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THE TOTAL DISTANCE FOR TOTALLY POSITIVE ALGEBRAIC INTEGERS

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Abstract

Let $P(x)$ be a polynomial of degree d with zeros $\alpha_1, \ldots, \alpha_d$. Stulov and Yang ['An elementary inequality about the Mahler measure', *Involve* **6**(4) (2013), 393–397] defined the *total distance of P* as $\mathrm{td}(P) = \sum_{i=1}^{d} ||\alpha_i| - 1|$. In this paper, using the method of explicit auxiliary functions, we study the spectrum of the total distance for totally positive algebraic integers and find its five smallest points. Moreover, for totally positive algebraic integers, we establish inequalities comparing the total distance with two standard measures and also the trace. We give numerical examples to show that our bounds are quite good. The polynomials involved in the auxiliary functions are found by a recursive algorithm.

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1. Introduction

1.1. The total distance. We recall the definitions of two standard measures of a polynomial $P = a_0x^d + a_1x^{d-1} + \cdots + a_d = a_0(x - \alpha_1)\cdots(x - \alpha_d)$ with complex coefficients $a_0, \ldots, a_d$ and zeros $\alpha_1, \ldots, \alpha_d$:

- the *length of P* is $\mathrm{L}(P) = \sum_{i=0}^{d} |a_i|$;
- the *Mahler measure of P* is $\mathrm{M}(P) = |a_0| \prod_{i=1}^{d} \max(1, |\alpha_i|)$.

In 2013, Stulov and Yang [SY13] defined a less common measure, namely,

- the *total distance of P*, which is $\mathrm{td}(P) = \sum_{i=1}^{d} ||\alpha_i| - 1|$.

If α is an algebraic integer then the Mahler measure of α is the Mahler measure of its minimal polynomial P in $\mathbb{Z}[x]$. If α is a nonzero algebraic integer and $\mathrm{M}(\alpha) = 1$ then a classical theorem of Kronecker [Kro57] tells us that α is a root of unity. This suggests the question: Is $\inf\{\mathrm{M}(\alpha) : \alpha \text{ not a root of unity}\} > 1$? This question is known as *Lehmer's problem*. It is still open. Another formulation of the problem is: does there exist an absolute constant $c > 0$ such that if $\mathrm{M}(\alpha) > 1$ then $\mathrm{M}(\alpha) \geq 1 + c$? The smallest known value of $\mathrm{M}(\alpha)$ greater than 1 is due to Lehmer himself and is $\mathrm{M}(P) = 1.176280\ldots$ where $P(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$.

In their paper, Stulov and Yang established the following inequalities for a complex polynomial P with $|a_d| = |a_0| = 1$:

$$\mathrm{m}(P) \le \mathrm{td}(P) \le 2(\mathrm{M}(P) - 1),$$

where $\mathrm{m}(P) = \log(\mathrm{M}(P))$. Thus, a third version of Lehmer's question can be formulated as follows: does there exist an absolute constant $c' > 0$ such that if $\mathrm{td}(\alpha) > 0$, with $|a_d| = |a_0| = 1$, then $\mathrm{td}(\alpha) \ge c'$?

In this paper, we are interested in totally positive algebraic integers α, that is, algebraic integers all of whose conjugates are positive real numbers. We define the *absolute total distance of* α as $\mathrm{Td}(\alpha) = \mathrm{td}(\alpha)/\deg(\alpha)$. As part of a study of the set $\mathcal{T}d$ of such quantities, we prove the following theorem.

THEOREM 1.1. *If α is a nonzero totally positive algebraic integer whose minimal polynomial is different from $x - 1$, $x^2 - 3x + 1$, $x^3 - 5x^2 + 6x - 1$, $x^5 - 9x^4 + 27x^3 - 32x^2 + 13x - 1$, $x^5 - 9x^4 + 28x^3 - 35x^2 + 15x - 1$ and $x^6 - 11x^5 + 45x^4 - 84x^3 + 70x^2 - 21x + 1$, then*

$$\mathrm{Td}(\alpha) \ge 1.318\,853.$$

The polynomials involved in this result can be read from Table 1.

TABLE 1. Polynomials Q_j involved in Theorem 1.1 with their coefficients c_j.

	c_j	Polynomial Q_j
1	0.106 557 00	x
2	0.712 622 51	$x - 1$
3	0.202 324 50	$x - 2$
4	0.115 672 45	$x^2 - 3x + 1$
5	0.008 661 76	$x^2 - 4x + 2$
6	0.005 009 40	$x^2 - 5x + 5$
7	0.002 539 41	$x^3 - 6x^2 + 9x - 1$
8	0.073 174 71	$x^3 - 5x^2 + 6x - 1$
9	0.004 342 34	$x^3 - 6x^2 + 9x - 3$
10	0.004 069 45	$x^4 - 7x^3 + 14x^2 - 8x + 1$
11	0.007 322 31	$x^5 - 9x^4 + 27x^3 - 32x^2 + 13x - 1$
12	0.014 947 44	$x^5 - 9x^4 + 28x^3 - 35x^2 + 15x - 1$
13	0.014 895 08	$x^5 - 9x^4 + 27x^3 - 31x^2 + 12x - 1$
14	0.000 525 01	$x^5 - 9x^4 + 26x^3 - 29x^2 + 11x - 1$
15	0.000 355 86	$x^6 - 11x^5 + 46x^4 - 90x^3 + 81x^2 - 27x + 1$
16	0.001 830 29	$x^6 - 11x^5 + 44x^4 - 79x^3 + 63x^2 - 18x + 1$
17	0.007 222 08	$x^6 - 11x^5 + 45x^4 - 84x^3 + 70x^2 - 21x + 1$
18	0.002 309 77	$x^8 - 14x^7 + 78x^6 - 221x^5 + 337x^4 - 267x^3 + 94x^2 - 8x - 1$
19	0.000 707 39	$x^8 - 15x^7 + 89x^6 - 268x^5 + 438x^4 - 384x^3 + 166x^2 - 29x + 1$
20	0.003 702 84	$x^8 - 14x^7 + 78x^6 - 222x^5 + 345x^4 - 289x^3 + 120x^2 - 21x + 1$
21	0.001 556 73	$2x^8 - 26x^7 + 136x^6 - 367x^5 + 544x^4 - 435x^3 + 171x^2 - 27x + 1$
22	0.002 179 93	$x^9 - 17x^8 + 119x^7 - 444x^6 + 954x^5 - 1188x^4 + 817x^3 - 276x^2 + 37x - 1$
23	0.001 604 14	$x^9 - 17x^8 + 119x^7 - 444x^6 + 954x^5 - 1187x^4 + 812x^3 - 268x^2 + 33x - 1$
24	0.000 677 00	$x^9 - 17x^8 + 119x^7 - 444x^6 + 953x^5 - 1181x^4 + 800x^3 - 259x^2 + 31x - 1$

