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# A TRANSLATION THEOREM FOR THE GENERALISED ANALYTIC FEYNMAN INTEGRAL ASSOCIATED WITH GAUSSIAN PATHS

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**Abstract**

In this paper, we establish a translation theorem for the generalised analytic Feynman integral of functionals that belong to the Banach algebra $\mathcal{F}(C_{a,b}[0,T])$.

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## 1. Introduction

Numerous constructions and applications of the translation theorem (Cameron–Martin theorem) for integrals on infinite-dimensional spaces have been found during the last seventy years. Most of the results in the literature are concentrated on Wiener space. The meaning of the translation theorem on Wiener space and its applications can be found in [1, 2].

In [5, 7, 9], the authors studied the generalised analytic Feynman integral and the generalised analytic Fourier–Feynman transform on the function space $C_{a,b}[0,T]$. The function space $C_{a,b}[0,T]$, induced by a generalised Brownian motion process, was introduced by Yeh in [10] and was used extensively in [5, 7–9]. The translation theorem on the function space $C_{a,b}[0,T]$ was established in [6] and was applied to establish a Cameron–Storvick-type theorem and integration by parts formulas involving the generalised analytic Feynman integral and the generalised analytic Fourier–Feynman transform in [5, 7].

In this paper, we establish a translation theorem for the generalised analytic Feynman integral with respect to Gaussian paths on the function space $C_{a,b}[0,T]$. The class of functionals on $C_{a,b}[0,T]$ that we work with throughout this paper is the Banach algebra $\mathcal{F}(C_{a,b}[0,T])$, which is somewhat analogous to the Banach algebra $\mathcal{S}(L^2[0,T])$ introduced by Cameron and Storvick in [3].

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## 2. Preliminaries

Let $a(t)$ be an absolutely continuous real-valued function on $[0, T]$ with $a(0) = 0$ and $a'(t) \in L^2[0, T]$, and let $b(t)$ be a strictly increasing, continuously differentiable real-valued function with $b(0) = 0$ and $b'(t) > 0$ for each $t \in [0, T]$. The generalised Brownian motion process $Y$ determined by $a(t)$ and $b(t)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s, t) = \min\{b(s), b(t)\}$.

We consider the function space $(C_{a,b}[0, T], \mathcal{W}(C_{a,b}[0, T]), \mu)$ induced by the generalised Brownian motion process, where $C_{a,b}[0, T]$ denotes the set of continuous sample paths of the generalised Brownian motion process $Y$ and $\mathcal{W}(C_{a,b}[0, T])$ is the $\sigma$-algebra of all Wiener measurable subsets of $C_{a,b}[0, T]$. For the precise procedure to construct this function space $C_{a,b}[0, T]$, we refer to the references [5–7, 9–11]. We note that the coordinate process defined by $e_t(x) = x(t)$ on $C_{a,b}[0, T] \times [0, T]$ is also the generalised Brownian motion process determined by $a(t)$ and $b(t)$, that is, for each $t \in [0, T]$,

$$e_t(x) \sim N(a(t), b(t))$$

and the process $\{e_t : 0 \le t \le T\}$ has nonstationary and independent increments.

A subset $B$ of $C_{a,b}[0, T]$ is said to be scale-invariant measurable provided that $\rho B$ is $\mathcal{W}(C_{a,b}[0, T])$-measurable for all $\rho > 0$, and a scale-invariant measurable set $N$ is said to be a scale-invariant null set provided that $\mu(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere.

