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DIRICHLET SERIES WITH POSITIVE REAL PART

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We consider the sequence $\Lambda = \{0 < \lambda_2 < \lambda_2 < \ldots\}$, for which $\lambda_n \to +\infty$. We denote by $PD(\Lambda)$ the class of Dirichlet's series having the form $F(s) = \sum_{n=0}^{\infty} a_n \exp\{-\lambda_n s\}(a_0 = 1)$ defined in the half plan $\operatorname{Re} s > 0$ converging absolutely and $\operatorname{Re} F \geqslant 0$. If $N_0 = \{0, 1, 2, \ldots\}$ then the class $PD(N_0)$ coincides with the Caratheodory's class P. In this paper some classical results holding for the class P are generalised in any class $PD(\Lambda)$. In special cases for the sequence Λ extreme problems are examined in the class $PD(\Lambda)$.

INTRODUCTION

We consider the sequence $\Lambda = \{0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots\}$, for which $\lambda_n \to +\infty$. We denote by

(a) $D(\Lambda)$ the class of Dirichlet's series having the form

$$F(s) = \sum_{n=0}^{\infty} \alpha_n \exp\{-\lambda_n s\} \qquad (\alpha_0 = 1)$$

defined in the half-plane $\operatorname{Re} s > 0$ and converging absolutely;

(b) $PD(\Lambda)$ the class

$$\{F \in D(\Lambda) : \operatorname{Re} F \geqslant 0\};$$

(c) D the union of all classes $D(\Lambda)$ and by PD the union of all classes $PD(\Lambda)$.

If we set $\exp\{-\operatorname{Re} s\} = r$, $-\operatorname{Im} s = t$ $(0 \leqslant r < 1,\ -\infty < t < +\infty)$ then every $F(s) \in D$ can be written in the form

$$\widetilde{F}(r, t) = 1 + \sum_{n=1}^{\infty} \alpha_n r^{\lambda_n} \exp\{i\lambda_n t\}.$$

If $N_0 = \{0, 1, 2, \ldots\}$, then the class $PD(N_0)$ coincides with the Caratheodory's class P.

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In [1] the inequality $|\alpha_n| \leqslant 2$, which is true for the class P, is generalised for the class PD.

In [2] it is shown that if $f \in P$, then f is an extreme point of the class P if and only if $|\alpha_1|$ takes the maximal possible value, that is $\alpha_1 = 2\exp\{i\varphi\}$, or, equivalently,

$$f(z) = (1 + \exp\{i\varphi\}z)(1 - \exp\{i\varphi\}z)^{-1}.$$

In the present paper some results holding for the class P are generalised in the class PD.

The form of extreme points of a class $D(\Lambda)$ is decisively affected by the structure of the sequence Λ, hence the solution of this problem is difficult in the general case. This assertion is also implied by Remark 2 of Theorem 2, Theorem 3 and Theorem 4.

Remark 2 of Theorem 2 shows how to find all the extreme elements of a class $PD(\Lambda)$, if the values of the sequence $\Lambda - \{0\}$ form a linearly independent set with respect to the field of rationals.

Theorems 3 and 4 examine, in some specific cases for the sequence Λ, the form of the series

$$\sum_{n=0}^{\infty} \alpha_n \exp\{-\lambda_n s\} \in PD(\Lambda),$$

when $|\alpha_1|$ takes the maximal possible value.

The following lemma from classical Harmonic analysis will be used in the proofs of the theorems.

Lemma 1. *If $f(x)$ is an integrable function in $\mathbb{R}$,*

$$\widehat{f}(t) = \int_{\mathbb{R}} f(x) \exp\{-itx\} dx$$

is the Fourier transform of f and $\operatorname{Re} f \geqslant 0$*; then*

$$\left|\widehat{f}(t) + \bar{\widehat{f}}(-t)\right| \leqslant 2 \operatorname{Re} \widehat{f}(0), \qquad \textit{for every } t \in \mathbb{R}.$$

The proof is obvious.

Theorem 2. *If*

$$F(s) = \sum_{n=0}^{\infty} \alpha_n \exp\{-\lambda_n s\} \in D$$

then the following are equivalent:

(i) $F(s) \in PD$.

