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Mathematical Instantons in Characteristic Two

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Abstract. On $\mathbf{P}^3$, we show that mathematical instantons in characteristic two are unobstructed. We produce upper bounds for the dimension of the moduli space of stable rank two bundles on $\mathbf{P}^3$ in characteristic two. In cases where there is a phenomenon of good reduction modulo two, these give similar results in characteristic zero. We also give an example of a nonreduced component of the moduli space in characteristic two.

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Introduction

The study of mathematical instantons on projective three space has been pursued partly because of the Atiyah–Ward–Drinfeld–Manin theorem which showed that the solutions of the self-dual Yang–Mills equations on S^4 could be described in algebraic terms as particular cases of mathematical instanton bundles on $\mathbf{P}^3_{\mathbb{C}}$. As a consequence, many workers have studied these bundles (and their generalizations to $\mathbf{P}^{2n+1}$) over the complex numbers and are still studying issues like the smoothness and irreducibility of the moduli spaces of these bundles. In this paper, we would like to discuss mathematical instanton bundles on $\mathbf{P}^3$ defined over fields of any characteristic. The question of whether such bundles are unobstructed is still unknown in general, though it has been verified in some special cases. In [N-T], unobstructedness is proved for all mathematical instantons with a section in degree one. In [R-2], it is proved for those with a jumping line of maximal order.

Over a field of characteristic two, we will find (Theorem 2.4) that the unobstructedness of all mathematical instantons on $\mathbf{P}^3$ is extremely easy to see. In fact, these are the only unobstructed stable rank two bundles on $\mathbf{P}^3$ (with $c_1 = 0$) in this characteristic. We also find a simple example (2.6) of a nonreduced component of the moduli space in characteristic two. We expect that most components will have this property of nonreducedness.

As a consequence, we show that in characteristic zero, a mathematical instanton is unobstructed if some pull-back of it by an automorphism of $\mathbf{P}^3$ reduces modulo 2 to a mathematical instanton (Theorem 3.2). However, this result does not answer the problem for all mathematical instantons in characteristic zero. In fact, there are examples, in characteristic zero, of such bundles for which no pull-back by an automorphism of $\mathbf{P}^3$ reduces modulo two to a mathematical instanton (Example 3.7).

These computations in characteristic two also allow us to bound the dimension of each component of the moduli space of stable rank two bundles on $\mathbf{P}^3$. The bound has order n^2 where n is the normalized second Chern class (Corollary 2.8). Once again this bound has a consequence in characteristic zero. Specifically, if N is a component of the moduli space which contains one bundle that reduces to a stable or semi-stable bundle in characteristic two, then the dimension of N is bounded above by a bound of the order n^2 (Theorem 3.8).

The first section contains a review of facts about bundles on $\mathbf{P}^3$ in any characteristic, and includes a definition of mathematical instantons (1.5). In Section two, some calculations in characteristic two are made. These arise from the relationship between the second symmetric power in characteristic two and Frobenius pull-backs. I would like to thank V. Mehta for pointing out to me this relationship. In the last section, applications to characteristic zero are made.

1. We first review some elementary facts about vector bundles over a projective space defined over an arbitrary field k. Many of the results in the literature are discussed for an algebraically closed field, and in the case of mathematical instanton bundles, even over the complex numbers. We observe that most of these conditions can be relaxed.

Consider $\mathbf{P}^n_k$, projective n-space defined over a field k. Let $\bar{k}$ be the algebraic closure of k. The notion of a (geometric) vector bundle over $\mathbf{P}^n_k$ and the notion of a locally free sheaf on $\mathbf{P}^n_k$ are equivalent ([H-1], II, 5.18). Let $\mathscr{E}$ be a vector bundle of rank r defined on $\mathbf{P}^n_k$. Let $\bar{\mathscr{E}} = \mathscr{E} \otimes_k \bar{k}$ be its pull-back to $\mathbf{P}^n_{\bar{k}}$.