In [5, 7, 9], the generalised analytic Feynman integral of functionals on $C_{a,b}[0, T]$ was investigated. The functionals considered in [5, 7] are associated to the separable Hilbert space $L^2_{a,b}[0, T]$, where

$$L^2_{a,b}[0, T] := \left\{ v : \int_0^T v^2(s)\, db(s) < +\infty \text{ and } \int_0^T v^2(s)\, d|a|(s) < +\infty \right\}$$

and where $|a|(\cdot)$ denotes the total variation function of $a(\cdot)$. The inner product on $L^2_{a,b}[0, T]$ is given by the formula

$$(u, v)_{a,b} := \int_0^T u(s)v(s)\, d[b(s) + |a|(s)].$$

Also, let

$$C'_{a,b}[0, T] := \left\{ w \in C_{a,b}[0, T] : w(t) = \int_0^t z(s)\, db(s) \text{ for some } z \in L^2_{a,b}[0, T] \right\}.$$

For a path $w \in C'_{a,b}[0, T]$, with $w(t) = \int_0^t z(s)\, db(s)$ for $t \in [0, T]$, define the operator $D : C'_{a,b}[0, T] \to L^2_{a,b}[0, T]$ by the formula

$$Dw(t) := z(t) = \frac{w'(t)}{b'(t)}.$$

Then $C'_{a,b} \equiv C'_{a,b}[0,T]$ with inner product

$$(w_1, w_2)_{C'_{a,b}} := \int_0^T Dw_1(t) Dw_2(t)\, db(t)$$

is a separable Hilbert space. One can see that $L^2_{a,b}[0,T]$ and $C'_{a,b}[0,T]$ are (topologically) homeomorphic under the operator $D$.

In this paper, in addition to the conditions put on $a(t)$ above, we now add the condition

$$\int_0^T |a'(t)|^2\, d|a|(t) < +\infty,$$

from which it follows that $a$ is an element of $C'_{a,b}[0,T]$.

For each $w \in C'_{a,b}[0,T]$ and $x \in C_{a,b}[0,T]$, we let

$$(w, x)^\sim := \int_0^T Dw(t)\, dx(t)$$

denote the Paley–Wiener–Zygmund stochastic integral [8]. We note that for each $w \in C'_{a,b}[0,T]$, $(w,x)^\sim$ is defined for scale-invariant almost everywhere $x \in C_{a,b}[0,T]$ and, if $Dw = z \in L^2_{a,b}[0,T]$ is of bounded variation on $[0,T]$, then the Paley–Wiener–Zygmund stochastic integral $(w,x)^\sim$ is equal to the Riemann–Stieltjes integral $\int_0^T z(t)\, dx(t)$. Furthermore, for each $w \in C'_{a,b}[0,T]$, $(w,x)^\sim$ is a Gaussian random variable with mean $(w,a)_{C'_{a,b}}$ and variance $\|w\|^2_{C'_{a,b}}$. We also note that for $w, x \in C'_{a,b}[0,T]$,

$$(w, x)^\sim = (w, x)_{C'_{a,b}}. \tag{2.1}$$

For more detailed studies of the Paley–Wiener–Zygmund stochastic integral, see [8].

## 3. Gaussian processes on $C_{a,b}[0,T]$

For each $t \in [0,T]$ and $k \in C'_{a,b}[0,T]$ with $Dk = h$ and $\|k\|_{C'_{a,b}} > 0$, let $\mathcal{Z}_k(x,t)$ be the Paley–Wiener–Zygmund stochastic integral

$$\mathcal{Z}_k(x,t) := (D^{-1}(h\chi_{[0,t]}), x)^\sim. \tag{3.1}$$

Then the stochastic process $\mathcal{Z}_k : C_{a,b}[0,T] \times [0,T] \to \mathbb{R}$ is Gaussian with mean function

$$\gamma_k(t) \equiv \int_{C_{a,b}[0,T]} \mathcal{Z}_k(x,t)\, d\mu(x) = \int_0^t h(u)\, da(u)$$

and covariance function

$$\int_{C_{a,b}[0,T]} (\mathcal{Z}_k(x,s) - \gamma_k(s))(\mathcal{Z}_k(x,t) - \gamma_k(t))\, d\mu(x) = \int_0^{\min\{s,t\}} h^2(u)\, db(u).$$

In addition, by [11, Theorem 21.1], $\mathcal{Z}_k(\cdot, t)$ is stochastically continuous in $t$ on $[0,T]$. If $h = Dk$ is of bounded variation on $[0,T]$, then, for all $x \in C_{a,b}[0,T]$, $\mathcal{Z}_k(x,t)$ is continuous in $t$. Of course, if $k(t) = b(t)$, then $\mathcal{Z}_b(x,t) = e_t(x) = x(t)$.