(ii) $\left|F(s) - A_k(s)\exp\{\lambda_k \operatorname{Re} s\} - \overline{A}_k(s)\exp\{-\lambda_k \operatorname{Re} s\}\right| \leqslant 2\operatorname{Re} A_k(s)$ *where*

$$A_k(s) = [F_k(s)\exp\{\lambda_k \operatorname{Re} s\} - F_k(-\overline{s})\exp\{-\lambda_k \operatorname{Re} s\}]\cdot [\exp\{\lambda_k \operatorname{Re} s\} - \exp\{-\lambda_k \operatorname{Re} s\}]^{-2},$$

$$F_k(s) = \sum_{n=0}^{k} \alpha_n \exp\{-\lambda_n s\}, \qquad k = 1, 2, \ldots.$$

(iii) *Proposition (ii) is true for at least one natural number* k.

(iv) $\operatorname{Re}\left[\sum_{n=0}^{k} \alpha_n(1 - \lambda_n/\lambda_k)\exp\{i\lambda_n t\}\right] \geqslant 0, \qquad k = 1, 2, \ldots, \quad t \in \mathbb{R}.$

PROOF: (i) $\Rightarrow$ (ii). Let $\sigma > 0$, $c > 0$ and

$$P(x) = \sum_{n=0}^{\infty} \alpha_n \exp\{-\lambda_n \sigma\}\left(\frac{\exp\{-i\lambda_n x\}}{c^2 + x^2}\right).$$

Since

$$\widehat{\left(\frac{1}{c^2 + x^2}\right)} = \frac{\pi}{c}\exp\{-c\,|t|\}$$

it follows that
$$\widehat{P}(t) = \frac{\pi}{c}\sum_{n=0}^{\infty} \alpha_n \exp\{-\lambda_n \sigma - c\,|t - \lambda_n|\}.$$

Applying Lemma 1, the function P, for $t \in [\lambda_k, \lambda_{k-1}]$ becomes

$$|F_k(\sigma - c)\exp\{-ct\} + F(\sigma + c)\exp\{ct\} - F_k(\sigma + c)\exp\{ct\} + \overline{F}(\sigma + c)\exp\{-ct\}| \leqslant 2\operatorname{Re} F(\sigma + c).$$

For $t = \lambda_k$ and $\sigma \to 0$ it becomes

$$(*) \qquad \left|F_k(-c)\exp\{-c\lambda_k\} + F(c)\exp\{c\lambda_k\} - F_k(c)\exp\{c\lambda_k\} + \overline{F}(c)\exp\{-c\lambda_k\}\right| \leqslant 2\operatorname{Re} F(c).$$

If we replace the absolute value with the real part we obtain the evaluation

$$\operatorname{Re} A_k(c) \geqslant 0.$$

Also, taking the square of $(*)$, we obtain

$$\left|F(c) - A_k(c)\exp\{c\lambda_k\} - \overline{A}_k(c)\exp\{-c\lambda_k\}\right|^2 \leqslant [2\operatorname{Re} A_k(c)]^2$$

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which, for $s = c$, is the required result.

The general case, where $s = c + i\tau\,(\operatorname{Re} s > 0)$, is immediately obtained by substituting $F_\tau \in PD(\Lambda)$ for F in the last inequality, where $F_\tau(\omega) = F(\omega + i\tau)$.

(iii) $\Rightarrow$ (i) By (iii), it is obvious that

$$\operatorname{Re} F(s) \geqslant (\exp\{\lambda_k \operatorname{Re} s\} + \exp\{-\lambda_k \operatorname{Re} s\} - 2) \operatorname{Re} A_k(s) \geqslant 0.$$

(i) $\Rightarrow$ (iv) The inequality $\operatorname{Re} A_k(c - it) \geqslant 0$ is equivalent to

$$\operatorname{Re}\{\sum_{n=0}^{k-1} \alpha_n \exp\{i\lambda_n t\} \frac{\exp\{c(\lambda_k - \lambda_n)\} - \exp\{c(\lambda_n - \lambda_k)}{\exp\{c\lambda_k\} - \exp\{-c\lambda_k\}}\} \geqslant 0$$

which, for $c \to 0$, gives the required result.

(iv) $\Rightarrow$ (i) If

$$f(z) = \sum_{n=0}^{k-1} \alpha_n \left(1 - \frac{\lambda_n}{\lambda_k}\right) \exp\{-\lambda_n (1+z)(1-z)^{-1}\}$$

then the function f is bounded in the disc $U = \{|z| < 1\}$, because

$$\operatorname{Re}[(1+z)(1-z)^{-1}] > 0, \qquad \text{for every } z \in U.$$

Furthermore, $\operatorname{Re} f(z) \geqslant 0$ almost everywhere in $\partial U = \{|z| = 1\}$ because

$$\operatorname{Re}[(1+z)(1-z)^{-1}] = 0 \quad \text{almost everywhere in} \quad \partial U = \{|z| = 1\}.$$

From the Poisson integral of the function f, it follows that $\operatorname{Re} f(z) > 0$, for every $z \in U$, or

$$\operatorname{Re}\left\{\sum_{n=0}^{k-1} \alpha_n \left(1 - \frac{\lambda_n}{\lambda_k}\right) \exp\{-\lambda_n s\}\right\} > 0, \quad \text{when } \operatorname{Re} s > 0.$$

For $k \to +\infty$, it follows that $F(s) \in PD$.

