C_3, then $k = 3, d_1 = 1, d_2 = 2, d_3 = 3$, and

$$\begin{aligned}
y^1 &= \mathrm{e}^{2\pi i x_4}(\xi_1 + \xi_2 + \xi_3);\\
y^2 &= \mathrm{e}^{4\pi i x_4}(\xi_1\xi_2 + \xi_1\xi_3 + \xi_2\xi_3);\\
y^3 &= \mathrm{e}^{6\pi i x_4}(\xi_1\xi_2\xi_3);\\
y^4 &= 2\pi i x_4,
\end{aligned}$$

where $\xi_1 = \mathrm{e}^{2\pi i x_1} + \mathrm{e}^{-2\pi i x_1}, \xi_2 = \mathrm{e}^{2\pi i(x_2-x_1)} + \mathrm{e}^{-2\pi i(x_2-x_1)}, \xi_3 = \mathrm{e}^{2\pi i(x_3-x_2)} + \mathrm{e}^{-2\pi i(x_3-x_2)}$. The metric $(\ ,\ )^{\sim}$ has the form

$$((dx_i, dx_j)^{\sim}) = \frac{1}{4\pi^2}\begin{pmatrix} 1 & 1 & 1 & 0\\ 1 & 2 & 2 & 0\\ 1 & 2 & 3 & 0\\ 0 & 0 & 0 & -\frac{1}{3}\end{pmatrix}.$$

The flat coordinates $t_1 = y^1, t_2 = y^2 - \frac{1}{6}(y^1)^2 + 3\,\mathrm{e}^{2y^4}, t_3 = y^3 + 2y^1\,\mathrm{e}^{2y^4}, t_4 = y^4$, the elements of the intersection form are given by

$$\begin{aligned}
g^{11} &= -\tfrac{1}{3}t_1^2 + 2t_2 + 6\,\mathrm{e}^{2t_4},\\
g^{12} &= 3t_3 + \tfrac{1}{18}t_1^3 - t_1t_2 + 3t_1\,\mathrm{e}^{2t_4},\\
g^{13} &= \tfrac{4}{3}t_1^2\,\mathrm{e}^{2t_4} + 8t_2\,\mathrm{e}^{2t_4},\\
g^{14} &= \tfrac{1}{3}t_1,\\
g^{22} &= -\tfrac{1}{54}t_1^4 + \tfrac{2}{9}t_1^2t_2 - \tfrac{2}{3}t_2^2 + 2t_1^2\,\mathrm{e}^{2t_4} + 4t_2\,\mathrm{e}^{2t_4} + 6\,\mathrm{e}^{4t_4},\\
g^{23} &= \tfrac{5}{9}t_1^3\,\mathrm{e}^{2t_4} + \tfrac{10}{3}t_1t_2\,\mathrm{e}^{2t_4} + 10t_1\,\mathrm{e}^{4t_4},\\
g^{24} &= \tfrac{2}{3}t_2,\\
g^{33} &= \tfrac{1}{9}t_1^4\,\mathrm{e}^{2t_4} + \tfrac{4}{3}t_1^2t_2\,\mathrm{e}^{2t_4} + 4t_2^2\,\mathrm{e}^{2t_4} + 8t_1^2\,\mathrm{e}^{4t_4} + 12\,\mathrm{e}^{6t_4},\\
g^{34} &= t_3,\\
g^{44} &= \tfrac{1}{3}.
\end{aligned}$$

The free energy

$$\begin{aligned}
F &= \tfrac{1}{3}t_1t_2t_3 + \tfrac{1}{2}t_3^2t_4 + \tfrac{1}{18}t_2^3 - \tfrac{1}{36}t_1^2t_2^2 + \tfrac{1}{648}t_1^4t_2 - \tfrac{1}{19440}t_1^6\\
&\quad + \tfrac{1}{6}t_1^2t_2\,\mathrm{e}^{2t_4} + \tfrac{1}{72}t_1^4\,\mathrm{e}^{2t_4} + \tfrac{1}{2}t_2^2\,\mathrm{e}^{2t_4} + \tfrac{1}{4}t_1^2\,\mathrm{e}^{4t_4} + \tfrac{1}{6}\,\mathrm{e}^{6t_4}
\end{aligned}$$

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and the Euler vector field is given by

$$E = \tfrac{1}{3}t_1\partial_1 + \tfrac{2}{3}t_2\partial_2 + t_3\partial_3 + \tfrac{1}{3}\partial_4.$$

3. Groups $\widetilde{W}^{(k)}(A_l)$ and the spaces of trigonometric polynomials

A trigonometric polynomial of one variable of bidegree (k, m) is a function of the form

$$\lambda(\phi) = a_0\,\mathrm{e}^{ik\phi} + a_1\,\mathrm{e}^{i(k-1)\phi} + \cdots + a_k + a_{k+1}\,\mathrm{e}^{-i\phi} + \cdots + a_{k+m}\,\mathrm{e}^{-im\phi},$$
$$a_0, \ldots, a_{k+m} \in \mathbf{C}, \quad a_0 a_{k+m} \neq 0.$$

We will usually normalize $\lambda(\phi)$ by the condition $a_0 = 1$.