1.1. The square

$$\begin{array}{ccc} \mathbf{P}^n_{\bar{k}} & \longrightarrow & \mathbf{P}^n_k \\ \downarrow & & \downarrow \\ \operatorname{Spec} \bar{k} & \longrightarrow & \operatorname{Spec} k \end{array}$$

is Cartesian with $k \to \bar{k}$ faithfully flat. Hence

$$H^i(\mathbf{P}^n_{\bar{k}}, \bar{\mathscr{E}}) = H^i(\mathbf{P}^n_k, \mathscr{E}) \otimes_k \bar{k}$$

and

$$H^i(\mathbf{P}^n_{\bar{k}}, \bar{\mathscr{E}}) = 0 \Leftrightarrow H^i(\mathbf{P}^n_k, \mathscr{E}) = 0.$$

1.2. The integers $c_i(\mathscr{E})$ can be defined as $c_i(\bar{\mathscr{E}})$. However since $\operatorname{Pic}(\mathbf{P}^n_k) \cong \mathbb{Z}$, there is an isomorphism $\wedge^r \mathscr{E} \cong \mathscr{O}_{\mathbf{P}^n_k}(c_1)$ which is defined over k.

1.3. Horrock's Theorem states that $\mathcal{E}$ is isomorphic over k to a sum of line bundles if and only if $H^i_*(\mathbf{P}^n_k, \mathcal{E}) = 0$ for all i between 1 and $n-1$. This is valid over any field. For example, consider the proof given in [O-S-S], which uses the complex numbers as the base field. Upon reading the proof, we see only one place where the argument does not work for arbitrary k. This is in the proof of Grothendieck's theorem, an auxiliary result needed in the proof. In this part, a section $s \in H^0(\mathbf{P}^1_k, \mathcal{E}(k_0))$ is chosen, where k_0 is the least integer for which a nonzero section s can be chosen and the claim is made that this section is nowhere vanishing. For us, in our context where k is arbitrary, this should mean that s has no zeros over $\bar{k}$. Indeed this is true, for if s has a zero in $\mathbf{P}^1_{\bar{k}}$, we would conclude that s comes from a section of $H^0(\mathbf{P}^1_{\bar{k}}, \mathcal{E}(k_0 - 1))$. But then $H^0(\mathbf{P}^1_k, \mathcal{E}(k_0 - 1))$ itself is nonzero by (1.1) contradicting our choices. Thus we still get

$$0 \to \mathcal{O}_{\mathbf{P}^1_k} \xrightarrow{s} \mathcal{E}(k_0) \to \mathcal{F} \to 0,$$

where $\mathcal{F}$ is a bundle on $\mathbf{P}^1_k$, and the proof in [O-S-S] continues without change.

1.4. Therefore the results of [B-H] and [R-1] on the construction of monads for a bundle $\mathcal{E}$ of rank two on $\mathbf{P}^3_k$ are valid in any characteristic. Let $\mathcal{E}$ have first Chern class c_1. Then there is an isomorphism $\mathcal{E}^\vee \cong \mathcal{E}(-c_1)$ which is defined over k. $M = H^1_*(\mathbf{P}^3_k, \mathcal{E})$ is a finite length module over $S = k[X_0, X_1, X_2, X_3]$. Let $L_0 \to M$ be a surjective homomorphism where L_0 is a sum of graded twists of S, picking out a set of minimal generators of M. Then there is a monad

$$0 \to \tilde{L}_0^\vee(c_1) \xrightarrow{\beta} \tilde{L}_1 \xrightarrow{\alpha} \tilde{L}_0 \to 0,$$

where L_1 is also a sum of twists of S and α, β are matrices of homogeneous polynomials in S. Furthermore, there is an isomorphism $H{:}L_1 \cong L_1^\vee(c_1)$ with H a matrix of homogeneous polynomials in S such that $H\beta = \alpha^\vee$. This gives an isomorphism between the monad and the dual monad which lifts the isomorphism between $\mathcal{E}^\vee$ and $\mathcal{E}(-c_1)$. If $\mathrm{Hom}(L_0, L_1) = 0$, then this H is unique and can be chosen so that $H^\vee = -H$.

