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OVERRINGS OF HALF-FACTORIAL DOMAINS

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ABSTRACT. An atomic integral domain D is a half-factorial domain (HFD) if for any irreducible elements $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$ of D with $\alpha_1 \cdots \alpha_n = \beta_1 \cdots \beta_m$, then $n = m$. In [3], Anderson, Anderson, and Zafrullah explore factorization problems in overrings of HFDs and ask whether a localization of a HFD is again a HFD. We construct an example of a Dedekind domain which is a HFD, but with a localization which is not a HFD. We also give an example of a Dedekind domain where each localization is a HFD, but with an overring which is not a HFD.

Much recent literature has been devoted to the study of factorization properties of atomic integral domains. An integral domain D is *atomic* if every nonzero nonunit of D can be factored as a product of irreducible elements of D. Of particular interest has been the study of half-factorial domains. Recall that an atomic integral domain D is a *half-factorial domain* (HFD) if for any irreducible elements $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$ of D with $\alpha_1 \cdots \alpha_n = \beta_1 \cdots \beta_m$, then $n = m$. In [3], Anderson, Anderson, and Zafrullah consider how HFDs behave under localization and directed unions and pose the question of whether a localization of a HFD is again a HFD. While the authors obtain several positive results for specific types of multiplicative sets (see Theorem 2.4, Corollary 2.5, and Theorem 3.3 in [3]), they never completely resolve their question. In this note, we will use some of the terminology and results of [5] to construct a Dedekind HFD with a localization which is not a HFD. We also give an example of a Dedekind HFD for which each localization is a HFD, but with an overring which is not a HFD (by an overring of R, we mean a subring of the quotient field of R that contains R). The interested reader is also directed to an additional study of overrings of HFDs in Zaks [10].

Define an integral domain D to be a *locally half-factorial domain* (LHFD) if each localization $S^{-1}D$ of D is a HFD and a *strong half-factorial domain* (SHFD) if every overring of D (including D itself) is a HFD. Clearly a SHFD is also a LHFD, and the converse holds if each overring is a localization. In particular, the two are equivalent when D is a Dedekind domain with torsion class group. A UFD is trivially a LHFD. We next use the $D + M$ construction to obtain some less trivial examples.

EXAMPLE 1. Let T be a UFD of the form $K + M$, where M is a nonzero maximal ideal of T and K is a subfield of T. Let D be a subring of K and $R = D + M$. Then R is a HFD if and only if D is a field (Proposition 3.1 of [2]). Let $D = k$ be a subfield of

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K. Suppose, in addition, that T is quasilocal. Then $R[1/m] = T[1/m]$ for each nonzero $m \in M$. Hence R is a LHFD, but not a UFD unless $k = K$.

Next, specialize to $T = K[[X]] = K + M$, where $M = XT$. In this case, $R = k + M$ is a quasilocal, one-dimensional HFD and each overring of R (except its quotient field) has the form $A = B + M$, where B is a subring of K containing k. If K/k is an algebraic extension of fields, then each such subring B is a field, so A is a HFD. Thus R is a SHFD. If, however, K/k is not an algebraic extension, then for $t \in K$ transcendental over k, $A = k[t] + M$ is not a HFD. Hence R is a LHFD, but not a SHFD. One could also use $T = K + XK[X]_{(X)}$. ■

In Theorem 5, we construct a Dedekind HFD which is not a LHFD (and hence not a SHFD), and in Example 7, we construct a Dedekind LHFD which is not a SHFD. For the convenience of the reader, we recall several well known facts about Dedekind domains. Given a Dedekind domain D, let $\mathrm{Cl}(D)$ denote its divisor (ideal) class group, S_D the set of nonzero ideal classes of D which contain prime ideals, and $\bar{I}$ the ideal class of I in $\mathrm{Cl}(D)$.