We denote $M_{k,m}$ the affine space of normalized trigonometric polynomials. Equivalently, $M_{k,m}$ coincides with certain covering of the space of rational functions with two poles of the orders k and m respectively. Geometry of these spaces was described in [D] as a part of general differential geometry of Hurwitz spaces of branched coverings over $\bar{\mathbf{C}}$ (our space $M_{k,m}$ in the notations of [D] is $\hat{M}_{0;k-1,m-1}$). Recall that, according to this paper, the space $M_{k,m}$ carries a natural structure of Frobenius manifold. The invariant inner product of two vectors ∂', ∂'' tangent to $M_{k,m}$ at a point $\lambda(\phi)$ equals

$$\langle \partial', \partial'' \rangle_\lambda = (-1)^{k+1} \sum_{|\lambda|<\infty} \operatorname*{res}_{\mathrm{d}\lambda=0} \frac{\partial'(\lambda(\phi)\,\mathrm{d}\phi)\ \partial''(\lambda(\phi)\,\mathrm{d}\phi)}{\mathrm{d}\lambda(\phi)}. \tag{3.1}$$

In this formula the derivatives $\partial'(\lambda(\phi)\,\mathrm{d}\phi), \partial''(\lambda(\phi)\,\mathrm{d}\phi)$ are to be calculated keeping ϕ fixed. The intersection form is given by the formula

$$(\partial', \partial'')_\lambda = -\sum_{|\lambda|<\infty} \operatorname*{res}_{\mathrm{d}\lambda=0} \frac{\partial'(\log\lambda(\phi)\,\mathrm{d}\phi)\ \partial''(\log\lambda(\phi)\,\mathrm{d}\phi)}{\mathrm{d}\log\lambda(\phi)}. \tag{3.2}$$

The discriminant $\Sigma \subset M_{k,m}$ consists of all functions $\lambda(\phi)$ which fail to have all simple roots of the equation $\lambda(\phi) = 0$. The intersection form is defined only outside Σ.

The formulae (3.1), (3.2) uniquely determine multiplication of tangent vectors on $M_{k,m}$ assuming that the Euler vector field E has the form

$$E = \sum_{j=1}^{k+m} \frac{j}{k} a_j \frac{\partial}{\partial a_j}. \tag{3.3}$$

For any three tangent vectors $\partial', \partial'', \partial'''$ to $M_{k,m}$ we obtain

$$\langle \partial' \cdot \partial'', \partial''' \rangle_\lambda = -\sum_{|\lambda|<\infty} \operatorname*{res}_{\mathrm{d}\lambda=0} \frac{\partial'(\lambda(\phi)\,\mathrm{d}\phi)\ \partial''(\lambda(\phi)\,\mathrm{d}\phi)\ \partial'''(\lambda(\phi)\,\mathrm{d}\phi)}{\mathrm{d}\lambda(\phi)\,\mathrm{d}\phi}. \tag{3.4}$$

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The canonical coordinates $u_1, \dots, u_{k+m}$ for this multiplication are the critical values of $\lambda(\phi)$

$$\frac{\partial}{\partial u_\alpha} \cdot \frac{\partial}{\partial u_\beta} = \delta_{\alpha\beta} \frac{\partial}{\partial u_\alpha}, \tag{3.5}$$

(see [D] for details).

In this section we will show that the space $M_{k,m}$ as a Frobenius manifold is isomorphic to the orbit space of our extended affine Weyl group $\widetilde{W}^{(k)}(A_l)$ for $l = k + m - 1$.

We start with factorizing the trigonometric polynomial

$$\lambda(\phi) = \mathrm{e}^{-im\phi} \prod_{\alpha=1}^{k+m} (\mathrm{e}^{i\phi} - \mathrm{e}^{i\phi_\alpha}). \tag{3.6}$$

LEMMA 3.1. *The map*

$$(x_1, \dots, x_{k+m}) \mapsto (\phi_1, \dots, \phi_{k+m}), \tag{3.7}$$

where

$$\begin{aligned}
\phi_1 &= 2\pi \left(x_1 + \frac{m}{k+m} x_{k+m} \right), \\
\phi_j &= 2\pi \left(x_j - x_{j-1} + \frac{m}{k+m} x_{k+m} \right), \\
\phi_{k+m} &= 2\pi \left(-x_{k+m-1} + \frac{m}{k+m} x_{k+m} \right), \\
j &= 2, \dots, k+m-1
\end{aligned} \tag{3.8}$$

establishes a diffeomorphism of the orbit space of the group $\widetilde{W}^{(k)}(A_{k+m-1})$ to the space of normalized trigonometric polynomials.

