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NORM CONVERGENCE OF T^n

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Introduction. Throughout this paper X will denote a complex Banach space and all operators T will be assumed to be continuous linear transformations from X into X. If T is an operator then $\sigma(T)$, $r(T)$, and $R(T)$ will denote the spectrum of T, the spectral radius of T, and range of T, respectively. This paper contains necessary and sufficient conditions for the (norm) convergence of $\{T^n\}$ when T is an operator on X. The results of this paper generalize results of Yosida and Kakutani [**10**] and of M. Lin [**7**]. Recall that T is quasi-compact if there exists a compact operator K and a positive integer n such that $||T^n - K|| < 1$. In [**10**, Theorem 4, p. 200] Yoshida and Kakutani have proved:

THEOREM 1. (Yosida and Kakutani). *If T is quasi-compact and if there exists a constant C such that $||T^n|| \leqq C$ for all $n = 1, 2, \ldots$ then $\sigma(T) \cap \{z: |z| = 1\} = \{\lambda_1, \ldots \lambda_k\}$, a finite set, where each λ_i is an eigenvalue of finite multiplicity. Furthermore, there exists compact operators $K_1, \ldots, K_m$ and a quasi-compact operator S such that*

$$T^n = \sum_{i=1}^{m} \lambda_i^n K_i + S^n, n = 1, 2, \ldots$$

and

$$||S^n|| \leqq \frac{M}{(1+\epsilon)^n}$$

for some $\epsilon > 0$.

Let S be a topological space and let $C(S)$ denote all bounded continuous scalar-valued functions on S with the sup norm. The following theorem is similar to Theorem 1 and is found in Dunford and Schwartz [**1**, Theorem VIII. 8.6].

THEOREM 2. *If T is a positive quasi-compact operator in $C(S)$ such that T^n/n converges to zero weakly, then the same conclusions found in Theorem 1 are valid.*

M. Lin [**7**, p. 337] has shown the following.

THEOREM 3. (M. Lin) *If T is an operator on X such that $||T^n/n|| \to 0$ then the following are equivalent:*

(1) *$T - I$ has closed range,*

(2) *$T - I$ has closed range and $X = \ker(T - I) \oplus R(T - I)$, and*

(3) *the sequence $\{N^{-1}\sum_{i=1}^{N} T^i\}$ (norm) converges.*

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Necessary and sufficient conditions for T^n to converge. It has already been shown by J. J. Koliha [**5**, Theorem 2.5] that T^n converges if and only if $\sup|\sigma(T) \sim \{1\}| < 1$ and 1 is a pole of $(T - \lambda I)^{-1}$ of order $\leqq 1$. If $\sigma(T) = \{1\}$ take $\sup|\sigma(T) \sim \{1\}| = 0$.

Koliha [**5**, Theorem 3.2] has shown that if (1) $\sup_n ||T^n|| < \infty$, (2) $T - I$ has closed range, (3) $T - I$ has finite descent, and if (4) $\lambda \in \sigma(T) \sim \{1\}$ implies $|\lambda| < 1$, then T^n converges. Using a result of M. Lin the following improvement is true.

THEOREM 4. *If* (1) $||T^n/n|| \to 0$, (2) $T - I$ *has closed range, and* (3) $\lambda \in \sigma(T) \sim \{1\}$ *implies* $|\lambda| < 1$, *then* T^n *converges.*

Proof. Since $||T^n/n|| \to 0$ and $T - I$ has closed range, $X = \ker(T - I) \oplus R(T - I)$ (see [**7**, p. 337]). Since $\ker(T - I)$ and $R(T - I)$ are invariant under T we may write $T = I \oplus A$. Since $T - I$ is invertible on $R(T - I)$, and since $1 \neq \lambda \in \sigma(T)$ implies $|\lambda| < 1$, $A - \lambda I$ is invertible for all $|\lambda| = 1$ so that $r(A) < 1$. $r(A) < 1$ implies $A^n \to 0$ so that $T^n = I \oplus A^n \to I \oplus 0$ and the proof is complete.

THEOREM 5. *If* (1) $X = \ker(T - I) \oplus M$, $T(M) \subseteq M$, (2) $T - I$ *has closed range, and* (3) $\lambda \in \sigma(T) \sim \{1\}$ *implies* $|\lambda| < 1$, *then* T^n *converges.*

Notice that in Theorem 5 the first hypothesis is weaker than the first hypothesis ($||T^n/n|| \to 0$) of Theorem 4. This is true since $||T^n/n|| \to 0$ and $T - I$ has closed range implies $X = \ker(T - I) \oplus R(T - I)$ [**7**, p. 337], but the converse is false (for example take $T = 2I$). Notice that for Theorem 5 there are no *a priori* bounds on $||T^n||$ but that it follows from Theorem 5 that $\sup_n ||T^n||$ is finite. Also notice that the third hypothesis of Theorem 5 allows 1 to be an accumulation point of $\sigma(T)$.

