BULL. AUSTRAL. MATH. SOC. 52C05, 52A40, 52A10
VOL. 58 (1998) [159–166]

# SOME INEQUALITIES FOR PLANAR CONVEX SETS CONTAINING ONE LATTICE POINT

M.A. HERNÁNDEZ CIFRE AND S. SEGURA GOMIS

We obtain two inequalities relating the diameter and the (minimal) width with the area of a planar convex set containing exactly one point of the integer lattice in its interior. They are best possible. We then use these results to obtain some related inequalities.

## 1. INTRODUCTION AND RESULTS

Let $K$ be a compact convex set in the Euclidean plane $E^2$, having area $A(K) = A$, (minimal) width $\omega(K) = \omega$ and diameter $d(K) = d$.

In [6], Scott established two "dual" inequalities, relating $d$, $\omega$ and $A$ for compact convex sets containing no points of the integer lattice $\mathbf{Z}^2$ in its interior:

$$(\omega - 1)A \leqslant \frac{1}{2}\omega^2$$

$$(d-1)A \leqslant \frac{1}{2}d^2, \text{ providing } d \leqslant 2.$$

In this paper we prove two similar inequalities, but in the case that $K$ contains one lattice-point of $\mathbf{Z}^2$ in its interior, and some related inequalities.

Let us define the family of the triangles $T_\varepsilon$:

$$\mathcal{T} = \left\{ conv\left\{(-2,-1),(1+\varepsilon,-1),\left(\frac{1-\varepsilon}{1+\varepsilon},\frac{2}{1+\varepsilon}\right)\right\} \Big/ \varepsilon \in \left[0,\sqrt{2}-1\right] \right\}.$$

Note that the triangles of the family $\mathcal{T}$ are the triangles "intermediate between" the isosceles triangles $T_0$ and $T_{\sqrt{2}-1}$, which are shown in Figure 1.

Received 19th February, 1998
The first author's work was supported by an FPPI Predoctoral Grant, Universidad de Murcia, 1996 and by Consejería de Cultura y Educación (C.A.R.M.) COM-05/96MAT. The second author's work was partially supported by a DGICYT Grant No. PB94-0750-C02-02 and by Consejería de Cultura y Educación (C.A.R.M.) COM-05/96MAT.

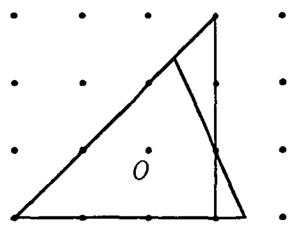


Figure 1: Extremal triangles

Now, we may establish the following theorems:

**Theorem 1.** *If $K$ contains the origin $O$, but no other point of the integer lattice in its interior, then*

$$\left(\omega - \sqrt{2}\right) A \leqslant \frac{\sqrt{2}}{2}\omega^2,$$

*with equality when and only when $K = T_\varepsilon \in \mathcal{T}$, for all $\varepsilon \in \left[0, \sqrt{2}-1\right]$ (up to congruence).*

**Theorem 2.** *If $K$ contains the origin $O$, but no other point of $\mathbf{Z}^2$ in its interior, and $d \leqslant 2\sqrt{2}$, then*

$$\left(d - \sqrt{2}\right) A \leqslant \frac{\sqrt{2}}{2}d^2,$$

*with equality when and only when $K$ is a square of side length 2.*

**Corollary 1.** *If $K$ contains the origin $O$, but no other point of $\mathbf{Z}^2$ in its interior, then*

$$\left(\omega - \sqrt{2}\right) d^2 \leqslant 2\sqrt{2}A,$$

*with equality when and only when $K = T_\varepsilon \in \mathcal{T}$, for all $\varepsilon \in \left[0, \sqrt{2}-1\right]$ (up to congruence).*