Proof. From the explicit formulae (1.6), (1.10), (2.1) and (3.8) it follows that in the coordinates $y^1, \dots, y^{k+m}$ the map (3.7) has the form

$$\begin{aligned}
a_1 &= -y^1, \\
&\dots \\
a_k &= (-1)^k y^k, \\
a_{k+1} &= (-1)^{k+1} y^{k+1} \exp(y^{k+m}), \\
&\dots \\
a_{k+m-1} &= (-1)^{k+m-1} y^{k+m-1} \exp((m-1) y^{k+m}), \\
a_{k+m} &= (-1)^{k+m} \exp(m y^{k+m}).
\end{aligned} \tag{3.9}$$

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Lemma is proved. □

According to this Lemma, our group $\widetilde{W}^{(k)}(A_l)$ describes monodromy of logarithms of the roots of a trigonometric polynomial along closed loops in the space of coefficients nonintersecting the discriminant Σ.

THEOREM 3.1. *The diffeomorphism* (3.7) *is an isomorphism of Frobenius manifolds.*

Proof. Since the Euler vector fields (2.26) and (3.3) coincide, it suffices to prove that the intersection form (3.2) coincides with the intersection form of the orbit spaces, and the metric (3.1) coincides with the metric (2.9).

Let's denote the roots of $\lambda'(\phi)$ by $\psi_j, 1 \leqslant j \leqslant k+m$. Then we have

$$\lambda'(\phi) = ki\,\mathrm{e}^{-im\phi}\prod_{\alpha=1}^{k+m}(\mathrm{e}^{i\phi}-\mathrm{e}^{i\psi_\alpha}). \tag{3.10}$$

We define $u_\alpha = \lambda(\psi_\alpha), 1 \leqslant \alpha \leqslant k+m$, then

$$\partial_{u_\alpha}\lambda(\phi)|_{\phi=\psi_\beta} = \delta_{\alpha\beta}. \tag{3.11}$$

By using (3.10), (3.11) and the Lagrange interpolation formula we obtain

$$\partial_{u_\alpha}\lambda(\phi) = \frac{i\,\mathrm{e}^{i\psi_\alpha}\lambda'(\phi)}{(\mathrm{e}^{i\phi}-\mathrm{e}^{i\psi_\alpha})\lambda''(\psi_\alpha)}. \tag{3.12}$$

Formulae (3.6) and (3.12) lead to

$$-\sum_{a=1}^{k+m} i\partial_{u_\alpha}\phi_a\,\mathrm{e}^{i\phi_a}\frac{\lambda(\phi)}{\mathrm{e}^{i\phi}-\mathrm{e}^{i\phi_a}} = \frac{i\,\mathrm{e}^{i\psi_\alpha}\lambda'(\phi)}{(\mathrm{e}^{i\phi}-\mathrm{e}^{i\psi_\alpha})\lambda''(\psi_\alpha)}. \tag{3.13}$$

By putting $\phi = \phi_\beta$ in the above formula we obtain

$$\partial_{u_\alpha}\phi_\beta = -\frac{i\,\mathrm{e}^{i\psi_\alpha}}{(\mathrm{e}^{i\phi_\beta}-\mathrm{e}^{i\psi_\alpha})\lambda''(\psi_\alpha)}. \tag{3.14}$$

We denote

$$\begin{aligned}&\theta_1 = x_1, \quad \theta_j = x_j - x_{j-1},\\ &j = 2,\dots,k+m-1, \quad \theta_{k+m} = x_{k+m}.\end{aligned} \tag{3.15}$$

It follows from (3.8) and (3.14) that

$$\begin{aligned}\frac{\partial\theta_\beta}{\partial u_\alpha} &= \frac{1}{2\pi i}\frac{\mathrm{e}^{i\psi_\alpha}}{(\mathrm{e}^{i\phi_\beta}-\mathrm{e}^{i\psi_\alpha})\lambda''(\psi_\alpha)}\\ &\quad -\frac{m}{k+m}\frac{\partial\theta_{k+m}}{\partial u_\alpha}, \quad 1 \leqslant \beta \leqslant k+m-1,\\ \frac{\partial\theta_{k+m}}{\partial u_\alpha} &= \frac{1}{2m\pi i}\sum_{a=1}^{k+m}\frac{\mathrm{e}^{i\psi_\alpha}}{(\mathrm{e}^{i\phi_a}-\mathrm{e}^{i\psi_\alpha})\lambda''(\psi_\alpha)}.\end{aligned} \tag{3.16}$$

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From (3.2) and (3.11) we obtain

$$\tilde{g}_{\alpha\beta} := (\partial_{u_\alpha}, \partial_{u_\beta})_\lambda = -\frac{\delta_{\alpha\beta}}{u_\alpha \lambda''(\psi_\alpha)}. \tag{3.17}$$

In a similar way we can compute the inner product of the vectors ∂_{u_α} w.r.t. the bilinear form (3.1), the result reads

$$\tilde{\eta}_{\alpha\beta} := \langle \partial_{u_\alpha}, \partial_{u_\beta} \rangle_\lambda = (-1)^{k+1} \frac{\delta_{\alpha\beta}}{\lambda''(\psi_\alpha)}. \tag{3.18}$$