The result in [R-1] says that we may take α as a minimal presentation of the S-module M. Furthermore, if $L_2 \to L_1 \to L_0 \to M \to 0$ is part of a minimal resolution for M, then the map $L_0^\vee(c_1) \xrightarrow{\beta} L_1$ will be a direct-summand of the map $L_2 \to L_1$ [H-R, 3.2].

1.5. We give the definition of a mathematical instanton of rank two on $\mathbf{P}^3_k$. The general definition due to Okonek and Spindler [O-S] of a mathematical instanton on $\mathbf{P}^{2n+1}$ has conditions on natural cohomology and trivial splittings on the general line. We will relax this part of their definition for our case of rank two on $\mathbf{P}^3$.

DEFINITION. An indecomposable rank two bundle $\mathcal{E}$ on $\mathbf{P}^3_k$ with $c_1 = 0, c_2 = n$ is called a mathematical instanton if $H^1(\mathbf{P}^3_k, \mathcal{E}(-2)) = 0$.

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Recall that for a rank two bundle with $c_1 = 0$ in characteristic zero, the condition that it splits trivially on the general line is equivalent to the condition of semi-stability. Likewise, natural cohomology in the range -3 to 0 also implies stability (if $H^1(\mathcal{E}) \neq 0$.) We will not assume *a priori* such conditions. However, stability will follow below. In arbitrary characteristic, we do not know if the splitting type of a bundle as defined above can be non-trivial on the general line. The following theorem is well known. We include a proof because, for example, the proof in [B-H] uses trivial splitting on the general line and the corresponding proof in [O-S] uses their condition of natural cohomology.

THEOREM. *Let $\mathcal{E}$ be a mathematical instanton on $\mathbf{P}^3_k$ with $c_1 = 0$, $c_2 = n$. Then*

(a) $H^1(\mathbf{P}^3_k, \mathcal{E}(-k)) = 0$ *for* $k \geqslant 2$.
(b) $M = H^1_*(\mathbf{P}^3_k, \mathcal{E})$ *has all its minimal generators in degree* -1.
(c) $H^0(\mathbf{P}^3_k, \mathcal{E}) = 0$ (*hence $\mathcal{E}$ is stable*).
(d) $n = h^1(\mathbf{P}^3_k, \mathcal{E}(-1)) > 0$.
(e) *$\mathcal{E}$ is the homology of a minimal monad*

$$0 \to n\mathcal{O}_{\mathbf{P}^3_k}(-1) \xrightarrow{\beta} (2n+2)\mathcal{O}_{\mathbf{P}^3_k} \xrightarrow{\alpha} n\mathcal{O}_{\mathbf{P}^3_k}(+1) \to 0.$$

(f) *Conversely, any bundle which is the homology of such a monad as in* (e) *is a mathematical instanton bundle with $c_1 = 0$, $c_2 = n$.*

Proof. Since most of the statements are about dimensions of cohomology, by (1.1) we will assume that k is algebraically closed so that geometric constructions work as usual. To prove (a), we will show that if $c_1(\mathcal{E}) = 0$ or -1 and $H^1(\mathbf{P}^3_k, \mathcal{E}(-m)) = 0$ for some $m > 0$, then $H^1(\mathbf{P}^3_k, \mathcal{E}(-k)) = 0$ for all $k \geqslant m$. For suppose $H^1(\mathbf{P}^3_k, \mathcal{E}(-k)) \neq 0$ with $k > m$ and let k be the least such. Let H be a general hyperplane in $\mathbf{P}^3_k$, and consider

$$0 \to \mathcal{E}(-k) \xrightarrow{H} \mathcal{E}(-k+1) \to \mathcal{E}_H(-k+1) \to 0. \qquad (*)$$