Let $C^*_{a,b}[0,T]$ be the set of functions $k$ in $C'_{a,b}[0,T]$ such that $Dk$ is continuous except for a finite number of finite jump discontinuities and of bounded variation on $[0,T]$.

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For any $w \in C'_{a,b}[0,T]$ and $k \in C^*_{a,b}[0,T]$, let the operation $\odot$ between $C'_{a,b}[0,T]$ and $C^*_{a,b}[0,T]$ be defined by

$$w \odot k := D^{-1}(DwDk), \quad \text{that is, } D(w \odot k) = DwDk,$$

where $DwDk$ denotes the pointwise multiplication of the functions $Dw$ and $Dk$. Then $(C^*_{a,b}[0,T], \odot)$ is a commutative algebra with the identity $b$.

For $w \in C'_{a,b}[0,T]$ and $k \in C^*_{a,b}[0,T]$, it follows that

$$\begin{aligned}(w, \mathcal{Z}_k(x,\cdot))^\sim &= \int_0^T Dw(t)\, d\Big(\int_0^t Dk(s)\, dx(s)\Big) \\ &= \int_0^T Dw(t)Dk(t)\, dx(t) = (w \odot k, x)^\sim \end{aligned} \tag{3.2}$$

for scale-invariant almost everywhere $x \in C_{a,b}[0,T]$. Thus, throughout the remainder of this paper, we require $k$ to be in $C^*_{a,b}[0,T]$ for each process $\mathcal{Z}_k$. This will ensure that the Lebesgue–Stieltjes integrals

$$\|w \odot k\|^2_{C'_{a,b}} = \int_0^T (Dw(t))^2 (Dk(t))^2\, db(t)$$

and

$$(w \odot k, a)_{C'_{a,b}} = \int_0^T Dw(t)Dk(t)Da(t)\, db(t) = \int_0^T Dw(t)Dk(t)\, da(t)$$

exist for all $w \in C'_{a,b}[0,T]$ and $k \in C^*_{a,b}[0,T]$.

## 4. Generalised analytic $\mathcal{Z}_k$-Feynman integral

We begin this section with the definition of the generalised analytic Feynman integral of functionals on $C_{a,b}[0,T]$.

Let $\mathcal{Z}_k$ be the Gaussian process given by (3.1). We define the $\mathcal{Z}_k$-function space integral (namely, the function space integral with respect to the Gaussian paths $\mathcal{Z}_k(x,\cdot)$) for functionals $F$ on $C_{a,b}[0,T]$ by the formula

$$I_k[F] \equiv I_{k,x}[F(\mathcal{Z}_k(x,\cdot))] := \int_{C_{a,b}[0,T]} F(\mathcal{Z}_k(x,\cdot))\, d\mu(x)$$

whenever the integral exists.

Let $\mathbb{C}$ denote the set of complex numbers. Let $\mathbb{C}_+ := \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ and $\widetilde{\mathbb{C}}_+ := \{\lambda \in \mathbb{C} : \lambda \neq 0 \text{ and } \mathrm{Re}(\lambda) \geq 0\}$. Let $F$ be a $\mathbb{C}$-valued scale-invariant measurable functional on $C_{a,b}[0,T]$ such that the $\mathcal{Z}_k$-function space integral

$$J_F(\mathcal{Z}_k; \lambda) := I_{k,x}[F(\lambda^{-1/2}\mathcal{Z}_k(x,\cdot))] = \int_{C_{a,b}[0,T]} F(\lambda^{-1/2}\mathcal{Z}_k(x,\cdot))\, d\mu(x)$$