We observe now that the vector field $e = \partial/\partial y^k$ in the coordinates $a_1, \ldots, a_{k+m}$ coincides with

$$e = (-1)^k \frac{\partial}{\partial a^k}.$$

This follows from (3.9). Shift

$$a_k \mapsto a_k + c$$

produces the correspondent shift

$$u_i \mapsto u_i + c, \quad i = 1, \ldots, k+m$$

of the critical values. This shift does not change the critical points ψ_α neither the values of the second derivative $\lambda''(\psi_\alpha)$. So

$$\mathcal{L}_e \tilde{g}^{\alpha\beta} = \mathcal{L}_e(-u_\alpha \lambda''(\psi_\alpha)\delta_{\alpha\beta}) = (-1)^{k+1} \lambda''(\psi_\alpha)\delta_{\alpha\beta} = \tilde{\eta}^{\alpha\beta}, \tag{3.19}$$

where $(\tilde{g}^{\alpha\beta}) = (\tilde{g}_{\alpha\beta})^{-1}$, $(\tilde{\eta}^{\alpha\beta}) = (\tilde{\eta}_{\alpha\beta})^{-1}$.

Now we proceed to the computation of the bilinear form (3.2) in the coordinates $x_1, \ldots, x_{k+m}$ of (3.7) (or, equivalently in the coordinates $\theta_1, \ldots, \theta_{k+m}$ of the form (3.15)). It turns out that this coincides with the form $(\ ,\ )^{\sim}$ defined in Section 2 above.

We will use the following identity

$$\sum_{\alpha=1}^{k+m} \frac{u_\alpha \, \mathrm{e}^{2i\psi_\alpha}}{(\mathrm{e}^{i\phi_a} - \mathrm{e}^{i\psi_\alpha})(\mathrm{e}^{i\phi_b} - \mathrm{e}^{i\psi_\alpha})\lambda''(\psi_\alpha)} = \delta_{ab} - \frac{1}{k}. \tag{3.20}$$

In fact the left-hand side of (3.20) equals

$$\sum_{\alpha=1}^{k+m} \operatorname*{res}_{\phi=\psi_\alpha} \frac{\lambda(\phi)\, \mathrm{e}^{2i\phi}}{(\mathrm{e}^{i\phi} - \mathrm{e}^{i\phi_a})(\mathrm{e}^{i\phi} - \mathrm{e}^{i\phi_b})\lambda'(\phi)}$$

$$= \sum_{\alpha=1}^{k+m} \operatorname*{res}_{v=\mathrm{e}^{i\psi_\alpha}} \frac{\lambda(-i\log v)\; v^2 \frac{1}{iv}}{(v - \mathrm{e}^{i\phi_a})(v - \mathrm{e}^{i\phi_b})\lambda'(-i\log v)}$$

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$$= \left(\operatorname*{res}_{v=\mathrm{e}^{i\phi_a}} + \operatorname*{res}_{v=\mathrm{e}^{i\phi_b}} + \operatorname*{res}_{v=\infty} \right) \frac{i\lambda(-i\log v)\, v}{(v-\mathrm{e}^{i\phi_a})(v-\mathrm{e}^{i\phi_b})\lambda'(-i\log v)}$$

$$= \delta_{ab} - \frac{1}{k}.$$

By using (3.16), (3.17) and (3.20) we obtain

$$(d\theta_{k+m}, d\theta_{k+m})_\lambda = \sum_{\alpha=1}^{k+m} \frac{1}{\tilde{g}_{\alpha\alpha}(u)} \frac{\partial\theta_{k+m}}{\partial u_\alpha} \frac{\partial\theta_{k+m}}{\partial u_\alpha}$$

$$= \sum_{\alpha=1}^{k+m} (-u_\alpha \lambda''(\psi_\alpha)) \frac{1}{(2m\pi i)^2} \sum_{a,b=1}^{k+m} \frac{\mathrm{e}^{2i\psi_\alpha}}{(\mathrm{e}^{i\phi_a}-\mathrm{e}^{i\psi_\alpha})(\mathrm{e}^{i\phi_b}-\mathrm{e}^{i\psi_\alpha})\lambda''(\psi_\alpha)^2}$$

$$= \frac{1}{4m^2\pi^2} \sum_{\alpha=1}^{k+m} \sum_{a,b=1}^{k+m} \frac{u_\alpha\, \mathrm{e}^{2i\psi_\alpha}}{(\mathrm{e}^{i\phi_a}-\mathrm{e}^{i\psi_\alpha})(\mathrm{e}^{i\phi_b}-\mathrm{e}^{i\psi_\alpha})\lambda''(\psi_\alpha)}$$

$$= \frac{1}{4m^2\pi^2} \sum_{a,b=1}^{k+m} \left(\delta_{ab} - \frac{1}{k} \right)$$