REMARK 1. From Part (ii) of Theorem 2, the following evaluation for $|F(s)|$ follows:

$$\begin{aligned} &\left|A_k(s)\exp\{\lambda_k \operatorname{Re} s\} + \overline{A}_k(s)\exp\{-\lambda_k \operatorname{Re} s\}\right| - 2\operatorname{Re} A_k(s) \leqslant |F(s)| \\ &\quad \leqslant \left|A_k(s)\exp\{\lambda_k \operatorname{Re} s\} + \overline{A}_k(s)\exp\{-\lambda_k \operatorname{Re} s\}\right| + 2\operatorname{Re} A_k(s), \quad k = 1, 2, \ldots. \end{aligned}$$

For $k = 1$ and

$$F(r, t) = \sum_{n=0}^{\infty} \alpha_n r^{\lambda_n} \exp\{i\lambda_n t\} \in PD$$

we have

$$\frac{1 - r^{\lambda_1}}{1 + r^{\lambda_1}} \leqslant |F(r, t)| \leqslant \frac{1 + r^{\lambda_1}}{1 - r^{\lambda_1}}.$$

This last inequality generalises the classical evaluation

$$(1-r)(1+r)^{-1} \leqslant |F(r, t)| \leqslant (1+r)(1-r)^{-1}$$

when $F \in P$ in case $F \in PD$.

REMARK 2. For $k = 1$, (iv) is equivalent to the inequality $|\alpha_1| \leqslant \lambda_2(\lambda_2 - \lambda_1)^{-1}$ which is stronger than $|\alpha_1| \leqslant 2$, in the case $\lambda_2 > 2\lambda$.

More generally, if for the natural number ρ, the numbers $\lambda_1, \lambda_2, \dots, \lambda_\rho$ are linearly independent with respect to the field of rational numbers, then (ii), for $k = \rho$, yields

$$\inf_{t\in\mathbb{R}} \sum_{n=0}^{\rho} \alpha_n \left(1 - \frac{\lambda_n}{\lambda_{\rho+1}}\right) \exp\{i\lambda_n t\} = 1 - \sum_{n=1}^{\rho} \left(1 - \frac{\lambda_n}{\lambda_{\rho+1}}\right) |\alpha_n| \geqslant 0$$

(see [3], p.181).

Suppose that the linear independence for the sequence Λ is true for every natural number ρ and $F \in D(\Lambda)$. The following proposition is obvious:

$$F(s) = \sum_{n=0}^{\infty} \alpha_n \exp\{-\lambda_n s\} \in PD(\Lambda) \text{ if and only if } \sum_{n=1}^{\infty} |\alpha_n| \leqslant 1.$$

If there exist two non-zero coefficients $\alpha_\rho = |\alpha_\rho| \exp\{i\vartheta\}$, $\alpha_k = |\alpha_k| \exp\{i\varphi\}$ and $0 < \varepsilon < \min\{|\alpha_\rho|, |\alpha_k|\}$,

$$|\alpha_\rho \pm \varepsilon \exp\{i\vartheta\}| + |\alpha_\lambda \mp \varepsilon \exp\{i\varphi\}| = |\alpha_\rho| + |\alpha_\lambda|.$$

Consequently, $F(s)$ is an extreme element of the class $PD(\Lambda)$ if and only if it has the form

$$F(s) = 1 + \alpha \exp\{-\lambda_k s\}$$

where $|\alpha| = 1$, $k = 1, 2, \dots$. □

THEOREM 3. *If for $\Lambda = \{0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots\}$ it is true that*

$$\lambda_4 + \lambda_1 \geqslant 2\lambda_2, \quad \lambda_{k+4} - \lambda_{k-1} \geqslant 2\lambda_2, \quad k = 1, 2, \dots$$

and
$$F(r, t) = \sum_{n=0}^{\infty} \alpha_n r^{\lambda_n} \exp\{i\lambda_n t\} \in PD(\Lambda)$$

then the following propositions are equivalent:

(i) $\alpha_1 = \lambda_2(\lambda_2 - \lambda_1)^{-1} \exp\{i\varphi\}$;