**Corollary 2.** *If $K$ contains the origin $O$, but no other point of $\mathbf{Z}^2$ in its interior, then*

$$\left(\omega - \sqrt{2}\right) A \leqslant \frac{3\sqrt{2}}{2} + 2,$$

*with equality when and only when $K = T_{\sqrt{2}-1}$ (up to congruence).*

## 2. Some preliminary results

We shall require the following lemmas:

**Lemma 1.** *Let $Q$ be the quadrilateral with vertices $XYZU$. Let $XZ = t$, and $\delta$ be the width of $Q$ in a direction perpendicular to $XZ$. Then, any square which is inscribed in $Q$ and has a vertex on each side of $Q$ has side length $s$, satisfying:*

$$s \geqslant \frac{t\delta}{t+\delta}.$$

Proof: Let $A$, $B$, $C$, $D$ be the vertices of the inscribed square, which lie on the sides $UX$, $XY$, $YZ$, $ZU$ respectively (see Figure 2).

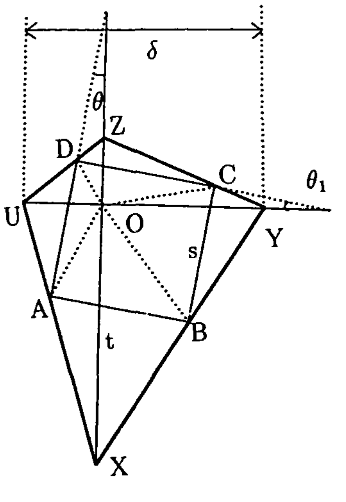


Figure 2

Let $AD$ make an angle $\theta$ with $XZ$, let $DC$ make an angle $\theta_1$ with $UY$, and let $XZ$ meet $UY$ in $O$.

Now, it is clear that the area of $Q$ is $t\delta/2$. But this area is also given by adding the areas of quadrilaterals $ODZC$, $OBXA$ to the areas of quadrilaterals $OCYB$, $OAUD$.

Then, we obtain:

$$\frac{1}{2}t\delta = \frac{1}{2}st\cos\theta + \frac{1}{2}s\overline{UY}\cos\theta_1 \leqslant \frac{1}{2}s\left(t\cos\theta + \delta\right).$$

So,

$$s \geqslant \frac{t\delta}{t\cos\theta + \delta} \geqslant \frac{t\delta}{t+\delta}.$$

∎

**Lemma 2.** *Let $A$, $B$, $C$, $D$ be the vertices of a square with side length $s$, and let $Q$ be the quadrilateral $XYZU$, with sides $XY$, $YZ$, $ZU$, $UX$, passing through $B$, $C$, $D$ $A$ respectively. Let $d_1 \leqslant d_2$ be its diagonals. Then, $d_1 \leqslant 2s$.*

https://doi.org/10.1017/S0004972700032093 Published online by Cambridge University Press

PROOF: Using Steiner symmetrisation (see [3], Lemma 3), it is easy to see that $Q$ can be transformed into a "kite" (a quadrilateral which is symmetric with respect to one of its diagonals) $Q'$, such that

(i) $Q'$ has its symmetry axis along the line $x = s/2$ (for a suitable choice of coordinate axis).

(ii) The sides of $Q'$ pass through $A$, $B$, $C$, $D$ respectively.

(iii) The minor diagonal of $Q'$, $d_1'$, satisfies $d_1' \geqslant d_1$.

It is clear that for $Q'$, $d_1' \leqslant 2s$, and hence, $d_1 \leqslant 2s$. ∎

## 3. Proof of Theorem 1

Let $r = r(K)$ be the inradius of $K$. It is shown in [5] that for any bounded convex set $K$,

$$(\omega - 2r)\,A \leqslant \frac{\sqrt{3}}{3}\omega^2 r.$$

So, it follows that if $r \leqslant \sqrt{2}/2$, then

$$\left(\omega - \sqrt{2}\right) A \leqslant (\omega - 2r)\,A \leqslant \frac{\sqrt{3}}{3}\omega^2 r \leqslant \frac{\sqrt{3}}{3}\frac{\sqrt{2}}{2}\omega^2 < \frac{\sqrt{2}}{2}\omega^2.$$

Hence, we may assume that $K$ contains a disc $D_\varepsilon$ of radius $\dfrac{\sqrt{2}}{2} + \varepsilon$, $\varepsilon > 0$.