$$= \frac{1}{4m^2\pi^2} \left(k+m - \frac{(k+m)^2}{k} \right)$$

$$= -\frac{1}{4\pi^2} \frac{k+m}{mk} = -\frac{1}{4\pi^2 d_k}. \tag{3.21}$$

For any $1 \leqslant \alpha \leqslant k+m-1$, it follows from (3.16), (3.17), (3.20) and (3.21) that

$$(d\theta_{k+m}, d\theta_\alpha)_\lambda = \sum_{a=1}^{k+m} \frac{1}{\tilde{g}_{aa}(u)} \frac{\partial\theta_{k+m}}{\partial u_a} \frac{\partial\theta_\alpha}{\partial u_a}$$

$$= \sum_{a=1}^{k+m} \frac{1}{\tilde{g}_{aa}(u)} \frac{\partial\theta_{k+m}}{\partial u_a} \left(\frac{1}{2\pi i} \frac{\mathrm{e}^{i\psi_a}}{(\mathrm{e}^{i\phi_\alpha}-\mathrm{e}^{i\psi_a})\lambda''(\psi_a)} - \frac{m}{k+m} \frac{\partial\theta_{k+m}}{\partial u_a} \right)$$

$$= -\frac{m}{k+m} \left(-\frac{1}{4\pi^2} \frac{k+m}{mk} \right) + \frac{1}{2\pi i} \sum_{a=1}^{k+m} \frac{1}{\tilde{g}_{aa}(u)}$$

$$\times \frac{\partial\theta_{k+m}}{\partial u_a} \frac{\mathrm{e}^{i\psi_a}}{(\mathrm{e}^{i\phi_\alpha}-\mathrm{e}^{i\psi_a})\lambda''(\psi_a)}$$

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$$= \frac{1}{4k\pi^2} + \frac{1}{4m\pi^2} \sum_{a,b=1}^{k+m} \frac{u_a \, \mathrm{e}^{2i\psi_a}}{(\mathrm{e}^{i\phi_\alpha} - \mathrm{e}^{i\psi_a})(\mathrm{e}^{i\phi_b} - \mathrm{e}^{i\psi_a})\lambda''(\psi_a)}$$

$$= \frac{1}{4k\pi^2} + \frac{1}{4m\pi^2} \sum_{b=1}^{k+m} \left(\delta_{\alpha b} - \frac{1}{k}\right)$$

$$= \frac{1}{4k\pi^2} + \frac{1}{4m\pi^2} \left(1 - \frac{k+m}{k}\right) = 0. \tag{3.22}$$

Finally, for any $1 \leqslant \alpha, \beta \leqslant k+m-1$, by using (3.16), (3.17) and (3.20)–(3.22) we obtain

$$(d\theta_\alpha, d\theta_\beta)_\lambda = \sum_{a=1}^{k+m} \frac{1}{\tilde{g}_{aa}(u)} \frac{\partial \theta_\alpha}{\partial u_a} \frac{\partial \theta_\beta}{\partial u_a}$$

$$= \sum_{a=1}^{k+m} \frac{1}{\tilde{g}_{aa}(u)} \left(\frac{1}{2\pi i} \frac{\mathrm{e}^{i\psi_a}}{(\mathrm{e}^{i\phi_\alpha} - \mathrm{e}^{i\psi_a})\lambda''(\psi_a)} - \frac{m}{k+m} \frac{\partial \theta_{k+m}}{\partial u_a} \right)$$

$$\times \left(\frac{1}{2\pi i} \frac{\mathrm{e}^{i\psi_a}}{(\mathrm{e}^{i\phi_\beta} - \mathrm{e}^{i\psi_a})\lambda''(\psi_a)} - \frac{m}{k+m} \frac{\partial \theta_{k+m}}{\partial u_a} \right)$$

$$= \sum_{a=1}^{k+m} \frac{1}{\tilde{g}_{aa}(u)} \frac{1}{(2\pi i)^2} \frac{\mathrm{e}^{2i\psi_a}}{(\mathrm{e}^{i\phi_\alpha} - \mathrm{e}^{i\psi_a})(\mathrm{e}^{i\phi_\beta} - \mathrm{e}^{i\psi_a})\lambda''(\psi_\alpha)^2}$$

$$- \frac{m}{2\pi i(k+m)} \sum_{a=1}^{k+m} \frac{1}{\tilde{g}_{aa}(u)} \frac{\mathrm{e}^{i\psi_a}}{(\mathrm{e}^{i\phi_\alpha} - \mathrm{e}^{i\psi_a})\lambda''(\psi_a)} \frac{\partial \theta_{k+m}}{\partial u_a}$$

$$= \frac{1}{4\pi^2} \left(\delta_{\alpha\beta} - \frac{1}{k}\right) - \frac{m}{2\pi i(k+m)} \frac{2\pi i}{4m\pi^2} \left(1 - \frac{k+m}{k}\right)$$

$$= \frac{1}{4\pi^2} \left(\delta_{\alpha\beta} - \frac{1}{k+m}\right). \tag{3.23}$$