(ii) $\lambda_k = k\lambda_1$, $\alpha_k = 2\exp\{ik\varphi\}$, $k = 1, 2, \dots$, *or*

$$F(r, t) = [1 + r^{\lambda_1} \exp\{i(t\lambda_1 + \varphi)\}][1 - r^{\lambda_1} \exp\{i(t\lambda_1 + \varphi)\}]^{-1}.$$

THEOREM 4. *If for $\Lambda = \{0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots\}$ it is true that $\lambda_k - \lambda_{k-1} \geqslant \lambda_1$, $k = 1, 2, \dots$ and*

$$F(r, t) = \sum_{n=0}^{\infty} \alpha_n r^{\lambda_n} \exp\{i\lambda_n t\} \in PD(\Lambda)$$

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then the following propositions are equivalent:

(i) $\alpha_1 = 2\exp\{i\varphi\}$;

(ii) $\lambda_k = k\lambda_1$, $\alpha_k = 2\exp\{ik\varphi\}$, *or*

$$F(r,\, t) = [1 + r^{\lambda_1}\exp\{i(t\lambda_1 + \varphi)][1 - r^{\lambda_1}\exp\{i(t\lambda_1 + \varphi)]^{-1}.$$

PROOF OF THEOREM 3: If we consider the function

$$F\left(r,\, t + \frac{\pi - \varphi}{\lambda_1}\right) \in PD(\Lambda)$$

then the general case is reduced to $\alpha_1 = -\lambda_2(\lambda_2 - \lambda_1)^{-1}$. If

$$h_r(x) = \frac{\sin^2 \delta x}{x^2} F(r,\, t), \quad P(t) = \pi\widehat{\left(\frac{\sin^2 \delta x}{x^2}\right)} = \sup(0,\, 2\delta - |t|)$$

then $$\widehat{h}_r(t) = \sum_{n=0}^{\infty} \alpha_n r^n P(t - \lambda_n)$$

and $$\lim_{r\to 0} \widehat{h}_r(0) = 0$$

whenever $2\delta = \lambda_2$. Applying Lemma 1 in the function h_r we have that

$$\lim_{r\to 1} \left|\widehat{h}_r(t) + \widehat{h}_r(-t)\right| = 0$$

or

$$(**) \qquad \sum_{n=0}^{\infty} \alpha_n P(t + \lambda_n) + \sum_{n=0}^{\infty} \overline{\alpha}_n P(t - \lambda_n) = 0.$$

The set $\{\varepsilon\colon 0 < \varepsilon < \lambda_1,\, \lambda_3 - \varepsilon > \lambda_2\}$ is an interval. Setting $t = \varepsilon$ in $(**)$, we have

$$2P(\varepsilon) + \alpha_1 P(\varepsilon + \lambda_1) + \overline{\alpha}_1 P(\lambda_1 - \varepsilon) + \overline{\alpha}_2 P(\lambda_2 - \varepsilon) = 0, \quad \text{or } \alpha_2 = 2.$$

From the inequalities $\lambda_2 \geqslant 2\lambda_1$ (since $|\alpha_1| \leqslant 2$), $\lambda_4 + \lambda_1 \geqslant 2\lambda_2$, it follows that the set

$$\{\varepsilon\colon \lambda_1 > \varepsilon > 0,\, 0 \leqslant \lambda_2 - 2\lambda_1 + \varepsilon < \lambda_2,\, \lambda_4 + \lambda_1 - \lambda_2 - \varepsilon > \lambda_2\}$$

is an interval.

Setting $t = \lambda_2 - \lambda_1 + \varepsilon$ in $(**)$ we have

$$2P(\lambda_2 - \lambda_1 + \varepsilon) + \overline{\alpha}_1 P(\lambda_2 - 2\lambda_1 + \varepsilon) + \overline{\alpha}_2 P(\lambda_1 - \varepsilon) + \overline{\alpha}_3 P(\lambda_3 - \lambda_2 + \lambda_1 - \varepsilon) = 0$$

or $$-\alpha_1\varepsilon + 2\lambda_1\alpha_1 + 2\lambda_2 + \alpha_3 P(\lambda_3 - \lambda_2 + \lambda_1 - \varepsilon) = 0.$$

From the last equality it follows that

$$P(\lambda_3 - \lambda_2 + \lambda_1 - \varepsilon) \neq 0$$

or $$P(\lambda_3 - \lambda_2 + \lambda_1 - \varepsilon) = 2\lambda_2 - \lambda_3 - \lambda_1 + \varepsilon, \quad \alpha_3 = \alpha_1 \quad \text{and} \quad \lambda_3 = 3\lambda_1.$$

In the same manner, if we set $t = \lambda_2 + \varepsilon$ in $(**)$, we obtain the relations $\alpha_4 = 2$ and $\lambda_4 = 2\lambda_2$.