It is no loss of generality to assume that the centre of $D_\varepsilon$ is interior to the square with vertices $Q_1 = (1,1)$, $Q_2 = (-1,1)$, $Q_3 = (-1,-1)$, $Q_4 = (1,-1)$ (we suppose that $int(K) \cap \mathbf{Z}^2 = \{O\}$).

Let $\mathcal{K}_\varepsilon$ denote the class of compact convex sets $K$ containing a disc $D_\varepsilon$ placed as above, with $int(K) \cap \mathbf{Z}^2 = \{O\}$. It will be sufficient to prove the theorem for $\mathcal{K}_\varepsilon$.

Let now $\varepsilon > 0$ be given. Since $Q_i$, $i = 1,2,3,4$ are not interior to the sets of $\mathcal{K}_\varepsilon$, we deduce that $\mathcal{K}_\varepsilon$ is uniformly bounded in the plane. As $\sqrt{2}(\omega - \sqrt{2})A/\omega^2$ is a continuous function of $\omega$, $A$ for $\omega > 0$, Blaschke's selection theorem [1] guarantees the existence of a maximal set in $\mathcal{K}_\varepsilon$, that is, a set $K \in \mathcal{K}_\varepsilon$ for which $\sqrt{2}(\omega - \sqrt{2})A/\omega^2$ is as large as possible.

Let $f(\omega) = \sqrt{2}\omega^2/2(\omega - \sqrt{2})$. As $f(\omega)$ is a decreasing function of $\omega$ (since $\omega \leqslant 1 + \sqrt{2} < 2\sqrt{2}$, see [7]), we seek to obtain $K$ by initially making $\omega$ and $A$ as large as possible.

Let $N$, $S$, $E$, $W$ denote the lattice points $(0,\pm 1)$, $(\pm 1, 0)$ respectively. As $N, S, E, W \notin int(K)$, and $K$ is convex, $K$ is bounded by lines through these points. These lines can form a convex quadrilateral $Q \supseteq K$ or determine a triangular region $T \supseteq K$. In any case, $A(Q) \geqslant A(K)$ and $A(T) \geqslant A(K)$.

THE TRIANGLE CASE:

Since in any triangle the width $\omega$ is in a direction perpendicular to the longest side, $A(T) = d\omega/2$.

In [8], it is proved that if $K$ contains no point of the rectangular lattice with basis $\{(u,0),(0,v)\}$, then

$$(1) \qquad (\omega - u)(d - v) \leqslant uv.$$

Let us consider the lattice $\Gamma$ formed by the points

$$\Gamma = \left\{(m,n) \in \mathbf{Z}^2 \,/\, m+n \text{ is an odd number}\right\},$$

which we can identify with the lattice $\sqrt{2}\mathbf{Z}^2$ with basis $\left\{\left(\sqrt{2},0\right),\left(0,\sqrt{2}\right)\right\}$. Then, $int(K) \cap \sqrt{2}\mathbf{Z}^2 = \emptyset$, and by (1), $\left(\omega - \sqrt{2}\right)\left(d - \sqrt{2}\right) \leqslant 2$. But this is easily seen to be equivalent to

$$(2) \qquad d \leqslant \sqrt{2}\frac{\omega}{\omega - \sqrt{2}}.$$

Since the triangle $T$ contains no points of $\sqrt{2}\mathbf{Z}^2$, then the inequality (2) holds for $T$. Then

$$A(T) = \frac{1}{2}d(T)\omega(T) \leqslant \frac{\sqrt{2}}{2}\frac{\omega(T)^2}{\omega(T) - \sqrt{2}}.$$