Recall that the *approximate point spectrum* of an operator A, $\sigma_\pi(A)$, is the set of all $\lambda \in \sigma(A)$ such that there exists $||x_n|| = 1$ such that $||(A - \lambda I)x_n|| \to 0$. In [**2**, Problem 63] a proof is given that for any (bounded) operator A on a Hilbert space that $\partial\sigma(A)$ is a subset of $\sigma_\pi(A)$. By appropriately modifying the proof given in [**2**, Problem 63] we have the following lemma for operators on a Banach space.

LEMMA. $\partial\sigma(A) \subseteq \sigma_\pi(A)$.

Proof. Let $\lambda \in \partial\sigma(A)$. Without loss of generality assume $\lambda = 0$. Since $0 \in \partial\sigma(A)$, there exists invertible $A_n \to A$. Suppose, to the contrary, that $0 \notin \sigma_\pi(A)$. Then there exists $\epsilon > 0$ such that $||Ax|| \geqq \epsilon||x||$ for all x. Therefore A is one-to-one and has closed range. Since A is not invertible, the closed set $R(A)$ is not dense in the Banach space X. Thus there exists $y \in X$ and $\delta > 0$ so that $||y - Ax|| \geqq \delta$ for all $x \in X$. Define $x_n = A_n^{-1}y/||A_n^{-1}y||$. Then $||x_n|| = 1$, $||A_nx_n - Ax_n|| \leqq ||A_n - A|| \to 0$, and

$$||A_nx_n - Ax_n|| = ||y/||A_n^{-1}y|| - Ax_n|| = ||y - A(x_n/||A_n^{-1}y||)||/||A_n^{-1}y|| \geqq \delta/||A_n^{-1}y||.$$

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Therefore $||A_n^{-1}y|| \to +\infty$. Hence

$$||Ax_n|| \leqq ||A_nx_n - Ax_n|| + ||A_nx_n|| \leqq ||A_n - A|| + ||y||/||A_n^{-1}y|| \to 0.$$

But this contradicts $||Ax|| \geqq \epsilon ||x||$ for all x and the proof of the lemma is complete.

Proof of Theorem 5. Since $X = \ker (T - I) \oplus M$, $T(M) \subseteq M$, $T = I \oplus A$ so that $T^n = I \oplus A^n$. Therefore, to show T^n converges it suffices to show $A^n \to 0$.

If $1 \notin \sigma(A)$, then since $\lambda \in \sigma(A) \sim \{1\}$ implies $|\lambda| < 1$ (this is true for A since it is true for T), $r(A) < 1$. Therefore, since $||A^n||^{1/n} \to r(A) < 1$, $A^n \to 0$.

Next suppose $1 \in \sigma(A)$. Then since $\partial\sigma(A) \subseteq \sigma_\pi(A)$, $1 \in \sigma_\pi(A)$. Thus there exists $||x_n|| = 1$ such that $||(A - I)x_n|| \to 0$. Since $T - I$ has closed range, $A - I$ has closed range. By construction $A - I$ is one-to-one. Thus $A - I$ is one-to-one and has closed range so there exists $\delta > 0$ such that $||(A - I)x|| \geqq \delta ||x||$ for all x. But this contradicts $||(A - I)x_n|| \to 0$. Therefore $1 \notin \sigma(A)$ and the proof of Theorem 5 is complete.

COROLLARY 1. *$T^n \to 0$ if and only if $r(T) < 1$.*

COROLLARY 2. *If* (1) *$\lambda \in \sigma(T) \sim \{1\}$ implies $|\lambda| < 1$,* (2) *$T - I$ has closed range, and* (3) *$T^n \to Q$ weakly, then $T^n \to Q$ (in norm).*

Proof. Since $Q^2 = Q = TQ = QT$, $X = N \oplus M$ where N and M are invariant under T, $N = R(Q)$, and $M = \ker Q$. It follows that $\ker (T - I) = R(Q)$ so that $X = \ker (T - I) \oplus M$, $T(M) \subseteq M$. Thus the corollary follows from Theorem 5.

Another variation of Theorem 5 is

THEOREM 6. *If* (1) *$||T^n(T - I)|| \to 0$,* (2) *$T - I$ has closed range and* (3) *$X = \ker (T - I) \oplus M$, $T(M) \subseteq M$, then T^n converges.*

Proof. As in the proof of Theorem 5 write $T = I \oplus A$. It follows that $A - I$ is one-to-one and has closed range. Hence there exists $\delta > 0$ such that $||(A - I)x|| \geqq \delta ||x||$ for all $x \in M$. Since $T^n(T - I) \to 0$, $A^n(A - I) \to 0$. Now $||A^n(A - I)x|| = ||(A - I)A^nx|| \geqq \delta ||A^nx||$ so that $\delta ||A^nx|| \leqq ||A^n(A - I)x||$ for all x. Hence $\delta ||A^n|| \leqq ||A^n(A - I)|| \to 0$ which implies $A^n \to 0$ and the proof is complete.