25	0.003 371 74	$x^9 - 17x^8 + 120x^7 - 456x^6 + 1011x^5 - 1324x^4 + 986x^3 - 376x^2 + 57x - 1$
26	0.000 315 96	$x^{10} - 18x^9 + 135x^8 - 550x^7 + 1330x^6 - 1959x^5 + 1737x^4 - 886x^3 + 238x^2 - 28x + 1$
27	0.007 538 97	$x^{10} - 18x^9 + 136x^8 - 562x^7 + 1388x^6 - 2104x^5 + 1937x^4 - 1036x^3 + 294x^2 - 36x + 1$
28	0.004 155 71	$x^{10} - 19x^9 + 152x^8 - 668x^7 + 1764x^6 - 2873x^5 + 2841x^4 - 1602x^3 + 446x^2 - 44x + 1$
29	0.001 522 77	$x^{10} - 18x^9 + 136x^8 - 562x^7 + 1387x^6 - 2096x^5 + 1913x^4 - 1002x^3 + 271x^2 - 30x + 1$
30	0.000 388 75	$x^{10} - 18x^9 + 135x^8 - 549x^7 + 1320x^6 - 1921x^5 + 1667x^4 - 820x^3 + 208x^2 - 23x + 1$
31	0.000 211 83	$x^{10} - 17x^9 + 121x^8 - 470x^7 + 1090x^6 - 1556x^5 + 1369x^4 - 734x^3 + 237x^2 - 43x + 3$
32	0.000 097 80	$x^{11} - 20x^{10} + 171x^9 - 819x^8 + 2418x^7 - 4560x^6 + 5502x^5 - 4135x^4 + 1822x^3 - 420x^2 + 40x - 1$
33	0.000 126 07	$x^{11} - 21x^{10} + 189x^9 - 954x^8 + 2967x^7 - 5879x^6 + 7412x^5 - 5760x^4 + 2571x^3 - 573x^2 + 46x - 1$
34	0.000 725 32	$x^{12} - 23x^{11} + 231x^{10} - 1332x^9 + 4878x^8 - 11854x^7 + 19398x^6 - 21250x^5 + 15190x^4 - 6738x^3 + 1695x^2 - 202x + 7$
35	0.000 305 07	$x^{12} - 22x^{11} + 210x^{10} - 1142x^9 + 3906x^8 - 8753x^7 + 12982x^6 - 12579x^5 + 7676x^4 - 2759x^3 + 523x^2 - 43x + 1$
36	0.000 167 45	$x^{12} - 23x^{11} + 230x^{10} - 1314x^9 + 4741x^8 - 11278x^7 + 17930x^6 - 18906x^5 + 12849x^4 - 5324x^3 + 1218x^2 - 127x + 4$
37	0.001 459 80	$x^{13} - 24x^{12} + 252x^{11} - 1523x^{10} + 5865x^9 - 15051x^8 + 26160x^7 - 30702x^6 + 23858x^5 - 11830x^4 + 3529x^3 - 575x^2 + 43x - 1$
38	0.000 837 11	$x^{14} - 26x^{13} + 300x^{12} - 2027x^{11} + 8911x^{10} - 26781x^9 + 56259x^8 - 82985x^7 + 85077x^6 - 59065x^5 + 26496x^4 - 7089x^3 + 981x^2 - 54x + 1$
39	0.000 404 96	$x^{14} - 26x^{13} + 299x^{12} - 2006x^{11} + 8720x^{10} - 25793x^9 + 53052x^8 - 76192x^7 + 75625x^6 - 50595x^5 + 21850x^4 - 5679x^3 + 790x^2 - 48x + 1$
40	0.000 312 72	$x^{18} - 35x^{17} + 560x^{16} - 5430x^{15} + 35664x^{14} - 167990x^{13} + 586104x^{12} - 1542459x^{11} + 3089299x^{10} - 4717066x^9 + 5466151x^8 - 4755434x^7 + 3050835x^6 - 1405114x^5 + 446641x^4 - 92341x^3 + 11292x^2 - 691x + 15$
41	0.000 115 11	$x^{19} - 35x^{18} + 560x^{17} - 5431x^{16} + 35692x^{15} - 168343x^{14} + 588756x^{13} - 1555701x^{12} + 3135698x^{11} - 4834464x^{10} + 5683528x^9 - 5050713x^8 + 3343089x^7 - 1612543x^6 + 549476x^5 - 126561x^4 + 18484x^3 - 1556x^2 + 65x - 1$
42	0.000 420 98	$x^{19} - 35x^{18} + 560x^{17} - 5431x^{16} + 35692x^{15} - 168343x^{14} + 588756x^{13} - 1555701x^{12} + 3135698x^{11} - 4834465x^{10} + 5683542x^9 - 5050795x^8 + 3343351x^7 - 1613042x^6 + 550057x^5 - 126968x^4 + 18645x^3 - 1587x^2 + 67x - 1$
43	0.000 343 12	$x^{19} - 35x^{18} + 561x^{17} - 5461x^{16} + 36098x^{15} - 171626x^{14} + 606463x^{13} - 1623033x^{12} + 3321706x^{11} - 5213487x^{10} + 6255933x^9 - 5688941x^8 + 3862060x^7 - 1913635x^6 + 669910x^5 - 158075x^4 + 23448x^3 - 1962x^2 + 77x - 1$
44	0.000 855 92	$x^{19} - 35x^{18} + 560x^{17} - 5431x^{16} + 35692x^{15} - 168344x^{14} + 588778x^{13} - 1555913x^{12} + 3136877x^{11} - 4838661x^{10} + 5693555x^9 - 5067109x^8 + 3361466x^7 - 1626464x^6 + 556386x^5 - 128692x^4 + 18858x^3 - 1588x^2 + 66x - 1$
45	0.001 245 61	$x^{19} - 35x^{18} + 561x^{17} - 5461x^{16} + 36099x^{15} - 171649x^{14} + 606698x^{13} - 1624441x^{12} + 3327206x^{11} - 5228221x^{10} + 6283668x^9 - 5725945x^8 + 3896983x^7 - 1936732x^6 + 680478x^5 - 161381x^4 + 24145x^3 - 2055x^2 + 83x - 1$
46	0.000 318 96	$x^{19} - 35x^{18} + 561x^{17} - 5460x^{16} + 36071x^{15} - 171298x^{14} + 604092x^{13} - 1611668x^{12} + 3283605x^{11} - 5121678x^{10} + 6095068x^9 - 5483972x^8 + 3674067x^7 - 1792070x^6 + 616291x^5 - 142721x^4 + 20802x^3 - 1722x^2 + 69x - 1$