But as $int(T) \cap \sqrt{2}\mathbf{Z}^2 = \emptyset$, we have [3]

$$\omega(T) \leqslant \sqrt{2}\frac{1}{2}\left(2 + \sqrt{3}\right) < 2\sqrt{2}$$

and hence,

$$A \leqslant A(T) \leqslant \frac{\sqrt{2}}{2}\frac{\omega(T)^2}{\omega(T) - \sqrt{2}} \leqslant \frac{\sqrt{2}}{2}\frac{\omega^2}{\omega - \sqrt{2}}.$$

And equality holds when and only when the equality holds in (2) and $K$ contains one lattice point in its interior, that is, when and only when $K$ is a triangle of diameter $d$ and width $\omega = \sqrt{2}d/(d - \sqrt{2})$, containing a unique lattice point in its interior. Hence, equality holds when and only when $K = T_\varepsilon \in \mathcal{T}$ (up to congruence).

THE QUADRILATERAL CASE:

Let quadrilateral $\mathcal{Q} \equiv XYZU$ have just one of the lattice points $N$, $S$, $E$, $W$ on each side. Let $XZ = t$ be its major diagonal, and let the width of $\mathcal{Q}$ in a direction perpendicular to $XZ$ be $\delta$.

Then, Lemma 1 assures us that $\sqrt{2} \geqslant t\delta/(t+\delta)$, which is equivalent to the inequality

$$t \leqslant \sqrt{2}\frac{\delta}{\delta - \sqrt{2}}.$$

Further, the area of $\mathcal{Q}$ is given by

$$A(\mathcal{Q}) = \frac{1}{2}t\delta \leqslant \frac{\sqrt{2}}{2}\frac{\delta^2}{\delta - \sqrt{2}}.$$

Note that $\delta \leqslant UY$ (minor diagonal), and applying Lemma 2, $UY \leqslant 2\sqrt{2}$. Hence, $\delta \leqslant 2\sqrt{2}$.

Since $\omega < \omega(\mathcal{Q}) < \delta \leqslant 2\sqrt{2}$ and $\delta^2/(\delta - \sqrt{2})$ is a decreasing function of $\delta$, we have finally:

$$A \leqslant A(\mathcal{Q}) < \frac{\sqrt{2}}{2}\frac{\omega(\mathcal{Q})^2}{\omega(\mathcal{Q}) - \sqrt{2}} \leqslant \frac{\sqrt{2}}{2}\frac{\omega^2}{\omega - \sqrt{2}}.$$

This inequality is strict for a non-degenerate quadrilateral. This completes the proof of Theorem 1. ∎

## 4. Proof of Theorem 2

Let $f(d) = \sqrt{2}d^2/2(d - \sqrt{2})$. Again, $f(d)$ is a decreasing function of $d$ for $d \leqslant 2\sqrt{2}$.

A set $K$ for which $\sqrt{2}(d - \sqrt{2})/d^2$ is as large as possible will be called a maximal set. The existence of such a maximal set is guaranteed by Blaschke's selection theorem [1], as the sets of diameter $d \leqslant 2\sqrt{2}$ can be placed in a bounded portion of the plane.

Let us consider again the lattice $\Gamma$ defined in the proof of Theorem 1. Then, $K$ contains no lattice-points of $\Gamma \equiv \sqrt{2}\mathbf{Z}^2$ in its interior. It is shown in [4] that $A \leqslant \sqrt{2}\lambda d$ ($\lambda \approx 1.144$), with equality when and only when $K = K^*$ is the intersection set of the disc $x^2 + y^2 \leqslant d^2/4$ and the square with side $\sqrt{2}\sqrt{2} = 2$.