Now the coincidence of $(dx_i, dx_j)_\lambda$ and $(dx_i, dx_j)^\sim$ follows easily from (3.21)–(3.23). Hence the intersection form (3.2) coincides with the intersection form of the orbit spaces. The coincidence of the metric (3.1) with the metric (2.9) follows from (3.19). Theorem is proved. □

We construct now the flat coordinates $t^1, \ldots, t^{k+m}$ on the space of trigonometric polynomials (essentially, following [D]). Let's define

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$$
\begin{aligned}
t^\mu &= (-1)^{\mu+1}\frac{ki}{\mu}\operatorname*{res}_{\mathrm{e}^{i\phi}=\infty}\lambda(\phi)^{\mu/k}\,\mathrm{d}\phi, \quad 1\leqslant\mu\leqslant k-1,\\
t^{k+m-\mu} &= (-1)^{\mu}\frac{mi}{\mu}\operatorname*{res}_{\mathrm{e}^{-i\phi}=\infty}[(-1)^{k+m}\lambda(\phi)]^{\mu/m}\,\mathrm{d}\phi, \quad 1\leqslant\mu\leqslant m, \qquad (3.24)\\
t^{k+m} &= y^{k+m}.
\end{aligned}
$$

From the above definition we have

$$
\begin{aligned}
&t^\mu = y^\mu + f_\mu(y^1,\dots,y^{\mu-1}), \quad 1\leqslant\mu\leqslant k-1,\\
&t^{k+m-\mu} = y^{k+m-\mu} + h_\mu(y^{k+m-1},\dots,y^{k+m-\mu+1}),\\
&\quad 1\leqslant i\leqslant m-1, \qquad (3.25)\\
&t^k = y^k, \quad t^{k+m} = y^{k+m},
\end{aligned}
$$

where f's and h's are some polynomials, and the relation between y's and a's is given in (3.9).

PROPOSITION 3.1. *The variables t^μ are the flat coordinates for the metric* (3.1).

Proof. Let's denote

$$\xi = \lambda(\phi)^{1/k}, \quad \eta = [(-1)^{k+m}\lambda(\phi)]^{1/m},$$

it follows from (3.24) that when $\mathrm{e}^{i\phi}$ tends to infinity we have

$$\phi = -i\log\xi - \frac{i}{k}\left[\frac{t^1}{\xi} - \frac{t^2}{\xi^2} + \cdots + (-1)^k\frac{t^{k-1}}{\xi^{k-1}}\right] + \mathcal{O}\left(\frac{1}{\xi^k}\right)$$

and when $\mathrm{e}^{-i\phi}$ tends to infinity we have

$$
\begin{aligned}
\phi \;=\; & i\log\eta - i\,t^{k+m}\\
& + \frac{i}{m}\left[\frac{t^{k+m-1}}{\eta} - \frac{t^{k+m-2}}{\eta^2} + \cdots + (-1)^{m-1}\frac{t^k}{\eta^m}\right] + \mathcal{O}\left(\frac{1}{\eta^m}\right).
\end{aligned}
$$

By using the 'thermodynamical identity' [D]

$$\partial_{t^\mu}(\lambda d\phi)_{\phi=\text{constant}} = -\partial_{t^\mu}(\phi d\lambda)_{\lambda=\text{constant}},$$

we obtain

$$\partial_{t^\alpha}(\lambda(\phi)\,\mathrm{d}\phi) = \begin{cases} (-1)^{\alpha+1}\,i\xi^{k-\alpha-1}\,\mathrm{d}\xi + \mathcal{O}(\frac{1}{\xi})\,\mathrm{d}\xi, & \mathrm{e}^{i\phi}\to\infty,\\ \mathcal{O}(\frac{1}{\eta})\,\mathrm{d}\eta, & \mathrm{e}^{-i\phi}\to\infty, \end{cases}$$

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$$\partial_{t^{k+m-\beta}}(\lambda(\phi)\,\mathrm{d}\phi) = \begin{cases} \mathcal{O}(\frac{1}{\xi})\,\mathrm{d}\xi, & \mathrm{e}^{i\phi} \to \infty, \\ (-1)^{k+m-\beta}\, i\eta^{m-\beta-1}\,\mathrm{d}\eta + \mathcal{O}(\frac{1}{\eta})\,\mathrm{d}\eta, & \mathrm{e}^{-i\phi} \to \infty, \end{cases}$$

$$\partial_{t^{k+m}}(\lambda(\phi)\,\mathrm{d}\phi) = \begin{cases} \mathcal{O}(\frac{1}{\xi})\,\mathrm{d}\xi, & \mathrm{e}^{i\phi} \to \infty, \\ (-1)^{k+m} mi\eta^{m-1}\,\mathrm{d}\eta + \mathcal{O}(\frac{1}{\eta})\,\mathrm{d}\eta, & \mathrm{e}^{-i\phi} \to \infty, \end{cases}$$