THEOREM 2. *Let D be a Dedekind domain with class group $\mathrm{Cl}(D)$ and suppose that R is an overring of D.*

1) *R is a Dedekind domain.*
2) *The map $\tau\colon \mathrm{Cl}(D) \longrightarrow \mathrm{Cl}(R)$ defined by $\tau(\bar{I}) = \overline{IR}$ is a surjective homomorphism with kernel generated by all classes $\bar{P}$, where P is a prime ideal of D with $PR = R$.*
3) *$S_R = \tau(S_D) \setminus \{0\}$.*
4) *The following statements are equivalent.*
   a) *Each overring of D is a localization of D.*
   b) *$\mathrm{Cl}(R)$ is a torsion group.*
   c) *Every prime ideal of D is the radical of a principal ideal of D.*

PROOF. 1) is Corollary 13.2 of [7]. 2) is a special case of Nagata's Theorem ([7, Theorem 7.1]) since each overring of a Dedekind domain is a subintersection. 3) follows from 2). 4) is a special case of Proposition 6.8 of [7]. ■

Notice that by 3), if every nonzero ideal class of a Dedekind domain D contains a prime ideal, then the same holds true for any overring R of D. A simple generalization of a well known Theorem of Carlitz [4] states that a Dedekind domain D with the property that each nonzero ideal class contains a prime ideal is a HFD if and only if $|\mathrm{Cl}(D)| \leq 2$. The ring of integers in a finite algebraic number field over the rationals is an example of a Dedekind domain which satisfies the condition of having a prime ideal in each ideal class. Combining Theorem 2 with Carlitz' Theorem leads us to the following observation.

THEOREM 3. *Let D be a Dedekind domain with class group $\mathrm{Cl}(D)$ such that every nonzero ideal class of D contains a prime ideal. Then D is a* SHFD *if and only if $|\mathrm{Cl}(D)| \leq 2$ (and hence if and only if D is a* HFD*).*

PROOF. ($\Rightarrow$) If D is a SHFD, then D itself is a HFD and Carlitz's Theorem implies that $|\mathrm{Cl}(D)| \leq 2$ ($\Leftarrow$) If $|\mathrm{Cl}(D)| \leq 2$ then by Theorem 2 part 2), every overring R of D

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has the property that $|\operatorname{Cl}(R)| \leq 2$. Carlitz's Theorem then implies that each overring is a HFD, and hence D is a SHFD. ∎

We now construct an example of a Dedekind domain which is a HFD, but not a LHFD. Let D be a Dedekind domain with class group $\mathbb{Z}_6$ and suppose that $S_D = \{1, 2, 3\}$ (such a Dedekind domain exists by Corollary 1.5 of Grams [8]). The domain D is a HFD by Theorem 3.8 in [5]. Let $\mathcal{Q} = \{Q \mid Q$ is a prime ideal of D which lies in the class 1 or 2$\}$ and fix P_3 to be a prime ideal of D which lies in class 3. We now claim the following:

LEMMA 4. $P_3 \not\subseteq \bigcup_{Q \in \mathcal{Q}} Q$.

PROOF. Assume that $P_3 \subseteq \bigcup_{Q \in \mathcal{Q}} Q$. Since D has the property that every prime ideal is the radical of a principal ideal, the main Theorem of [9] implies that $P_3 = Q$ for some $Q \in \mathcal{Q}$. But P_3 and Q are in different ideal classes in $\operatorname{Cl}(D)$ (for any $Q \in \mathcal{Q}$), a contradiction. Hence $P_3 \not\subseteq \bigcup_{Q \in \mathcal{Q}} Q$. ∎

Notice in the proof that it is necessary to cite the Theorem from [9] rather than the usual finite covering condition since the number of prime ideals in the set $\mathcal{Q}$ may be infinite.

