ON SCHUR'S SECOND PARTITION THEOREM

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(Received 20 December, 1966)

1. Introduction. In 1926, I. J. Schur proved the following theorem on partitions [3].

THEOREM 1. *The number of partitions of n into parts congruent to* $\pm 1 \pmod 6$ *is equal to the number of partitions of n of the form* $b_1 + \ldots + b_s = n$, *where* $b_i - b_{i+1} \geqq 3$ *and, if* $3 \mid b_i$, *then* $b_i - b_{i+1} > 3$.

Schur's proof was based on a lemma concerning recurrence relations for certain polynomials. In 1928, W. Gleissberg gave an arithmetic proof of a strengthened form of Schur's theorem [2]; however, the combinatorial reasoning in Gleissberg's paper becomes very intricate.

Although claims of simplicity of proof are highly subjective, we shall in §2 give a proof of Schur's theorem which is shorter than the two previous proofs and seems to exhibit the crucial steps more clearly. This new proof depends on Appell's Comparison Theorem [1, p. 101]. In §3, we generalize our technique and prove a new partition theorem of which the following is a special case.

THEOREM 4. *Let* $A(n)$ *denote the number of partitions of n into parts congruent to* 0, 2, 3, 4, 7 (mod 8). *Let* $B(n)$ *denote the number of partitions of n of the form* $n = b_1 + \ldots + b_s$, *where* $b_s \geqq 2$, $b_i \geqq b_{i+1}$, *and, if* b_i *is odd*, $b_i - b_{i+1} \geqq 3$. *Then* $A(n) = B(n)$.

For example, if $n = 15$, the nineteen partitions enumerated by $A(15)$ are 15, 12+3, 11+4, 11+2+2, 10+3+2, 8+7, 8+4+3, 8+3+2+2, 7+4+4, 7+4+2+2, 7+3+3+2, 7+2+2+2+2, 4+4+4+3, 4+4+3+2+2, 4+3+3+3+2, 4+3+2+2+2+2, 3+3+3+3+3, 3+3+3+2+2+2, 3+2+2+2+2+2+2. The nineteen partitions enumerated by $B(15)$ are 15, 13+2, 12+3, 11+4, 11+2+2, 10+5, 10+3+2, 9+6, 9+4+2, 9+2+2+2, 8+7, 8+5+2, 8+4+3, 7+4+4, 7+4+2+2, 7+2+2+2+2, 6+6+3, 6+5+2+2, 4+4+4+3.

Finally in §4, we show how Schur's lemma concerning recurrence relations for certain polynomials is actually a direct corollary of the q-analogue of Gauss's theorem for hypergeometric series.

2. Proof of Theorem 1. Let $\pi(n)$ denote the number of partitions of n of the form $n = b_1 + \ldots + b_s$ with $b_i - b_{i+1} \geqq 3$ and $b_i - b_{i+1} > 3$ if $3 \mid b_i$. Let $\pi_m(n)$ denote the number of partitions just described, with the added condition that $b_1 \leqq m$. By breaking the set of partitions enumerated by $\pi_m(n)$ into two sets, those with largest part less than m and those with largest part equal to m, we see that

$$\pi_{3m+1}(n) = \pi_{3m}(n) + \pi_{3m-2}(n-3m-1), \tag{2.1}$$

$$\pi_{3m+2}(n) = \pi_{3m+1}(n) + \pi_{3m-1}(n-3m-2), \tag{2.2}$$

$$\pi_{3m+3}(n) = \pi_{3m+2}(n) + \pi_{3m-1}(n-3m-3). \tag{2.3}$$

† Partially supported by National Science Foundation Grant GP–5593.

If

$$d_m(q) = 1 + \sum_{n=1}^{\infty} \pi_m(n) q^n,$$

and

$$d(q) = 1 + \sum_{n=1}^{\infty} \pi(n) q^n,$$

then for $|q| < 1$, $d_m(q) \to d(q)$ as $m \to \infty$, since

$$|d(q) - d_m(q)| \leqq \sum_{n=m}^{\infty} p(n) |q|^n,$$

where $p(n)$ is the ordinary partition function. From (2.1), (2.2) and (2.3) we deduce