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We deduce from this result the first five points of the spectrum of Td:
$1.118\,034\ldots = \mathrm{Td}(x^2 - 3x + 1)$,
$1.201\,291\ldots = \mathrm{Td}(x^3 - 5x^2 + 6x - 1)$,
$1.291\,483\ldots = \mathrm{Td}(x^5 - 9x^4 + 28x^3 - 35x^2 + 15x - 1)$,
$1.312\,968\ldots = \mathrm{Td}(x^6 - 11x^5 + 45x^4 - 84x^3 + 70x^2 - 21x + 1)$,
$1.317\,559\ldots = \mathrm{Td}(x^5 - 9x^4 + 27x^3 - 32x^2 + 13x - 1)$.

We conjecture that the next smallest point of the spectrum has minimal polynomial $x^9 - 17x^8 + 120x^7 - 456x^6 + 1011x^5 - 1324x^4 + 986x^3 - 376x^2 + 57x - 1$ and absolute total distance $1.329\,113\ldots$.

It is natural to compare the total distance with other standard measures. We establish the following inequalities.

THEOREM 1.2. *If α is a nonzero totally positive algebraic integer of degree d whose minimal polynomial is different from $x - 1$, $x - 2$, $x^2 - 3x + 1$ and $x^2 - 5x + 5$, then*

$$0.227\,804\,\mathrm{m}(\alpha) + 1.077\,638d \le \mathrm{td}(\alpha) \le 86.546\,972\,\mathrm{m}(\alpha) - 43.686\,533d.$$

THEOREM 1.3. *If α is a nonzero totally positive algebraic integer of degree d whose minimal polynomial is different from $x - 1$, $x - 2$, $x - 3$ and $x^2 - 3x + 1$, then*

$$0.026\,162\,\mathrm{td}(\alpha) + 0.689\,186d \le \log \mathrm{L}(\alpha) \le 1.084\,996\,\mathrm{td}(\alpha) - 0.149\,578d.$$

Let $P = x^d + a_1 x^{d-1} + \cdots + a_d = (x - \alpha_1)\cdots(x - \alpha_d)$ be a monic polynomial with complex coefficients.The *trace* of P is defined by

$$\mathrm{trace}(P) = \sum_{i=1}^{d} \alpha_i = -a_1.$$

Although the trace is not a measure in general, it may be considered as a measure for minimal polynomials of totally positive algebraic integers. We can compare the trace and the toal distance to obtain the following inequalities.

THEOREM 1.4. *If α is a nonzero totally positive algebraic integer of degree d whose minimal polynomial is different from $x - 1$, $x - 2$, $x^2 - 3x + 1$, $x^2 - 4x + 2$, $x^3 - 6x^2 + 8x - 2$ and $x^3 - 6x^2 + 8x - 1$, then*

$$0.236\,643\,\mathrm{td}(\alpha) + 1.307\,819d \le \mathrm{trace}(\alpha) \le 2.211\,970\,\mathrm{td}(\alpha) - 0.635\,402d.$$

In Tables 2–4 we give numerical examples where we can see that our bounds are quite good. In particular, Table 2 shows that our bounds are better than those of Stulov and Yang for the set of polynomials that we consider. For reciprocal polynomials P, that is, $P(x) = x^{\deg(P)}P(1/x)$, Stulov and Yang proved the following stronger lower bound: $\mathrm{td}(P) \ge 2\mathrm{m}(P)$. We give in Table 5 a set of reciprocal polynomials for which our lower bound is even better than their bound.