Now, let $t \in P_3 \setminus \bigcup_{Q \in \mathcal{Q}} Q$. Set $T = \{1, t, t^2, \ldots\}$ and $R = T^{-1}D = D[1/t]$. We claim that R is the desired example. The important property of the element t just selected is that if $(t^k) = M_1 \cdots M_j$, where $M_1, \ldots, M_j$ are prime ideals of D, then each M_i must be an ideal in class 3 (otherwise, if M_w is not in class 3, for some $1 \leq w \leq j$, then M_w is in $\mathcal{Q}$, and hence $t^k \in M_w$ implies $t \in M_w$, which contradicts the fact that $t \notin \bigcup_{Q \in \mathcal{Q}} Q$). We are now ready to prove the claim.

THEOREM 5. *The localization R of D is a Dedekind domain with class group $\mathbb{Z}_3$ such that every nonzero ideal class of R contains a prime ideal. Hence, R is not a* HFD, *and thus D is not a* LHFD.

PROOF. By Theorem 2 part 3), the kernel of τ is generated by primes Q with $Q \cap T \neq \emptyset$. Thus, if Q is such a prime, then there exists a positive integer k such that $t^k \in Q$. Thus Q is in the prime ideal factorization of (t^k), and by the property discussed just prior to this Theorem, Q is in class 3. Thus, $\tau\colon \mathbb{Z}_6 \longrightarrow \operatorname{Cl}(R)$ with $\ker \tau = \{0, 3\}$. Hence $\operatorname{Cl}(R) \cong \mathbb{Z}_3$. Notice that if P_1 and P_2 are prime ideals of D taken from the ideal classes 1 and 2 (of $\mathbb{Z}_6$) respectively, then P_1R and P_2R will lie in different non-trivial ideal classes in R. Consequently, each nonzero ideal class of R will contain a prime ideal, and hence R is not a HFD. ∎

We offer an alternate view of the last result, this time considering the factorizations of elements directly. Let P_1, P_2, and P_3 be the prime ideals of D already cited above. Notice that there exist irreducible elements α, β, γ, and δ of D such that $(\alpha) = P_1P_2P_3$, $(\beta) = P_1^6$, $(\gamma) = P_2^3$, and $(\delta) = P_3^2$. In D we have that

$$\alpha^6 = u_1 \beta \gamma^2 \delta^3, \tag{1}$$

where u_1 is a unit of D. Since $P_3R = R$ and $(\delta) = P_3^2$, we also have that $\delta R = (P_3R)^2 = R$, so δ is a unit in R. Now, $\alpha R = P_1P_2P_3R = (P_1R)(P_2R)$. Since P_1R and P_2R are in nonprincipal ideal classes in R, α is an irreducible element of R. A similar argument works for γ. However, β is neither a unit nor irreducible in R. To see this, note that P_1^3R is principal in R and setting $P_1^3R = \mu R$ (where μ is irreducible in R) we have that $\beta R = (P_1R)^6 = [(P_1R)^3]^2 = \mu^2R$. Thus, in R, (1) reduces to

$$\alpha^6 = u_2 \cdot \mu^2 \cdot \gamma^2,$$

where u_2 is some unit of R. Notice also that

$$\alpha^3 = u_3 \cdot \mu \cdot \gamma,$$

where u_3 is some unit of R. Considering either of the above factorizations, R is clearly not a HFD.

The example we have constructed is minimal in the sense that if D is any Dedekind HFD such that $|\mathrm{Cl}(D)| < 6$, then D is a SHFD (and hence a LHFD). This of course follows since any proper overring R of D would have either

i) $\mathrm{Cl}(R) = \mathrm{Cl}(D)$ with the same distribution of prime ideals in $\mathrm{Cl}(R)$ and $\mathrm{Cl}(D)$, hence forcing R to be a HFD, or

ii) $|\mathrm{Cl}(R)| \leq 2$, in which case the before mentioned Theorem of Carlitz forces R to be a HFD.

We state this result in terms of class numbers.

THEOREM 6. *Let D be a Dedekind domain with $|\mathrm{Cl}(D)| \leq 5$. The following statements are equivalent:*

1) D is a HFD.