exists and is finite for all $\lambda > 0$. Let $\Lambda$ be a domain in $\mathbb{C}_+$ such that $(0,+\infty) \cap \Lambda$ is an open interval of positive real numbers. If there exists a function $J^*_F(\mathcal{Z}_k; \lambda)$ analytic on $\Lambda$ such that $J^*_F(\mathcal{Z}_k; \lambda) = J_F(\mathcal{Z}_k; \lambda)$ for all $\lambda \in (0,+\infty) \cap \Lambda$, then $J^*_F(\mathcal{Z}_k; \lambda)$ is defined to be the analytic $\mathcal{Z}_k$-function space integral of $F$ over $C_{a,b}[0,T]$ with parameter $\lambda$.

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In that case, for $\lambda \in \Lambda$ we write

$$I_k^{\mathrm{an}_\lambda}[F] \equiv I_{k,x}^{\mathrm{an}_\lambda}[F(\mathcal{Z}_k(x,\cdot))] := J_F^*(\mathcal{Z}_k;\lambda).$$

Let $q$ be a nonzero real number and let $\Gamma$ be a connected neighbourhood of $-iq$ in $\widetilde{\mathbb{C}}_+$ such that $(0,+\infty)\cap\Gamma$ is an open interval of positive real numbers. Next let $F$ be a measurable functional whose analytic $\mathcal{Z}_k$-function space integral exists for all $\lambda$ in $\mathrm{Int}(\Gamma)$, the interior of $\Gamma$ in $\widetilde{\mathbb{C}}_+$. If the following limit exists, we call it the generalised analytic $\mathcal{Z}_k$-Feynman integral (the generalised analytic Feynman integral with respect to the Gaussian paths $\mathcal{Z}_k(x,\cdot)$) of $F$ with parameter $q$, and we write

$$I_k^{\mathrm{anf}_q}[F] \equiv I_{k,x}^{\mathrm{anf}_q}[F(\mathcal{Z}_k(x,\cdot))] := \lim_{\substack{\lambda\to -iq\\ \lambda\in\mathrm{Int}(\Gamma)}} I_{k,x}^{\mathrm{an}_\lambda}[F(\mathcal{Z}_k(x,\cdot))].$$

In the definition of the generalised analytic $\mathcal{Z}_k$-Feynman integral, for each $\lambda\in\mathbb{C}$, $\lambda^{1/2}$ denotes the principal square root of $\lambda$, that is, $\lambda^{1/2}$ is always chosen to have nonnegative real part.

Remark 4.1. Note that if $k\equiv b$ on $[0,T]$, then the generalised analytic $\mathcal{Z}_b$-Feynman integral $I_b^{\mathrm{anf}_q}[F]$ agrees with the previous definitions of the generalised analytic Feynman integral $E^{\mathrm{anf}_q}[F]$ stated in [5, 7].

Let $\mathcal{M}(C'_{a,b}[0,T])$ be the Banach algebra of $\mathbb{C}$-valued, countably additive (and hence finite) Borel measures on $C'_{a,b}[0,T]$. The Fresnel-type class $\mathcal{F}(C_{a,b}[0,T])$ consists of those functionals $F$ on $C_{a,b}[0,T]$ expressible in the form

$$F(x) := \int_{C'_{a,b}[0,T]} \exp\{i(w,x)^{\sim}\}\,df(w) \tag{4.1}$$

for scale-invariant almost everywhere $x\in C_{a,b}[0,T]$, where the associated measure $f$ is an element of $\mathcal{M}(C'_{a,b}[0,T])$. The class $\mathcal{F}(C_{a,b}[0,T])$ is a Banach algebra with norm

$$\|F\| := \|f\| = \int_{C'_{a,b}[0,T]} d|f|(w).$$

For a more detailed study of the Banach algebra $\mathcal{F}(C_{a,b}[0,T])$, see [9].