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1.2. The explicit auxiliary function. The *absolute trace* of a totally positive algebraic integer α of degree $d \geq 2$ whose conjugates are $\alpha_1 = \alpha, \ldots, \alpha_d$ is $\mathcal{T}race(\alpha) = \mathrm{trace}(\alpha)/d = \sum_{i=1}^{d} \alpha_i/d$. The *Schur–Siegel–Smyth problem* is the following: fix $\rho < 2$, then show that all but finitely many totally positive algebraic integers α have $\mathcal{T}race(\alpha) > \rho$. The problem was solved by Schur in 1918 for $\rho < \sqrt{e}$ [Sch18], then by Siegel in 1945 for $\rho < 1.7337$ [Sie45]. In 1984, Smyth developed the method of explicit auxiliary functions and solved the problem for $\rho < 1.7719$ [Smy84]. All subsequent results were obtained thanks to this method. For a complete survey of this problem, see [ABP06].

Until 2003, the polynomials involved in the auxiliary functions were found by heuristic methods. In 2003, when studying the linear independence measure of logarithms of rational numbers, Wu [Wu03] developed an algorithm which finds suitable polynomials by a systematic search. The author [Fla09] improved this algorithm in 2006 so that suitable polynomials can be found by recursion. The best result to date for the trace problem is for $\rho < 1.792\,519$ [Fla].

TABLE 2. Comparison of the bounds of Stulov and Yang and the author's bounds. Here P denotes the minimal polynomial of a totally positive algebraic integer of degree d.

	The lower bound of Stulov and Yang	The author's lower bound		The author's upper bound	The upper bound of Stulov and Yang
P	$\mathrm{m}(P)$	$0.227\,804\,\mathrm{m}(P) + 1.077\,638d$	$\mathrm{td}(P)$	$86.546\,972\,\mathrm{m}(P) - 43.686\,533d$	$2\mathrm{M}(P) - 1$
P_1	2.167 873 7	4.804 402 3	5.330 704 3	12.876 775	15.479 363
P_2	2.087 985 7	4.786 203 5	5.456 219 4	5.962 705 3	14.137 292
P_3	2.810 228 7	6.028 371 3	6.587 798 4	24.784 124	31.227 436
P_4	2.883 792 7	6.045 129 5	6.457 414 8	31.150 865	33.763 933
P_5	2.809 678 7	6.028 246 1	6.710 971 3	24.736 521	31.209 166
P_6	2.748 125 1	6.014 223 9	6.761 143 1	19.409 241	29.226 662
P_7	3.460 063 8	7.254 044 4	7.985 274 7	37.338 850	61.638 016
P_8	3.532 515 8	7.270 549 2	7.877 810 3	43.609 351	66.419 852
P_9	3.369 281 9	7.233 363 9	8.437 066 1	29.481 952	56.115 309
P_{10}	4.064 714 8	8.469 424 3	10.133 815	45.983 024	114.496 58
P_{11}	4.134 475 5	8.485 316 1	9.976 043 4	52.020 603	122.913 65
P_{12}	4.648 035 2	9.679 945 0	10.973 399	52.781 105	206.759 39
P_{13}	4.458 649 7	9.636 802 2	10.172 583	36.390 369	170.741 61
P_{14}	5.485 819 7	10.948 434	12.480 484	81.602 289	480.493 24
P_{15}	5.417 525 9	10.932 876	11.962 019	75.691 664	448.641 92
P_{16}	5.490 440 8	10.949 486	12.464 721	82.002 232	482.728 05
P_{17}	5.506 405 0	10.953 123	12.519 260	83.383 879	490.528 41
P_{18}	6.020 437 6	12.147 860	13.500 532	84.185 315	821.517 49
P_{19}	5.660 547 1	12.065 875	13.366 620	53.037 882	572.611 57
P_{20}	5.571 342 6	12.045 554	13.433 823	45.317 500	523.573 35

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P_{21}	5.566 146 0	12.044 370	13.488 010	44.867 748	520.849 22
P_{22}	6.825 716 5	14.486 582	16.412 179	66.506 699	1840.472 5
P_{23}	6.846 512 1	14.491 319	16.464 912	68.306 499	1879.189 0
P_{24}	7.649 419 8	15.751 862	17.801 539	94.109 190	4196.854 2
P_{25}	8.217 148 2	16.958 831	18.935 244	99.557 834	7405.848 9
P_{26}	8.941 950 9	19.279 220	21.778 250	74.914 243	15290.198
P_{27}	11.016 533	22.984 732	25.502 603	123.403 42	121742.50