2) D is a SHFD.

3) D is a LHFD. ∎

The Grams result [8], along with results from [5], [6] and Theorems 3 and 6 above, allow us to characterize the Dedekind domains with $|\mathrm{Cl}(D)| \leq 5$ which satisfy the three equivalent conditions of Theorem 6. They are as follows:

1) any Dedekind domain D with trivial class group or $\mathrm{Cl}(D) \cong \mathbb{Z}_2$ (by Theorem 3).

2) any Dedekind domain D with $\mathrm{Cl}(D) \cong \mathbb{Z}_3$ or $\mathbb{Z}_5$ with all the nonprincipal prime ideals of D in one ideal class (Corollary 3.3 of [6]).

3) any Dedekind domain D with $\mathrm{Cl}(D) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $S_D \neq \{(1,0),(0,1),(1,1)\}$ (Theorem 4.8 of [5]).

4) any Dedekind domain D with $\mathrm{Cl}(D) \cong \mathbb{Z}_4$ with S_D equal to one of the following: $\{1\}, \{3\}, \{1,2\}, \{2,3\}$ (Corollary 4.7 of [5]).

Using similar techniques, we can characterize the Dedekind domains with class group $\mathbb{Z}_6$ which are SHFDs (and hence LHFDs since $\mathbb{Z}_6$ is a torsion group). The Dedekind domains D with $\mathrm{Cl}(D) \cong \mathbb{Z}_6$ which are HFDs are those with sets S_D of the form: $\{1\}$, $\{5\}$, $\{1,2\}$, $\{5,4\}$, $\{1,3\}$, $\{5,3\}$, $\{2,3\}$, $\{4,3\}$, $\{1,2,3\}$, or $\{5,4,3\}$. We have shown that the HFDs above with $S_D = \{1,2,3\}$ are not SHFDs. A similar argument works when $S_D = \{5,4,3\}$. The remaining possibilities for S_D all produce Dedekind domains which are SHFDs. To see this, notice that the class group of any overring R of such a domain will either be trivial, $\mathbb{Z}_2$, or $\mathbb{Z}_3$. In the last case all the nonprincipal prime ideals of R will be in one ideal class, and again by Corollary 3.3 of [6], R is a HFD.

We next give an example of a Dedekind LHFD R which is not a SHFD (notice, as remarked earlier, that such an R cannot have torsion class group).

EXAMPLE 7. Let R be the Dedekind domain constructed in [1, Example 3.4] via Claborn's Theorem. Using the notation from that example, $\mathrm{Cl}(R) = \langle \overline{e_3} \rangle \cong \mathbb{Z}$. Let $\overline{e_3} = -1 \in \mathbb{Z}$. Then $S_R = \{-1,1,4,6\}$. By Theorem 4.9 of [6], R is HFD. Since each proper localization of R is a PID, R is a LHFD. The only overrings of R which are not PIDs are the three subintersections R_1, R_2, and T (see [1] for the description of these overrings). Note that T is a HFD since $\mathrm{Cl}(T) \cong \mathbb{Z}_2$. However, R_1 and R_2 are not HFDs. $\mathrm{Cl}(R_1) \cong \mathbb{Z}_4$ and $S_{R_1} = \{3,2,1\}$ by part 3) of Theorem 2. Hence R_1 is not a HFD by Carlitz's Theorem. By the discussion before this example, R_2 is not a HFD since $\mathrm{Cl}(R_2) \cong \mathbb{Z}_6$ and $S_{R_2} = \{5,1,4\}$. Thus R is a Dedekind LHFD which is not a SHFD. ∎

We close by mentioning a problem which previous work and the results of this note clearly suggest.

PROBLEM. Let G be a abelian group. Characterize the subsets $S \subseteq G - \{0\}$ such that any Dedekind domain with $\mathrm{Cl}(D) \cong G$ and $S_D = S$ is a HFD, a LHFD, or a SHFD.

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