$$d_{3m+1}(q) = d_{3m}(q) + q^{3m+1} d_{3m-2}(q), \tag{2.4}$$

$$d_{3m+2}(q) = d_{3m+1}(q) + q^{3m+2} d_{3m-1}(q), \tag{2.5}$$

$$d_{3m+3}(q) = d_{3m+2}(q) + q^{3m+3} d_{3m-1}(q). \tag{2.6}$$

Let

$$\alpha_m(q) = d_{3m+2}(q).$$

Then, by (2.6),

$$d_{3m+3}(q) = \alpha_m(q) + q^{3m+3} \alpha_{m-1}(q). \tag{2.7}$$

By (2.5),

$$d_{3m+1}(q) = \alpha_m(q) - q^{3m+2} \alpha_{m-1}(q). \tag{2.8}$$

Hence, by substituting (2.7) and (2.8) into (2.4), we obtain

$$\alpha_m(q) = (1 + q^{3m+1} + q^{3m+2}) \alpha_{m-1}(q) + q^{3m}(1 - q^{3m}) \alpha_{m-2}(q). \tag{2.9}$$

We note that $\alpha_m(q)$ is uniquely determined by (2.9) and the two initial values $\alpha_{-1}(q) = 1$, $\alpha_0(q) = 1 + q + q^2$.

Now, for $|x| < 1$, $|q| < 1$, define $s_n(q)$ by

$$\prod_{n=0}^{\infty} (1 + xq^{3n+1})(1 + xq^{3n+2})(1 - xq^{3n})^{-1} = \sum_{n=0}^{\infty} s_n(q) x^n, \tag{2.10}$$

and let

$$S_n(q) = \prod_{j=1}^{n} (1 - q^{3j}) \, . \, s_n(q).$$

Calling the expression on the left-hand side of (2.10) $f(x; q)$, we have

$$(1 - x) f(x; q) = (1 + xq)(1 + xq^2) f(xq^3; q). \tag{2.11}$$

Hence $s_0(q) = 1$, $s_1(q) = (1 - q)^{-1}$ and, for $n > 1$,

$$s_n(q) - s_{n-1}(q) = q^{3n} s_n(q) + q^{3n-2} s_{n-1}(q) + q^{3n-1} s_{n-1}(q) + q^{3n-3} s_{n-2}(q).$$

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Thus

$$(1-q^{3n})s_n(q) = (1+q^{3n-2}+q^{3n-1})s_{n-1}(q)+q^{3n-3}s_{n-2}(q).$$

Therefore

$$S_n(q) = (1+q^{3n-2}+q^{3n-1})S_{n-1}(q)+q^{3n-3}(1-q^{3n-3})S_{n-2}(q),$$

and

$$S_0(q) = 1, \quad S_1(q) = 1+q+q^2.$$

Hence, by the remarks following (2.6), $S_{n+1}(q) = \alpha_n(q)$.

Thus, for $|x| < 1$, $|q| < 1$,

$$\prod_{n=0}^{\infty} (1+xq^{3n+1})(1+xq^{3n+2})(1-xq^{3n})^{-1} = \sum_{m=0}^{\infty}\left(\alpha_{m-1}(q)x^m \Big/ \prod_{j=1}^{m}(1-q^{3j})\right). \tag{2.12}$$

Hence, by Appell's comparison theorem [**1**, p. 101, with $p_n = 1$],

$$\begin{aligned}\prod_{n=0}^{\infty} (1+q^{3n-1})(1+q^{3n+2})(1-q^{3n+3})^{-1} &= \lim_{x\to 1}(1-x)\sum_{m=0}^{\infty}\left(\alpha_{m-1}(q)x^m \Big/ \prod_{j=1}^{m}(1-q^{3j})\right)\\ &= \lim_{m\to\infty} \alpha_{m-1}(q)\prod_{j=1}^{m}(1-q^{3j})^{-1}\\ &= d(q)\prod_{n=1}^{\infty}(1-q^{3n})^{-1}.\end{aligned}$$

Hence

$$\begin{aligned}d(q) &= \prod_{n=0}^{\infty}(1+q^{3n+1})(1+q^{3n+2})\\ &= \prod_{n=0}^{\infty}(1-q^{6n+1})^{-1}(1-q^{6n+5})^{-1}.\end{aligned} \tag{2.13}$$

Consequently, comparing coefficients of q^N on both sides of (2.13), we see that $\pi(N)$ is also the number of partitions of N into parts congruent to $\pm 1 \pmod 6$.

A slight refinement of the above argument will yield Gleissberg's generalization of Schur's theorem [**2**, p. 374].