Suppose that for $n \leqslant k+3$ the equalities $\alpha_n = \alpha_{n-2}$, $\lambda_n = n\lambda_1$ when n is odd and $n = (n/2)\lambda_2$ when n is even, hold. We will examine the case $n = k+4$, when k is even.

First, the following inequalities are true:

$$0 < \lambda_{k+2} - \lambda_{k+1} < \lambda_2 \text{ because } \lambda_{k+2} = \lambda_k + \lambda_2$$
$$0 < \lambda_{k+3} - \lambda_{k+2} < \lambda_2 \text{ because } \lambda_{k+2} = \frac{1}{2}(k+2)\lambda_2,\ \lambda_{k+3} = (k+3)\lambda_1,\ \lambda_2 \geqslant 2\lambda_1$$
$$\lambda_2 < \lambda_{k+5} - \lambda_{k+2} \quad \text{because } \lambda_{k+5} - \lambda_k > 2\lambda_2.$$

The above inequalities assure us that the set

$$\{\varepsilon > 0,\ 0 < \lambda_{k+2} - \lambda_{k+1} - \varepsilon < \lambda_2,\ 0 < \lambda_{k+3} - \lambda_{k+2} - \varepsilon < \lambda_2 < \lambda_{k+5} - \lambda_{k+2} - \varepsilon\}$$

is an interval.

If we set $t = \lambda_2 + \lambda_k + \varepsilon = \lambda_{k+2} + \varepsilon$ in the relation $(**)$, then

$$\alpha_{k+1}P(\lambda_{k+2} - \lambda_{k+1} + \varepsilon) + \alpha_{k+2}P(\varepsilon) + \alpha_{k+3}P(\lambda_{k+3} - \lambda_{k+2} - \varepsilon)$$
$$+\alpha_{k+4}P(\lambda_{k+4} - \lambda_{k+2} - \varepsilon) = 0$$

or $$-2\varepsilon + \alpha_{k+4}P(\lambda_{k+4} - \lambda_{k+2} - \varepsilon) = 0.$$

The last equality says that

$$P(\lambda_{k+4} - \lambda_{k+2} - \varepsilon) \neq 0$$

or $$P(\lambda_{k+4} - \lambda_{k+2} - \varepsilon) = \lambda_2 - \lambda_{k+4} + \lambda_{k+2} + \varepsilon,$$
$$\alpha_{k+4} = 2, \qquad \lambda_{k+4} = \lambda_{k+2} + \lambda_2 = \frac{1}{2}(k+4)\lambda_2.$$

In case k is odd we can prove in the same manner that $\lambda_{k+4} = (k+4)\lambda_1$ and $\alpha_{k+4} = \alpha_1$.

By the inequality

$$k\lambda_2 < (2k+1)\lambda_1 < (k+1)\lambda_2, \qquad k = 1, 2, \ldots,$$

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it follows that $\lambda_2 = 2\lambda_1$. □

PROOF OF THEOREM 4: If we consider the function

$$F\left(r, \frac{t-\varphi-\pi}{\lambda_1}\right) \in PD(\Lambda)$$

then Theorem 4 is reduced to the case where $\alpha_1 = -2$, $\lambda_1 = 1$.

From the relation

$$|\alpha_1| \leqslant \lambda_2(\lambda_2 - \lambda_1)^{-1}$$

it follows that $\lambda_2 = 2\lambda_1 = 2$.

If we set $t = \lambda_1$ in $(**)$ of Theorem 3, then we have that $\alpha_2 = 2$.

Suppose that for $k = k_0$ it is true that $\lambda_k = k$ and $\alpha_k = 2(-1)^k$. If we set $t = \lambda_k$ in $(**)$ we have

$$\alpha_k P(0) + \alpha_{k-1} P(\lambda_k - \lambda_{k-1}) + \alpha_{k+1} P(\lambda_{k+1} - \lambda_k) = 0$$

or

$$2(-1)^k + \alpha_{k+1} P(\lambda_{k+1} - \lambda_k) = 0$$

or

$$\alpha_{k+1}[2 - (\lambda_{k+1} - \lambda_k)] = 2(-1)^k.$$

Combining the last equality with the inequalities

$$|\alpha_{k+2}| \leqslant 2, \qquad \lambda_{k+1} - \lambda_k \geqslant 1$$

we have

$$\lambda_{k+1} = \lambda_k + 1 = k \quad \text{and} \quad \alpha_{k+1} = 2(-1)^{k+1}.$$

□

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