Using the notation of the Figure 3, and noting that $d = 2\sec\theta$, we have:

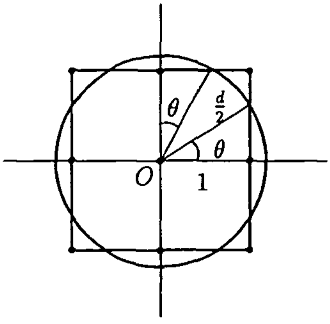


Figure 3

$$A(K^*) = 2\sec^2\theta\left(\frac{\pi}{2} - 2\theta + \sin 2\theta\right), \text{ where } 0 \leqslant \theta \leqslant \frac{\pi}{4}.$$

Since $f(d)$ is a decreasing function of $d$ for $d \leqslant 2$, it will be sufficient to show that $A(K^*) \leqslant f(2\sqrt{2}) = 4$, because then, $A \leqslant A(K^*) \leqslant f(2\sqrt{2}) \leqslant f(d)$.

https://doi.org/10.1017/S0004972700032093 Published online by Cambridge University Press

But $A(K^*) \leqslant 4$ is equivalent to

$$g(\theta) = \frac{\pi}{2} - 2\theta + \sin 2\theta - 2\cos^2\theta \leqslant 0.$$

Since $g'(\theta) > 0$ for the given range of $\theta$, we have $g(\theta) \leqslant g(\pi/4) = 0$.

It follows that $A \leqslant \sqrt{2}d^2/2(d - \sqrt{2})$, with equality when and only when $d = 2\sqrt{2}$, and this occurs when $K = K^*$ is a square of side 2.

It is easy to see that Theorem 2 fails for $d > 2\sqrt{2}$. We can take, for example, the intersection set of the isosceles triangle $T_0$ (shown in Figure 1), and the disc $x^2 + y^2 \leqslant r^2$, for any $r \in \left(\alpha, \sqrt{5}\right)$ $(\alpha \approx 1.4813)$.

## 5. Proof of the Corollaries

Corollary 2 is an immediate consequence of Theorem 1 and of inequality [**7**] $\omega \leqslant 1 + \sqrt{2}$, so we shall prove Corollary 1:

It is known [**2**] that for any convex set $K$,

$$d\omega \leqslant 2A \tag{3}$$

and if $2\omega \leqslant \sqrt{3}d$, then equality holds when $K$ is a triangle with basis $d$ and height $\omega$.

Corollary 1 now follows immediately from (3), since

$$\left(\omega - \sqrt{2}\right) d^2 \leqslant \left(\omega - \sqrt{2}\right) \frac{4A^2}{\omega^2} \leqslant 2\sqrt{2}A.$$

The equality occurs here when and only when $K = T_\varepsilon \in \mathcal{T}$, since it is easily seen that for all $T_\varepsilon \in \mathcal{T}$, the inequality $2\omega(T_\varepsilon) \leqslant \sqrt{3}d(T_\varepsilon)$ holds. $\square$

## References


[1] W. Blaschke, *Kreis und Kugel* (Chelsea, New York, 1948).

[2] T. Bonnesen and W. Fenchel, *Theorie der Konvexen Körper* (Springer, Berlin, 1934).

[3] P.R. Scott, 'A lattice problem in the plane', *Mathematika* **20** (1973), 247–252.

[4] P.R. Scott, 'Area-Diameter relations for two-dimensional lattices', *Math. Mag.* **47** (1974), 218–221.

[5] P.R. Scott, 'A family of inequalities for convex sets', *Bull. Austral. Math. Soc.* **20** (1979), 237–245.

[6] P.R. Scott, 'Area, width and diameter of planar convex sets with lattice point constraints', *Indian J. Pure Appl. Math.* **14** (1983), 444–448.

[7] P.R. Scott, 'On planar convex sets containing one lattice point', *Quart. J. Math. Oxford* **36** (1985), 105–111.

[8] P.R. Scott and P.W. Awyong, 'Width-diameter relations for planar convex sets with lattice point constraints', *Bull. Austral. Math. Soc.* **53** (1996), 469–478.

Departamento de Matemáticas
Universidad de Murcia
30100-Murcia
Spain
e-mail: mhcifre@fcu.um.es
salsegom@fcu.um.es