where $1 \leqslant \alpha \leqslant k-1$, $1 \leqslant \beta \leqslant m$. Thus for $1 \leqslant \alpha, \beta \leqslant k-1$ we have

$$\begin{aligned} \langle \partial_{t^\alpha}, \partial_{t^\beta} \rangle_\lambda &= (-1)^k \left(\operatorname*{res}_{\mathrm{e}^p=\infty} + \operatorname*{res}_{\mathrm{e}^{-p}=\infty} \right) \frac{\partial_{t^\alpha}(\lambda(\phi)\,\mathrm{d}\phi)\,\partial_{t^\beta}(\lambda(\phi)\,\mathrm{d}\phi)}{\mathrm{d}\lambda(\phi)} \\ &= (-1)^k (-1)^{\alpha+\beta+1} \operatorname*{res}_{\mathrm{e}^p=\infty} \frac{\xi^{k-\alpha-1}\,\mathrm{d}\xi\,\xi^{k-\beta-1}\,\mathrm{d}\xi}{\mathrm{d}\xi^k} = \frac{1}{k}\delta_{\alpha+\beta,k}; \end{aligned}$$

similarily, we have

$$\langle \partial_{t^\alpha}, \partial_{t^{k+m-\beta}} \rangle_\lambda = 0, \quad 1 \leqslant \alpha \leqslant k-1, 1 \leqslant \beta \leqslant m,$$

$$\langle \partial_{t^{k+m-\alpha}}, \partial_{t^{k+m-\beta}} \rangle_\lambda = \frac{1}{m}\delta_{\alpha+\beta,m}, \quad 1 \leqslant \alpha, \beta \leqslant m,$$

$$\langle \partial_{t^{k+m}}, \partial_{t^\alpha} \rangle_\lambda = \delta_{\alpha,k}, \quad 1 \leqslant \alpha \leqslant k+m.$$

Proposition is proved. □

We conclude this section with an example of topological application of Theorem 3.1. Let $M^0_{k,m}$ be the subspace consisting of all trigonometric polynomials having $k+m$ pairwise distinct critical values $u_1, \ldots, u_{k+m}$. What is the topology of $M^0_{k,m}$? Particularly, how to compute the number $N(k,m)$ of trigonometric polynomials of the bidegree (k,m) with given pairwise distinct nondegenerate critical values?

More generally, for a n-dimensional complex Frobenius manifold M satisfying semi-simplicity condition (i.e., the algebra on $\mathrm{T}_t M$ is semisimple for a generic t) we may consider the open subset M^0 consisting of all points t such that the eigenvalues $u_1(t), \ldots, u_n(t)$ of the operator of multiplication by the Euler vector field are pairwise distinct. According to [D], on M^0 the eigenvalues can serve as local coordinates. They are called *canonical coordinates* since the multiplication table of tangent vectors takes the very simple form (3.5) in the coordinates $(u_1, \ldots, u_n)$. [As we have explained above, for the space $M_{k,m}$ of trigonometric polynomials $\lambda(\phi)$ the canonical coordinates coincide with the critical values of $\lambda(\phi)$]. According to [D], the map

$$M^0 \to (\mathbf{C}^n \backslash \text{diagonals})/S_n,$$

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$$t \mapsto (u_1(t), \dots, u_n(t)) \text{ modulo permutations}$$

establishes an equivalence between M^0 and the space of isomonodromy deformations of certain linear differential operator with rational coefficients. What is the topology of this space? Particularly, how to compute the number $N(M)$ of the points $t \in M^0$ with given pairwise distinct canonical coordinates $u_1, \dots, u_n$?

To find this number we will compute the degree of the map

$$M^0 \to \mathbf{C}^n$$

given by the formula

$$t \mapsto (b_1(t), \dots, b_n(t)), \tag{3.26}$$

where $b_1(t), \dots, b_n(t)$ are coefficients of the characteristic polynomial

$$\begin{aligned}(-1)^n &\left[u^n + b_1(t)\, u^{n-1} + \cdots + b_n(t)\right] \\ &:= \det((E(t)\cdot) - u) = (\det \langle\ ,\ \rangle_t)^{-1} \\ &\quad \times \det((\ ,\ )_t - u\langle\ ,\ \rangle_t) = \prod_{i=1}^{n} (u_i(t) - u),\end{aligned}$$

here $(\ ,\ )_t$ and $\langle\ ,\ \rangle_t$ are the intersection form and the invariant metric respectively considered as bilinear forms on $\mathrm{T}^*_t M$. We call it generalized Looijenga–Lyashko map (LL-map) of the Frobenius manifold (cf. [A3, Lo, Ly]).