where

$$P_1 = x^4 - 7x^3 + 14x^2 - 8x + 1$$
$$P_2 = x^4 - 7x^3 + 13x^2 - 7x + 1$$
$$P_3 = x^5 - 9x^4 + 27x^3 - 32x^2 + 13x - 1$$
$$P_4 = x^5 - 9x^4 + 28x^3 - 35x^2 + 15x - 1$$
$$P_5 = x^5 - 9x^4 + 27x^3 - 31x^2 + 12x - 1$$
$$P_6 = x^5 - 9x^4 + 26x^3 - 29x^2 + 11x - 1$$
$$P_7 = x^6 - 11x^5 + 44x^4 - 79x^3 + 63x^2 - 18x + 1$$
$$P_8 = x^6 - 11x^5 + 45x^4 - 84x^3 + 70x^2 - 21x + 1$$
$$P_9 = x^6 - 11x^5 + 43x^4 - 72x^3 + 51x^2 - 14x + 1$$
$$P_{10} = x^7 - 13x^6 + 63x^5 - 143x^4 + 157x^3 - 78x^2 + 16x - 1$$
$$P_{11} = x^7 - 13x^6 + 64x^5 - 150x^4 + 172x^3 - 89x^2 + 18x - 1$$
$$P_{12} = x^8 - 15x^7 + 89x^6 - 268x^5 + 438x^4 - 384x^3 + 166x^2 - 29x + 1$$
$$P_{13} = x^8 - 14x^7 + 78x^6 - 221x^5 + 337x^4 - 267x^3 + 94x^2 - 8x - 1$$
$$P_{14} = x^9 - 17x^8 + 119x^7 - 444x^6 + 954x^5 - 1188x^4 + 817x^3 - 276x^2 + 37x - 1$$
$$P_{15} = x^9 - 17x^8 + 120x^7 - 456x^6 + 1011x^5 - 1324x^4 + 986x^3 - 376x^2 + 57x - 1$$
$$P_{16} = x^9 - 17x^8 + 119x^7 - 444x^6 + 954x^5 - 1187x^4 + 812x^3 - 268x^2 + 33x - 1$$
$$P_{17} = x^9 - 17x^8 + 119x^7 - 444x^6 + 953x^5 - 1181x^4 + 800x^3 - 259x^2 + 31x - 1$$
$$P_{18} = x^{10} - 19x^9 + 152x^8 - 668x^7 + 1764x^6 - 2873x^5 + 2841x^4 - 1602x^3 + 446x^2 - 44x + 1$$
$$P_{19} = x^{10} - 18x^9 + 135x^8 - 549x^7 + 1320x^6 - 1920x^5 + 1662x^4 - 813x^3 + 206x^2 - 24x + 1$$
$$P_{20} = x^{10} - 18x^9 + 134x^8 - 538x^7 + 1273x^6 - 1822x^5 + 1560x^4 - 766x^3 + 200x^2 - 24x + 1$$
$$P_{21} = x^{10} - 18x^9 + 134x^8 - 537x^7 + 1265x^6 - 1798x^5 + 1526x^4 - 743x^3 + 194x^2 - 24x + 1$$
$$P_{22} = x^{12} - 22x^{11} + 208x^{10} - 1108x^9 + 3667x^8 - 7851x^7 + 10995x^6 - 9977x^5 + 5702x^4 - 1952x^3 + 368x^2 - 33x + 1$$
$$P_{23} = x^{12} - 22x^{11} + 208x^{10} - 1108x^9 + 3666x^8 - 7840x^7 + 10948x^6 - 9877x^5 + 5589x^4 - 1885x^3 + 349x^2 - 31x + 1$$
$$P_{24} = x^{13} - 24x^{12} + 252x^{11} - 1523x^{10} + 5865x^9 - 15051x^8 + 26160x^7 - 30702x^6 + 23858x^5 - 11830x^4 + 3529x^3 - 575x^2 + 43x - 1$$
$$P_{25} = x^{14} - 26x^{13} + 299x^{12} - 2006x^{11} + 8720x^{10} - 25793x^9 + 53052x^8 - 76192x^7 + 75625x^6 - 50595x^5 + 21850x^4 - 5679x^3 + 790x^2 - 48x + 1$$

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$$P_{26} = x^{16} - 29x^{15} + 375x^{14} - 2859x^{13} + 14320x^{12} - 49711x^{11} + 123001x^{10} - 219813x^{9} + 284591x^{8} - 265614x^{7} + 176517x^{6} - 81849x^{5} + 25677x^{4} - 5199x^{3} + 630x^{2} - 40x + 1$$

$$P_{27} = x^{19} - 35x^{18} + 560x^{17} - 5431x^{16} + 35692x^{15} - 168343x^{14} + 588756x^{13} - 1555701x^{12} + 3135698x^{11} - 4834465x^{10} + 5683542x^{9} - 5050795x^{8} + 3343351x^{7} - 1613042x^{6} + 550057x^{5} - 126968x^{4} + 18645x^{3} - 1587x^{2} + 67x - 1$$

TABLE 3. Comparison of the trace and the total distance of some polynomials P which are minimal polynomials of totally positive algebraic integers of degree d.

P	$0.236\,643\ \mathrm{td}(P) + 1.077\,638d$	trace(P)	$2.211\,970\ \mathrm{td}(P) - 0.635\,402d$
P_1	6.492 749 8	7	9.753 839 6
P_2	6.522 452 1	7	10.113 062
P_3	8.098 051 4	9	11.975 985
P_4	8.067 197 0	9	11.602 829
P_5	8.127 199 4	9	12.328 504
P_6	8.139 072 2	9	12.472 095
P_7	9.736 573 4	11	14.599 902
P_8	9.711 142 7	11	14.292 340
P_9	9.843 486 6	11	15.892 924
P_{10}	11.552 829	13	19.373 356
P_{11}	11.515 494	13	18.921 816
P_{12}	13.059 330	15	20.400 591
P_{13}	12.869 823	14	18.108 666
P_{14}	14.723 790	17	23.338 209
P_{15}	14.601 099	17	21.854 367
P_{16}	14.720 060	17	23.293 094
P_{17}	14.732 966	17	23.449 186
P_{18}	16.272 996	19	24.881 930
P_{19}	16.241 307	18	24.498 675
P_{20}	16.257 210	18	24.691 011
P_{21}	16.270 033	18	24.846 094
P_{22}	19.577 655	22	30.463 744
P_{23}	19.590 134	22	30.614 665
P_{24}	21.214 257	24	33.064 432
P_{25}	22.790 359	26	34.933 438
P_{26}	26.078 774	29	40.318 803
P_{27}	30.883 573	35	46.850 926

The method of explicit auxiliary functions has also been used to get the greatest lower bound of the absolute measure of an algebraic integer α, all of whose conjugates lie in a sector $S_\theta = \{z \in \mathbb{C} : |\arg z| \le \theta\}$ with $0 < \theta < \pi$. Langevin [Lan86] proved that there exists a function $c(\theta)$ on $[0, \pi)$, always greater than 1, such that if $\alpha \neq 0$ is not a root of unity and its conjugates all lie in S_θ, then $\Omega(\alpha) = \mathrm{M}(\alpha)^{1/d} \ge c(\theta)$. Note that this result is not explicit. Rhin and Smyth [RS95] succeeded in finding the exact value of $c(\theta)$ for θ in nine subintervals of $[0, 2\pi/3]$ and conjectured that $c(\theta)$ is a 'staircase'

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TABLE 4. Comparison of the length and the total distance of some polynomials P which are minimal polynomials of totally positive algebraic integers of degree d.