For a positive real number $q_0$, let

$$\Gamma_{q_0} := \Big\{\lambda\in\widetilde{\mathbb{C}}_+ : |\mathrm{Im}(\lambda^{-1/2})| < \frac{1}{\sqrt{2q_0}}\Big\}. \tag{4.2}$$

Then one can observe the following facts.

(i) The set $\Gamma_{q_0}$ is an unbounded domain in the topological subspace $\widetilde{\mathbb{C}}_+$ of $\mathbb{C}$.

(ii) For a real $q$ with $|q|>q_0$, $\Gamma_{q_0}$ is a connected neighbourhood of $-iq$ in $\widetilde{\mathbb{C}}_+$ such that $(0,+\infty)\subset\Gamma_{q_0}$.

(iii) For all $\lambda\in\Gamma_{q_0}$,

$$\exp\{|\mathrm{Im}(\lambda^{-1/2})|\,\|w\odot k\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\} < \exp\Big\{\frac{1}{\sqrt{2q_0}}\|Dk\|_\infty\|w\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\Big\},$$

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because

$$\|w\odot k\|_{C'_{a,b}} = (w\odot k, w\odot k)^{1/2}_{C'_{a,b}} = \left[\int_0^T \{Dw(t)\}^2\{Dk(t)\}^2\,db(t)\right]^{1/2}$$
$$\le \|Dk\|_\infty\left[\int_0^T \{Dw(t)\}^2\,db(t)\right]^{1/2} = \|Dk\|_\infty\|w\|_{C'_{a,b}}$$

for $w \in C'_{a,b}[0,T]$ and $k \in C^*_{a,b}[0,T]$, where $\|\cdot\|_\infty$ denotes the essential supremum norm.

We note that for all real $q$ with $|q| > q_0$,

$$(-iq)^{-1/2} = (i/q)^{1/2} = (\sqrt{2|q|})^{-1}(1 + i\,\mathrm{sign}(q)).$$

By a close examination of (4.2), it can be seen that $-iq$ is an element of the domain $\Gamma_{q_0}$ in $\widetilde{\mathbb{C}}_+$. More precisely, we have $-iq \in \{\lambda \in \widetilde{\mathbb{C}}_+ : |\lambda| > q_0\} \subset \Gamma_{q_0}$ for all real $q$ with $|q| > q_0$.

For a positive real number $q_0$ and an element $k$ of $C^*_{a,b}[0,T]$, we define a subclass $\mathcal{F}^{q_0}_k$ of $\mathcal{F}(C_{a,b}[0,T])$ by $F \in \mathcal{F}^{q_0}_k$ if and only if

$$\int_{C'_{a,b}[0,T]} \exp\left\{\frac{1}{\sqrt{2q_0}}\|Dk\|_\infty\|w\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\right\} d|f|(w) < +\infty, \tag{4.3}$$

where $f$ and $F$ are related by (4.1).

Theorem 4.2 below is a simple modification of the results [7, Equation (4.3)] and [9, Equations (40) and (49)]. The condition (4.3) implies the existence of the right-hand side of (4.4) below.

Theorem 4.2. *Let $k$ be a nonzero element of $C^*_{a,b}[0,T]$, let $q_0$ be a positive real number and let $F \in \mathcal{F}^{q_0}_k$ be given by (4.1). Then, for all real $q$ with $|q| > q_0$, the generalised analytic $\mathcal{Z}_k$-Feynman integral of $F$, $I^{\mathrm{anf}_q}_k[F]$, exists and is given by*

$$I^{\mathrm{anf}_q}_k[F] = \int_{C'_{a,b}[0,T]} \exp\left\{-\frac{i}{2q}\|w\odot k\|^2_{C'_{a,b}} + i(-iq)^{-1/2}(w\odot k, a)_{C'_{a,b}}\right\} df(w). \tag{4.4}$$

## 5. Translation theorem for the generalised analytic $\mathcal{Z}_k$-Feynman integral

The following observation will be very useful in the development of our translation theorem for the generalised analytic $\mathcal{Z}_k$-Feynman integral.