THEOREM 7. *If $T^n \to Q$ then* (1) *$(T - zI)^{-1}$ has a pole of order $\leqq 1$ at $z = 1$,* (2) *$\sup |\sigma(T) \sim \{1\}| < 1$,* (3) *$T - I$ has closed range,* (4) *$X = \ker (T - I) \oplus R(T - I)$, and* (5)

$$Q = -\frac{1}{2\pi i} \int_{|z-1|=\epsilon} (T - zI)^{-1} dz$$

for some $\epsilon > 0$ sufficiently small.

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Proof. (1) and (2) have been proved by J. J. Koliha [**5**, Theorem 2.5]. Since $T^n \to Q$ implies $||T^n/n|| \to 0$ and $||N^{-1} \sum_{n=0}^{N-1} T^n - Q|| \to 0$, we may apply a result of M. Lin (see Theorem 3) to conclude that (3) and (4) are true. To prove (5), choose $\epsilon > 0$ so that $\{z: |z - 1| \leqq \epsilon\} \cap \sigma(T) \subseteq \{1\}$ and define

$$E = -\frac{1}{2\pi i} \int_{|z-1|=\epsilon} (T - zI)^{-1} dz.$$

Then $TE = ET = E = E^2$ [**8**, p. 421]. By (1) $(z - 1)(T - zI)^{-1}$ is analytic in a neighborhood of $z = 1$ so that

$$TE - E = (T - I)E = -\frac{1}{2\pi i} \int_{|z-1|=\epsilon} (z - 1)(T - zI)^{-1} dz = 0.$$

Thus $TE = E$. By (2) there exists $0 < p < 1$ so that $\sigma(T) \sim \{1\} \subseteq \{z:|z| < p\}$. Then

$$I - E = -\frac{1}{2\pi i} \int_{|z|=p} (T - zI)^{-1} dz$$

so that

$$||T^n(I - E)|| = \left\| -\frac{1}{2\pi i} \int_{|z|=p} z^n (T - zI)^{-1} dz \right\|$$

$$\leqq p^{n+1} \sup_{|z|=p} ||(T - zI)^{-1}|| \to 0.$$

Therefore $T^n = T^nE + T^n(I - E) = E + T^n(I - E) \to E$ and the proof is complete.

We conclude this paper with three examples:

Let $X = \mathbf{R}^2$ and $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then $\sigma(T) = \{1\}$ and $T - I$ has closed range. Suppose $X = \ker (T - I) \oplus M$, $T(M) \subseteq M$. One checks that $\ker (T - I) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ and hence $M \neq \{0\}$. Let $x \in M$, $x \neq 0$. Since $T(M) \subseteq M$ and since M has dimension one, $Tx = \lambda x$ for some λ. Since $\sigma(T) = \{1\}$, $\lambda = 1$ and $Tx = x$, i.e. $x \in \ker (T - I)$, a contradiction. Therefore hypothesis (1) cannot be omitted from Theorem 5. If we let $S = -T$ then $r(S) = 1$ and $S - I$ is invertible but $||S^n/n|| \not\to 0$. This shows that in Theorem 3 $r(T) = 1$ cannot replace $||T^n/n|| \to 0$. By letting $X = l_2$ and $T = \text{diag}\,(0, 1/2, 2/3, 3/4, 4/5, \ldots)$ one easily sees that hypotheses (2) and (3) of Theorem 5 cannot be deleted.

This next example shows that Theorem 5 is false if we omit hypothesis (2), i.e. $T - I$ has closed range. Let $X = l_2$. Let $x_1, x_2, \ldots$ be the canonical orthonormal basis for l_2 and define $Ax_n = a_n x_{n+1}$ where $a_n \downarrow 0$, $a_n > 0$ for all n. Then from [**2**, Problem 80] $\sigma(A) = \{0\}$, $\sigma_p(A) = \emptyset$, and A does not have closed range. Let $T = I + A$ so that $\sigma(T) = \{1\}$, $T - I$ does not have closed range, and since ker $(T - I) = \ker A = \{0\}$, $X = \ker (T - I) \oplus X$ and hypothesis (1) is satisfied trivially. One computes that

$$T^n x_k - x_k = a_k a_{k+1} \dots a_{k+n-1} x_{k+n} + n\, a_k a_{k+1} \dots a_{k+n-2} x_{k+n-1} + n\, a_k a_{k+1} \dots a_{k+n-3} x_{k+n-2} + \dots + n\, a_k a_{k+1}.$$

Hence $||(T^n - I)x_k|| \geqq n a_k$. If we let $a_k = 1/k$ for $k = 1, 2, \dots$ then $||(T^n - I)x_n|| \geqq 1$ for all n. Suppose $\{T^n\}$ converged (in norm). Then by Theorem 7, $T^n \to E$ where

$$E = -\frac{1}{2\pi i} \int_{|z-1|=\epsilon} (T - zI)^{-1} dz.$$

Since $\sigma(T) = \{1\}$, $E = I$. Thus, if $\{T^n\}$ converged then $T^n \to I$. But $||(T^n - I)x_n|| \geqq 1$ for all n, $||x_n|| = 1$. Hence $\{T^n\}$ does not converge.

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