3. Generalizations. We may extend our previous argument to prove the following theorem.

THEOREM 2. *Let q be real with $0 < q < 1$, and $a_i \geqq 0$ for $1 \leqq i \leqq r$. If $t_0 = 1$, $t_n = 0$ for $n < 0$, and for $n > 0$*

$$t_n = (1+a_1q^n)t_{n-1} + \sum_{j=2}^{r} a_j q^n t_{n-j}\prod_{s=1}^{j-1}(1-q^{n-s}), \tag{3.1}$$

then

$$\lim_{n\to\infty} t_n = \prod_{m=1}^{\infty}(1+a_1q^m+a_2q^{2m}+\ldots+a_rq^{rm}).$$

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Proof. If here we let

$$f_r(x;q) = \prod_{m=1}^{\infty}(1+a_1xq^m+a_2xq^{2m}+\ldots+a_rxq^{rm})(1-xq^{m-1})^{-1}, \tag{3.2}$$

and write

$$f_r(x;q) = \sum_{n=0}^{\infty}\beta_n(q)x^n, \tag{3.3}$$

then from

$$(1-x)f_r(x;q) = (1+a_1xq+a_2xq^2+\ldots+a_rxq^r)f_r(xq;q) \tag{3.4}$$

we deduce that

$$t_n = \beta_n(q)\prod_{j=1}^{n}(1-q^j). \tag{3.5}$$

Now $t_0 = 1 > 0$. Suppose that, for $0 \leqq n < m$, $t_n > 0$; then

$$t_m - t_{m-1} = a_1q^mt_{m-1} + \sum_{j=2}^{r} a_jq^mt_{m-j}\prod_{s=1}^{j-1}(1-q^{m-s})$$
$$\geqq 0. \tag{3.6}$$

Thus, by mathematical induction, $t_m\,(m > 0)$ is a non-decreasing sequence of positive numbers.

Consequently,

$$t_m \leqq (1+a_1q^m)t_{m-1} + \sum_{j=2}^{r} a_jq^mt_{m-1}$$
$$= (1+(a_1+\ldots+a_r)q^m)t_{m-1}. \tag{3.7}$$

Hence, for all $m \geqq 0$,

$$t_m \leqq \prod_{n=0}^{\infty}(1+(a_1+\ldots+a_r)q^n). \tag{3.8}$$

Thus t_m is a non-decreasing bounded sequence of positive terms, and therefore t_m converges to a limit L.

Hence, by Appell's comparison theorem [**1**, p. 101 with $p_n = 1$], we deduce as in Theorem 1 that

$$\text{iim}_{n\to 0}\, t_n = L = \prod_{m=1}^{\infty}(1+a_1q^m+a_2q^{2m}+\ldots+a_rq^{rm}). \tag{3.9}$$

Thus Theorem 2 is proved.

As an example of Theorem 2, we prove the following partition theorem.

THEOREM 3. *Let $r \geqq 2$ be an integer. Let $P_1(n)$ denote the number of partitions of n into parts which are either even and not congruent to $4r-2 \pmod{4r}$ or odd and congruent to $2r-1, 4r-1 \pmod{4r}$. Let $P_2(n)$ denote the number of partitions of n of the form $n = b_1+\ldots+b_s$, where $b_i \geqq b_{i+1}$, and for b_i odd, $b_i - b_{i+1} \geqq 2r-1$ $(1 \leqq i \leqq s$, where $b_{s+1} = 0)$. Then $P_1(n) = P_2(n)$.*

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Proof. Let $p(n, m)$ denote the number of partitions of n of the type enumerated by $P_2(n)$, with the added restriction that $b_1 \leqq 2m$. Let

$$B_m(q) = 1 + \sum_{n=1}^{\infty} p(n, m) q^n.$$

First we shall prove that

$$p(n, m) - p(n, m-1) = p(n-2m, m) + p(n-2m+1, m-r). \tag{3.10}$$

Now $p(n, m) - p(n, m-1)$ denotes the number of partitions of the type enumerated by $p(n, m)$ with the added restriction that either $2m$ or $2m-1$ is the largest part. If $2m$ is the largest part, remove it. This yields a partition of the type enumerated by $p(n-2m, m)$. If $2m-1$ is the largest part, then the next largest part does not exceed $2m-2r$. Hence, if $2m-1$ is removed from the partition under consideration, we obtain a partition of the type enumerated by $p(n-2m+1, m-r)$. Thus the above procedure establishes a one-to-one correspondence between those partitions enumerated by $p(n, m) - p(n, m-1)$ and the totality of partitions which are enumerated either by $p(n-2m, m)$ or by $p(n-2m+1, m-r)$. Thus (3.10) is established.