Let $q_0 > 0$, $k \in C^*_{a,b}[0,T]$ and $F \in \mathcal{F}^{q_0}_k$. Given $\theta \in C'_{a,b}[0,T]$ and $q \in \mathbb{R}\setminus\{0\}$, let

$$F^{q\theta}(x) := F(x)\exp\{-iq(\theta, x)^\sim\}. \tag{5.1}$$

Using (4.1), we can rewrite $F^{q\theta}(x)$ as follows:

$$F^{q\theta}(x) = \int_{C'_{a,b}[0,T]} \exp\{i(w - q\theta, x)^\sim\}\,df(w) = \int_{C'_{a,b}[0,T]} \exp\{i(r, x)^\sim\}\,df^{q\theta}(r) \tag{5.2}$$

for scale-invariant almost everywhere $x \in C_{a,b}[0,T]$, where $f^{q\theta}$ is a complex measure in $\mathcal{M}(C'_{a,b}[0,T])$ such that

$$f^{q\theta}(B) \equiv f(B+q\theta) \quad \text{for all } B \in \mathcal{B}(C'_{a,b}[0,T]). \tag{5.3}$$

For all real $q$ with $|q| > q_0$, $F^{q\theta}$ is an element of $\mathcal{F}_k^{q_0}$, because

$$\begin{aligned}
&\int_{C'_{a,b}[0,T]} \exp\Big\{\frac{1}{\sqrt{2q_0}}\|Dk\|_\infty \|r\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\Big\}\, d|f^{q\theta}|(r) \\
&\quad = \int_{C'_{a,b}[0,T]} \exp\Big\{\frac{1}{\sqrt{2q_0}}\|Dk\|_\infty \|w-q\theta\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\Big\}\, d|f|(w) \\
&\quad \le \exp\Big\{\frac{|q|}{\sqrt{2q_0}}\|Dk\|_\infty \|\theta\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\Big\}\int_{C'_{a,b}[0,T]} \exp\Big\{\frac{1}{\sqrt{2q_0}}\|Dk\|_\infty \|w\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\Big\}\, d|f|(w) \\
&\quad < +\infty.
\end{aligned}$$

We are now ready to present a translation theorem for the generalised analytic $\mathcal{Z}_k$-Feynman integral of functionals in $\mathcal{F}(C'_{a,b}[0,T])$.

Theorem 5.1. *Let $k$, $q_0$ and $F$ be as in Theorem 4.2. Given a function $\theta$ in $C'_{a,b}[0,T]$, let $x_0 \in C'_{a,b}[0,T]$ be given by $x_0 = (\theta \odot k)$. Then, for all real $q$ with $|q| > q_0$,*

$$I_k^{\mathrm{anf}_q}[F(\cdot + \mathcal{Z}_k(x_0,\cdot))] = \exp\Big\{\frac{iq}{2}\|\theta\odot k\|^2_{C'_{a,b}} - (-iq)^{1/2}(\theta\odot k, a)_{C'_{a,b}}\Big\} I_k^{\mathrm{anf}_q}[F^{q\theta}], \tag{5.4}$$

*where $F^{q\theta}$ is given by (5.1) above.*

Proof. First, using (3.2) with $x$ replaced by $x_0$ and (2.1),

$$(w, \mathcal{Z}_k(x_0,\cdot))^\sim = (w\odot k, x_0)^\sim = (w\odot k, x_0)_{C'_{a,b}} = (w\odot k, \theta\odot k)_{C'_{a,b}}. \tag{5.5}$$