The degree of LL-map can be computed easily in the case of polynomial Frobenius manifolds. More precisely, let the free energy F defining the Frobenius structure in the flat coordinates $t^1, \dots, t^n$ has the form

$$F = \text{cubic term} + G(t^1, \dots, t^p, q^1, \dots, q^r),$$

where G is a polynomial,

$$p + r = n, \quad q^i = \exp t^{p+i}, \quad i = 1, \dots, r$$

and the degrees of the variables $t^{p+1}, \dots, t^n$ are equal to zero. Let us assume that all the degrees

$$\deg t^i, \quad 1 \leqslant i \leqslant p, \quad \deg q^i, \quad 1 \leqslant i \leqslant r$$

are positive. Then LL-map is a polynomial map $M \to \mathbf{C}^n$. We recall that the degrees are normalized in such a way that $\deg t^1 = 1$. Then the degrees of the canonical coordinates $u_1, \dots, u_n$ are equal to 1 [D]. Hence the weighted degrees of the functions $b_1(t), \dots, b_n(t)$ are equal to $1, \dots, n$ respectively.

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Thus to compute the degree of the LL-map (3.26) we can use the graded Bezout theorem. We obtain

$$N(M) = \deg \mathrm{LL} = \frac{n!}{\deg t^1 \cdots \deg t^p \deg q^1 \cdots \deg q^r}. \tag{3.27}$$

In the case of the orbit space of the group $\widetilde{W}^{(k)}(A_{k+m-1})$ the degrees of the variables $t^1, \ldots, t^{k+m-1}$ are expressed via inner products of fundamental weights (i.e., via entries of inverse of the A_{k+m-1} Cartan matrix)

$$\deg t^j = \frac{(\omega_j, \omega_k)}{(\omega_k, \omega_k)}, \quad j = 1, \ldots, k+m-1$$

and

$$\deg \exp t^{k+m} = \frac{1}{(\omega_k, \omega_k)}.$$

We arrive at the following formula for the degree of LL-map of the Frobenius manifold $M_{k,m}$

$$\deg \mathrm{LL} = \frac{(k+m)!\,(\omega_k, \omega_k)^{k+m}}{\prod_{j=1}^{k+m-1}(\omega_j, \omega_k)}.$$

Using the explicit expression for (ω_j, ω_k) (see Table II above) we derive the formula for the number $N(k, m)$ (obtained first by Arnold in [A2])

$$N(k,m) = k^k m^m \frac{(k+m-1)!}{(k-1)!\,(m-1)!}.$$

We hope that our extended affine Weyl groups could be useful for other problems arising in topological study of spaces of rational functions.

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References

[A1] Arnold, V. I.: The Calculus of snakes and the combinatorics of Bernoulli, Euler and Springer numbers of Coxeter groups, *Russ. Math. Surv*. 47 (1992) 1–51.
[A2] Arnold, V. I.: Topological classification of complex trigonometric polynomials and combinatorics of graphs with equal numbers of vertices and links, *Funct. Anal. Appl*. 30 (1996).
[A3] Arnold, V. I.: Critical points of functions and classification of caustics, *Uspekhi Mathem. Nauk* 29(3) (1974) 243–244 (in Russian).
[B] Bourbaki, N.: Groupes et Algèbres de Lie, Chapitres 4, 5 et 6, Masson, Paris-New York-Barcelone-Milan-Mexico-Rio de Janeiro, (1981).
[D] Dubrovin, B.: Geometry of 2D topological field theories. In: Springer *Lecture Notes in Math*. 1620 (1996) 120–348.
[H] Humphreys, J. E.: Introduction to Lie algebras and representation theory, *Graduate texts in mathematics* 9, Springer-Verlag, New York, Heidelberg, Berlin, (1972).
[Lo] Looijenga, E.: The complement of the bifurcation variety of a simple singularity, *Invent. Math*. **23** (1974) 105–116.
[Ly] Lyashko, O. V.: Geometry of bifurcation diagrams, *J. Sov. Math*. 27 (1984) 2736–2759.
[SYS] Saito, K., Yano, T. and Sekiguchi, J.: On a certain generator system of the ring of invariants of a finite reflection group, *Comm. in Algebra* 8(4) (1980) 373–408.
[S] Saito, K.: On a linear structure of a quotient variety by a finite reflection group, Preprint RIMS-288 (1979), Publ. RIMS, Kyoto Univ. 29 (1993) 535–579.
[S1] Saito, K.: Extended affine root systems I, II, Publ. RIMS, Kyoto Univ. 21 (1985) 75–179, 26 (1990) 15–78.

Note added. In January of ’97 after the article has been submitted to the journal, the authors received an interesting paper of P. Slodowy ‘A remark on a recent paper by B. Dubrovin and Y. Zhang’. In this paper it is shown that our analogue of Chevalley theorem for extended affine Weyl groups can be derived from the results of K. Wirthmüller ‘Torus embeddings and deformations of simple space curves’, *Acta Mathematica* **157** (1986) 159–241. This raises a natural question (already formulated by P. Slodowy) to extend (if possible) our construction of Frobenius structures to the more general setting of Wirthmüller. We hope to address the problem in subsequent publications.