Equation (3.10) implies that

$$(1 - q^{2m}) B_m(q) = B_{m-1}(q) + q^{2m-1} B_{m-r}(q). \tag{3.11}$$

Now in Theorem 2 replace q by q^2, then set $a_1 = a_2 = \ldots = a_{r-1} = 0$, $a_r = q^{-1}$. This yields

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} P_2(n) q^n &= \lim_{m \to \infty} B_m(q) \\ &= \prod_{j=1}^{\infty} (1 + q^{2rj-1})(1 - q^{2j})^{-1} \\ &= \prod_{j=1}^{\infty} (1 - q^{4rj-2})(1 - q^{2j})^{-1}(1 - q^{2rq-1})^{-1} \\ &= 1 + \sum_{n=1}^{\infty} P_1(n) q^n. \end{aligned} \tag{3.12}$$

Comparing coefficients on both sides of (3.12), we obtain Theorem 3.

Theorem 4 (stated in the introduction) is obtained from Theorem 3 directly; set $r = 2$ in Theorem 3.

4. Schur's recurrence lemma. The following theorem is a strengthened form of the result Schur originally used to prove Theorem 1. We shall show that the result is a consequence of the q-analogue of Gauss's theorem for hypergeometric series [**4**, p. 97, (3.3.2.5)].

THEOREM 5. *If* $P_0 = 1$,

$$P_n = \prod_{j=1}^{n} (1 + \alpha q^j + z q^{2j}),$$

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and D_n is defined by $D_0 = 1$, $D_1 = 1+\alpha q$,

$$D_n = (1+\alpha q^n)D_{n-1} + zq^n(1-q^{n-1})D_{n-2} \quad (n > 1),$$

then

$$D_n = \sum_{m=0}^{n} (-z)^m q^{m(n+1)-\frac{1}{2}m(m-1)} \begin{bmatrix} n \\ m \end{bmatrix} P_{n-m}, \tag{4.1}$$

where

$$\begin{bmatrix} n \\ m \end{bmatrix} = \prod_{j=1}^{m} (1-q^{n-j+1})(1-q^j)^{-1}.$$

Proof. Let β_1 and β_2 be the roots of the equation $x^2+\alpha x+z = 0$. Then, by (3.3) and (3.4),

$$\prod_{n=1}^{\infty} (1-\beta_1 xq^n)(1-\beta_2 xq^n)(1-xq^{n-1})^{-1} = \sum_{n=0}^{\infty} D_n x^n \prod_{j=1}^{n} (1-q^j)^{-1}. \tag{4.2}$$

But, by the q-analogue of Gauss's theorem [**4**, p. 97, (3.3.2.5)],

$$\begin{aligned}
&\prod_{n=1}^{\infty} (1-\beta_1 xq^n)(1-\beta_2 xq^n)(1-xq^{n-1})^{-1} \\
&= \sum_{N=0}^{\infty} x^N \prod_{j=1}^{N} (1-\beta_1 q^j)(1-\beta_2 q^j)(1-q^j)^{-1} \prod_{h=0}^{\infty} (1-xzq^{h+N+2}) \\
&= \sum_{N=0}^{\infty} \sum_{k=0}^{\infty} P_N x^N (-1)^k z^k q^{(N+2)k+\frac{1}{2}k(k-1)} x^k \prod_{m=1}^{N} (1-q^m)^{-1} \prod_{j=1}^{k} (1-q^j)^{-1} \\
&= \sum_{n=0}^{\infty} \left(\sum_{N+k=n} P_N (-z)^k q^{(N+2)k+\frac{1}{2}k(k-1)} \prod_{m=1}^{N} (1-q^m)^{-1} \prod_{j=1}^{k} (1-q^j)^{-1} \right) x^n,
\end{aligned} \tag{4.3}$$

where the penultimate expression is obtained by expanding the infinite product in the sum and by applying Euler's theorem [**4**, p. 92, (3.2.2.15)].

Comparing coefficients of x^n in the series expansion of (4.2) and (4.3), we obtain

$$\begin{aligned}
D_n &= \sum_{N+k=n} (-z)^k P_N q^{(N+2)k+\frac{1}{2}k(k-1)} \begin{bmatrix} n \\ k \end{bmatrix} \\
&= \sum_{k=0}^{n} (-z)^k q^{(n-k+2)k+\frac{1}{2}k(k-1)} \begin{bmatrix} n \\ k \end{bmatrix} P_{n-k} \\
&= \sum_{k=0}^{n} (-z)^k q^{k(n+1)-\frac{1}{2}k(k-1)} \begin{bmatrix} n \\ k \end{bmatrix} P_{n-k}.
\end{aligned}$$

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