Given the functional $F \in \mathcal{F}_k^{q_0}$ and the associated measure $f$ of $F$, we consider the functional

$$\begin{aligned}
F_{\theta,k}(x) &:= \int_{C'_{a,b}[0,T]} \exp\{i(w,x)^\sim + i(w\odot k, \theta\odot k)_{C'_{a,b}}\}\, df(w) \\
&\equiv \int_{C'_{a,b}[0,T]} \exp\{i(w,x)^\sim\}\, df_{\theta,k}(w),
\end{aligned} \tag{5.6}$$

where $f_{\theta,k}$ is the complex measure in $\mathcal{M}(C'_{a,b}[0,T])$ defined by

$$f_{\theta,k}(B) := \int_B \exp\{i(w\odot k, \theta\odot k)_{C'_{a,b}}\}\, df(w) \quad \text{for all } B \in \mathcal{B}(C'_{a,b}[0,T]). \tag{5.7}$$

Then $F_{\theta,k}$ is an element of the class $\mathcal{F}_k^{q_0}$, because

$$\begin{aligned}
\|f_{\theta,k}\| &= \int_{C'_{a,b}[0,T]} d|f_{\theta,k}|(w) = \int_{C'_{a,b}[0,T]} |\exp\{i(w\odot k, \theta\odot k)_{C'_{a,b}}\}|\, d|f|(w) \\
&= \int_{C'_{a,b}[0,T]} d|f|(w) = \|f\| < +\infty
\end{aligned}$$

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and

$$\int_{C'_{a,b}[0,T]} \exp\Big\{\frac{1}{\sqrt{2q_0}}\|Dk\|_\infty\|w\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\Big\}\,d|f_{\theta,k}|(w) = \int_{C'_{a,b}[0,T]} \exp\Big\{\frac{1}{\sqrt{2q_0}}\|Dk\|_\infty\|w\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\Big\}\,d|f|(w) < +\infty.$$

Using (4.1), (5.5), (5.6), (4.4) with $F$ replaced by $F_{\theta,k}$ and (5.7),

$$\begin{aligned} I_k^{\mathrm{anf}_q}&[F(\cdot + \mathcal{Z}_k(x_0,\cdot))] \\ &= I_{k,x}^{\mathrm{anf}_q}\Big[\int_{C'_{a,b}[0,T]} \exp\{i(w,\mathcal{Z}_k(x,\cdot))^\sim + i(w,\mathcal{Z}_k(x_0,\cdot))^\sim\}\,df(w)\Big] \\ &= I_{k,x}^{\mathrm{anf}_q}\Big[\int_{C'_{a,b}[0,T]} \exp\{i(w,\mathcal{Z}_k(x,\cdot))^\sim + i(w\odot k,\theta\odot k)_{C'_{a,b}}\}\,df(w)\Big] \\ &= I_{k,x}^{\mathrm{anf}_q}\Big[\int_{C'_{a,b}[0,T]} \exp\{i(w,\mathcal{Z}_k(x,\cdot))^\sim\}df_{\theta,k}(w)\Big] \\ &= \int_{C'_{a,b}[0,T]} \exp\Big\{-\frac{i}{2q}\|w\odot k\|^2_{C'_{a,b}} + i(-iq)^{-1/2}(w\odot k,a)_{C'_{a,b}}\Big\}\,df_{\theta,k}(w) \\ &= \int_{C'_{a,b}[0,T]} \exp\Big\{i(w\odot k,\theta\odot k)_{C'_{a,b}} - \frac{i}{2q}\|w\odot k\|^2_{C'_{a,b}} + i(-iq)^{-1/2}(w\odot k,a)_{C'_{a,b}}\Big\}\,df(w). \end{aligned}$$