P	$0.236\,643\,\mathrm{td}(P) + 1.077\,638d$	trace(P)	$2.211\,970\,\mathrm{td}(P) - 0.635\,402d$
P_1	2.896 205 9	3.433 987 2	5.185 480 8
P_2	2.899 489 6	3.367 295 8	5.321 664 2
P_3	3.618 280 0	4.418 840 6	6.399 844 9
P_4	3.614 868 9	4.488 636 4	6.258 379 3
P_5	3.621 502 4	4.394 449 2	6.533 487 0
P_6	3.622 815 0	4.343 805 4	6.587 923 2
P_7	4.344 026 8	5.379 897 4	7.766 523 1
P_8	4.341 215 3	5.451 038 5	7.649 924 6
P_9	4.355 846 5	5.262 690 2	8.256 715 0
P_{10}	5.089 422 9	6.156 979 0	9.948 102 7
P_{11}	5.085 295 2	6.230 481 4	9.776 921 2
P_{12}	5.800 574 1	7.237 778 2	10.709 470
P_{13}	5.779 623 1	6.926 577 0	9.840 587 7
P_{14}	6.529 188 4	8.256 866 8	12.195 073
P_{15}	6.515 624 3	8.377 701 2	11.632 541
P_{16}	6.528 776 0	8.252 185 4	12.177 970
P_{17}	6.530 202 9	8.244 334 0	12.237 145
P_{18}	7.245 060 9	9.250 618 2	13.152 243
P_{19}	7.241 557 5	8.802 221 7	13.006 949
P_{20}	7.243 315 7	8.754 160 7	13.079 865
P_{21}	7.244 733 3	8.738 895 7	13.138 657
P_{22}	8.699 607 4	10.642 683	16.012 213
P_{23}	8.700 987 0	10.634 051	16.069 428
P_{24}	9.425 1419	11.690 352	17.370 085
P_{25}	10.143 988	12.678 190	18.450 572
P_{26}	11.596 739	14.038 836	21.236 066
P_{27}	13.761 733	17.100 651	24.828 240

function of θ, which is constant except for finitely many left discontinuities in any closed subinterval of $[0, \pi)$. The polynomials involved in the auxiliary functions were found by heuristic methods. In 2004, thanks to Wu's algorithm [Wu03], Rhin and Wu [RW04] gave the exact value of $c(\theta)$ for four new subintervals of $[0, \pi)$ and extended four existing subintervals. In 2013, the author and Rhin [FR] found for the first time a complete subinterval and a fourteenth subinterval. These improvements are due to the recursive algorithm. The author [Fla08] computed the greatest lower bound $c(\theta)$ of the absolute trace for θ belonging to seven subintervals of $[0, \pi/2)$.

As a final example, Wu and Wang [WW14] used the method of auxiliary functions to improve the irrationality measure of the number log 3.

In Section 2 we explain the method of explicit auxiliary functions. We link it with a generalisation of the classical integer transfinite diameter. Then we detail how our recursive algorithm enables us to get the lower bound in Theorem 1.1 for the total distance. In Section 3 we only give the auxiliary functions needed to obtain

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the inequalities in Theorems 1.2–1.4. For the rest of the proof, we proceed as in Theorem 1.1. All the computations have been done on a MacBookPro with the languages Pari and Pascal.

TABLE 5. Comparison of the bounds of Stulov and Yang and the author's for some reciprocal polynomials P which are minimal polynomials of totally positive reciprocal algebraic integers of degree d.

P	The improved lower bound of Stulov and Yang $2\,\mathrm{m}(P)$	The author's lower bound $0.227\,804\,\mathrm{m}(P) + 1.077\,638d$	$\mathrm{td}(P)$	The author's upper bound $86.546\,972\,\mathrm{m}(P) - 43.686\,533d$	The upper bound of Stulov and Yang $2\,\mathrm{M}(P) - 1$
P_1	4.175 971 3	5.213 489 5	5.456 219 4	5.962 705 3	14.137 292
P_2	4.452 345 4	5.428 552 2	6.441 205 3	17.922 373	16.528 681
P_3	6.476 695 3	7.985 778 3	8.701 710 2	18.149 987	48.983 133
P_4	6.524 108 0	8.022 672 8	9.639 090 6	20.201 699	50.206 196
P_5	8.585 745 2	10.608 914	11.898 492	22.042 862	144.352 75
P_6	8.793 221 3	10.770 363	12.885 760	31.021 073	160.350 54
P_7	8.671 494 9	10.675 641	11.919 817	25.753 549	150.764 06
P_8	8.712 173 2	10.707 295	12.864 449	27.513 840	153.902 96
P_9	13.076 541	16.067 387	18.359 886	41.629 114	1380.180 5
P_{10}	15.285 923	18.768 598	21.581 474	49.863 733	4169.825 2
P_{11}	15.332 635	18.804 946	21.602 688	51.885 084	4268.407 2
P_{12}	15.254 839	18.744 409	21.582 923	48.518 583	4105.486 2
P_{13}	15.265 027	18.752 337	21.585 202	48.959 471	4126.463 9
P_{14}	17.355 512	21.361 027	24.785 407	52.048 999	11 739.718
P_{15}	17.451 738	21.435 906	24.801 640	56.213 016	12 318.456
P_{16}	24.134 495	29.582 033	34.490 281	83.280 024	348 150.16
P_{17}	28.588 126	35.011 585	40.931 773	101.258 01	3227 471.5