From this representation, using (5.2), (4.4) with $F$ replaced by $F^{q\theta}$ and (5.3),

$$\begin{aligned} I_k^{\mathrm{anf}_q}&[F^{q\theta}] \\ &= \int_{C'_{a,b}[0,T]} \exp\Big\{-\frac{i}{2q}\|r\odot k\|^2_{C'_{a,b}} + i(-iq)^{-1/2}(r\odot k,a)_{C'_{a,b}}\Big\}\,df^{q\theta}(r) \\ &= \int_{C'_{a,b}[0,T]} \exp\Big\{-\frac{i}{2q}\|(w-q\theta)\odot k\|^2_{C'_{a,b}} + i(-iq)^{-1/2}((w-q\theta)\odot k,a)_{C'_{a,b}}\Big\}\,df(w) \\ &= \exp\Big\{-\frac{iq}{2}\|\theta\odot k\|^2_{C'_{a,b}} - iq(-iq)^{-1/2}(\theta\odot k,a)_{C'_{a,b}}\Big\} \\ &\quad\times \int_{C'_{a,b}[0,T]} \exp\Big\{i(w\odot k,\theta\odot k)_{C'_{a,b}} - \frac{i}{2q}\|w\odot k\|^2_{C'_{a,b}} \\ &\qquad\qquad + i(-iq)^{-1/2}(w\odot k,a)_{C'_{a,b}}\Big\}\,df(w) \\ &= \exp\Big\{-\frac{iq}{2}\|\theta\odot k\|^2_{C'_{a,b}} + (-iq)^{1/2}(\theta\odot k,a)_{C'_{a,b}}\Big\} I_k^{\mathrm{anf}_q}[F(\cdot + \mathcal{Z}_k(x_0,\cdot))]. \end{aligned} \tag{5.8}$$

Equation (5.8) now yields (5.4). □

Taking $k(t) = b(t)$ in (5.4), we have the following translation theorem for the generalised analytic Feynman integral $E^{\mathrm{anf}_q}[F] \equiv I_b^{\mathrm{anf}_q}[F]$ (see Remark 4.1 above).

COROLLARY 5.2. *Setting $k(t) = b(t)$ in Theorem 5.1 yields $x_0 = \theta$ and the formula*

$$\begin{aligned} E^{\mathrm{anf}_q}[F(\cdot + x_0)] &\equiv I_b^{\mathrm{anf}_q}[F(\cdot + \mathcal{Z}_b(x_0, \cdot))] \\ &= \exp\Big\{\frac{iq}{2}\|\theta\|^2_{C'_{a,b}} - (-iq)^{1/2}(\theta, a)_{C'_{a,b}}\Big\} E^{\mathrm{anf}_q}[F^{q\theta}] \end{aligned}$$

*for all real $q$ with $|q| > q_0$, where $F^{q\theta}$ is given by (5.1) above.*

If $a(t) \equiv 0$ and $b(t) = t$ on $[0, T]$, then the function space $C_{a,b}[0, T]$ reduces to the classical Wiener space $C_0[0, T]$. In this case, the Gaussian process given by (3.1) with $k(t) = t$ is an ordinary Wiener process. We thus have the following translation theorem for the analytic Feynman integral on Wiener space $(C_0[0, T], m_w)$. This result subsumes a similar result obtained by Cameron and Storvick in [4].

COROLLARY 5.3. *Setting $a(t) \equiv 0$ and $b(t) = t$ in Corollary 5.2 yields the formula*

$$\int_{C_0[0,T]}^{\mathrm{anf}_q} F(x + x_0)\, dm_w(x) = \exp\Big\{\frac{iq}{2}\|\theta\|^2_{C'_0}\Big\} \int_{C_0[0,T]}^{\mathrm{anf}_q} F(x) \exp\{-iq(\theta, x)^{\sim}\}\, dm_w(x)$$

*for all real $q \in \mathbb{R}\backslash\{0\}$.*

REMARK 5.4. Note that $\mathcal{F}(C_{a,b}[0, T])$ is a very rich class because it contains many functionals which appear in quantum mechanics. For instance, see [8, Examples 18 and 19].

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