where

$$P_1 = x^4 - 7x^3 + 13x^2 - 7x + 1$$
$$P_2 = x^4 - 8x^3 + 15x^2 - 8x + 1$$
$$P_3 = x^6 - 11x^5 + 41x^4 - 63x^3 + 41x^2 - 11x + 1$$
$$P_4 = x^6 - 12x^5 + 44x^4 - 67x^3 + 44x^2 - 12x + 1$$
$$P_5 = x^8 - 15x^7 + 83x^6 - 220x^5 + 303x^4 - 220x^3 + 83x^2 - 15x + 1$$
$$P_6 = x^8 - 16x^7 + 91x^6 - 244x^5 + 337x^4 - 244x^3 + 91x^2 - 16x + 1$$
$$P_7 = x^8 - 15x^7 + 84x^6 - 225x^5 + 311x^4 - 225x^3 + 84x^2 - 15x + 1$$
$$P_8 = x^8 - 16x^7 + 90x^6 - 239x^5 + 329x^4 - 239x^3 + 90x^2 - 16x + 1$$
$$P_9 = x^{12} - 23x^{11} + 218x^{10} - 1118x^9 + 3438x^8 - 6651x^7 + 8271x^6 - 6651x^5 + 3438x^4 - 1118x^3 + 218x^2 - 23x + 1$$
$$P_{10} = x^{14} - 27x^{13} + 309x^{12} - 1977x^{11} + 7871x^{10} - 20564x^9 + 36261x^8 - 43749x^7 + 36261x^6 - 20564x^5 + 7871x^4 - 1977x^3 + 309x^2 - 27x + 1$$

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$$P_{11} = x^{14} - 27x^{13} + 309x^{12} - 1979x^{11} + 7894x^{10} - 20668x^9 + 36503x^8 - 44067x^7 + 36503x^6 - 20668x^5 + 7894x^4 - 1979x^3 + 309x^2 - 27x + 1$$

$$P_{12} = x^{14} - 27x^{13} + 308x^{12} - 1963x^{11} + 7790x^{10} - 20307x^9 + 35763x^8 - 43131x^7 + 35763x^6 - 20307x^5 + 7790x^4 - 1963x^3 + 308x^2 - 27x + 1$$

$$P_{13} = x^{14} - 27x^{13} + 308x^{12} - 1964x^{11} + 7800x^{10} - 20348x^9 + 35853x^8 - 43247x^7 + 35853x^6 - 20348x^5 + 7800x^4 - 1964x^3 + 308x^2 - 27x + 1$$

$$P_{14} = x^{16} - 31x^{15} + 413x^{14} - 3141x^{13} + 15261x^{12} - 50187x^{11} + 115410x^{10} - 189036x^9 + 222621x^8 - 189036x^7 + 115410x^6 - 50187x^5 + 15261x^4 - 3141x^3 + 413x^2 - 31x + 1$$

$$P_{15} = x^{16} - 31x^{15} + 415x^{14} - 3177x^{13} + 15538x^{12} - 51389x^{11} + 118680x^{10} - 194903x^9 + 229733x^8 - 194903x^7 + 118680x^6 - 51389x^5 + 15538x^4 - 3177x^3 + 415x^2 - 31x + 1$$

$$P_{16} = x^{22} - 43x^{21} + 832x^{20} - 9625x^{19} + 74627x^{18} - 412074x^{17} + 1680988x^{16} - 5187203x^{15} + 12299616x^{14} - 22643382x^{13} + 32578283x^{12} - 36764041x^{11} + 32578283x^{10} - 22643382x^9 + 12299616x^8 - 5187203x^7 + 1680988x^6 - 412074x^5 + 74627x^4 - 9625x^3 + 832x^2 - 43x + 1$$

$$P_{17} = x^{26} - 51x^{25} + 1191x^{24} - 16934x^{23} + 164453x^{22} - 1160191x^{21} + 6175143x^{20} - 25425203x^{19} + 82394564x^{18} - 212746984x^{17} + 441495979x^{16} - 740812263x^{15} + 1009086208x^{14} - 1118311827x^{13} + 1009086208x^{12} - 740812263x^{11} + 441495979x^{10} - 212746984x^9 + 82394564x^8 - 25425203x^7 + 6175143x^6 - 1160191x^5 + 164453x^4 - 16934x^3 + 1191x^2 - 51x + 1$$

2. The total distance of totally positive algebraic integers

2.1. The explicit auxiliary function. Let α be a totally positive algebraic integer, $\alpha = \alpha_1, \ldots, \alpha_d$ be its conjugates and P be its minimal polynomial. The auxiliary function involved in the proof of Theorem 1.1 is of the following type:

$$\text{for } x > 0, \quad f(x) = |x - 1| - \sum_{0 \le j \le J} c_j \log |Q_j(x)| \tag{2.1}$$

where the c_j are positive real numbers and the polynomials Q_j are nonzero polynomials in $\mathbb{Z}[x]$. Then

$$\sum_{i=1}^{d} f(\alpha_i) \ge md$$

where m denotes the minimum of the function f. Thus,

$$\mathrm{td}(\alpha) \ge md + \sum_{1 \le j \le J} c_j \log \left| \prod_{i=1}^{d} Q_j(\alpha_i) \right|.$$

We assume that P does not divide any Q_j, and then $\prod_{i=1}^{d} Q_j(\alpha_i)$ is a nonzero integer because it is the resultant of P and Q_j. Therefore, if α is not a root of Q_j, then

$$\mathrm{Td}(\alpha) \ge m.$$

The main problem is to find a good list of polynomials Q_j which gives a value of m as large as possible. To do this, we link the auxiliary function with a generalisation of the integer transfinite diameter in order to find the polynomials with our recursive algorithm.

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2.2. Auxiliary functions and integer transfinite diameter. In this section, we need the following definition. Let K be a compact subset of $\mathbb{C}$. If φ is a positive function defined on K, the *φ-integer transfinite diameter of K* is defined as

$$t_{\mathbb{Z},\varphi}(K) = \liminf_{\substack{n\geq 1\\ n\to\infty}} \inf_{\substack{P\in\mathbb{Z}[X]\\ \deg(P)=n}} \sup_{x\in K} |P(x)|^{1/n}\varphi(x).$$

This weighted version of the integer transfinite diameter was introduced by Amoroso [Amo93] and is an important tool in the study of rational approximations of logarithms of rational numbers. Inside the auxiliary function (2.1), we replace the numbers c_j by rational numbers a_j/q where q is a common denominator of the c_j for $1 \leq j \leq J$. Then we may write,

$$\text{for } x > 0, \quad f(x) = |x-1| - \frac{t}{r}\log|Q(x)| \tag{2.2}$$

where $Q = \prod_{j=1}^{J} Q_j^{a_j} \in \mathbb{Z}[X]$ is of degree $r = \sum_{j=1}^{J} a_j \deg Q_j$ and $t = \sum_{j=1}^{J} c_j \deg Q_j$. We search for a polynomial $Q \in \mathbb{Z}[X]$ such that

$$\sup_{x>0} |Q(x)|^{t/r} \exp(-|x-1|) \leq e^{-m},$$

where m is as large as possible. If we suppose that t is fixed, it is clear that we need an effective upper bound for the quantity

$$t_{\mathbb{Z},\varphi}((0,\infty)) = \liminf_{\substack{r\geq 1\\ r\to\infty}} \inf_{\substack{P\in\mathbb{Z}[X]\\ \deg(P)=r}} \sup_{x>0} |P(x)|^{t/r}\varphi(x)$$

where we use the weight $\varphi(x) = \exp(-|x-1|)$. Even if we replace the compact subset K by the half line $(0, \infty)$, the weight φ ensures that the quantity $t_{\mathbb{Z},\varphi}((0, \infty))$ is finite.

2.3. Construction of the auxiliary function. The polynomials involved in the auxiliary function are found by our recursive algorithm developed in [Fla08] from Wu's algorithm [Wu03]. It replaces the heuristic search by a systematic search by induction for suitable polynomials. Suppose that we have $Q_1, Q_2, \ldots, Q_J$. Then we use semi-infinite linear programming (introduced into number theory by Smyth [Smy84]) to optimise f for this set of polynomials to get the greatest possible m. We obtain the positive real numbers $c_1, c_2, \ldots, c_J$ and then f in the form (2.2) as above.

For several values of k, we search for a polynomial $R(x) = \sum_{l=0}^{k} a_l x^l \in \mathbb{Z}[x]$ such that

$$\sup_{x>0} |Q(x)R(x)|^{t/(r+k)} \exp(-|x-1|) \leq e^{-m},$$

that is, such that

$$\sup_{x>0} |Q(x)R(x)| \exp\left(-|x-1|(r+k)/t\right)$$

is as small as possible.

Next, we apply the LLL lattice basis reduction algorithm to the linear forms

$$Q(x_i)R(x_i)\exp(-|x_i-1|(r+k)/t)$$

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where the x_i are control points in the half line $(0, \infty)$. After several trials, we find that a set of 70 points uniformly distributed in the interval $(0, 70)$ is convenient for the first step of LLL. After semi-infinite linear programming, this set will be augmented by the points where f has its least local minima. We get a polynomial R whose factors R_j are good candidates to enlarge the set of polynomials $(Q_1, Q_2, \dots, Q_J)$. We only keep the polynomials Q_j which have a nonzero coefficient in the newly optimised auxiliary function f. After optimisation, some previous polynomials Q_j may have a zero coefficient and so are removed.

In order to get the lower bound of Theorem 1.1, we successively take k from 4 to 20.

3. Comparison of measures

The method of explicit auxiliary functions has already been used to establish inequalities involving the Mahler measure, the length and the trace of totally positive polynomials; see [Fla14].

The auxiliary function used for the lower bound (upper bound) of Theorem 1.2 is of the following type:

$$\text{for } x > 0, \quad f(x) = \pm|x-1| \mp c_0 \log \max(1, x) - \sum_{1 \le j \le J} c_j \log |Q_j(x)|,$$

where the c_j are positive real numbers and the polynomials Q_j are nonzero polynomials in $\mathbb{Z}[x]$.

The auxiliary function used for the lower bound (upper bound) of Theorem 1.3 is of the following type:

$$\text{for } x > 0, \quad f(x) = \pm\log(x+1) \mp c_0|x-1| - \sum_{1 \le j \le J} c_j \log |Q_j(x)|,$$

where the c_j are positive real numbers and the polynomials Q_j are nonzero polynomials in $\mathbb{Z}[x]$.

The auxiliary function used for the lower bound (upper bound) of Theorem 1.4 is of the following type:

$$\text{for } x > 0, \quad f(x) = \pm x \mp c_0|x-1| - \sum_{1 \le j \le J} c_j \log |Q_j(x)|,$$

where the c_j are positive real numbers and the polynomials Q_j are nonzero polynomials in $\mathbb{Z}[x]$.

In our recursive algorithm, we take k from 4 to 10 only in order to have a small number of exceptions to the validity of the inequalities.

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