Then $\mathcal{E}_H(-k+1)$ clearly gets at least one global section t. Consider

$$0 \to \mathcal{E}(-k+1) \xrightarrow{H} \mathcal{E}(-k+2) \to \mathcal{E}_H(-k+2) \to 0.$$

Since $H^1(\mathcal{E}(-k+1)) = 0$, the multiples of t in $H^0(\mathcal{E}_H(-k+2))$ arise from sections of $\mathcal{E}(-k+2)$. So $\mathcal{E}(-k+2)$ has at least three global sections. The sections of $\mathcal{E}$ in degrees less than or equal to 0 (if they exist) are all multiples of a single section s and this section s is the unique section in degrees $\leqslant 0$ whose zero-scheme has codimension two. Since $-k+2 \leqslant 0$, there is a single section, say s, of $\mathcal{E}(-l)$ for some $l > 0$. This induces a nonzero section s' of $\mathcal{E}_H(-l)$. If H is chosen generally, the zero-scheme of s' in H has codimension two. Hence all sections of $\mathcal{E}_H$ in degrees $\leqslant 0$ must be multiples of s'. It is evident from the long exact sequence of cohomology of $(*)$ that the section t of $\mathcal{E}_H(-k+1)$ is not a multiple of s'. This is a contradiction. Hence the result is proved.

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To prove (b), we study the minimal monad of $\mathcal{E}$. The monad will be $0 \to \tilde{L}_0^\vee \to \tilde{L}_1 \to \tilde{L}_0 \to 0$ as before. If M has any generators in degrees 0 or 1 etc., it means that L_0 has summands like $S(0)$ or $S(-1)$ etc. By the minimality of the monad, L_1 must have summands like $S(-1)$ or $S(-2)$ etc. But $L_1^\vee \cong L_1$, hence L_1 has summands like $S(1)$ or $S(2)$ etc. Such summands must map to zero in L_0 by degree considerations and minimality. However, the map $L_1 \to L_0$ is a minimal presentation of M, hence no summands can map to zero. Hence all generators of M are in degree -1.

To prove (c), we now know that L_0 is a sum of $S(1)$'s. Hence L_1 cannot have any summands like $S(1)$ or $S(2)$, etc. Since L_1 is selfdual, L_1 can contain only $S(0)$'s. Hence the minimal resolution of M which looks like $\to L_2 \to L_1 \to L_0 \to M \to 0$ must have L_2 without any $S(0)$ or $S(1)$, etc. Now $L_2 = L_0^\vee \oplus L_2'$ and there is a surjection of $\tilde{L}_2' \to \mathcal{E}$ which induces a surjection of global sections in all twists. This shows that $\mathcal{E}$ has no sections in degree $\leqslant 0$.

(d) is merely a Riemann–Roch computation now. $n > 0$ because, for example, if $n = 0$, then we get $M = 0$ by (b). Hence, by Serre duality, $H^2_*(\mathcal{E}) = 0$ as well. So $\mathcal{E}$ is decomposable by Horrock's theorem. Therefore $n > 0$.

Of course, by now (e) has been demonstrated. (f) is quite obvious from the display of the monad. □

1.6. The coarse moduli scheme $\mathcal{M}_{\mathbf{P}^3_k}(c_1, c_2)$ of stable rank 2 bundles on $\mathbf{P}^3_k$ with Chern classes c_1, c_2 exists and is a k-scheme ([M], Theorem 5.6). The fact that this moduli scheme is quasi-projective has been proved by Maruyama and also discussed in [H-2]. This scheme behaves well under base change ([M], Remark 5.9) so that for example if $\bar{k}$ is the algebraic closure of k, then $\mathcal{M}_{\mathbf{P}^3_{\bar{k}}}(c_1, c_2) \cong \mathcal{M}_{\mathbf{P}^3_k}(c_1, c_2) \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$. If $\mathcal{E}$ is a bundle on $\mathbf{P}^3_k$ giving a k-valued point on $\mathcal{M}_{\mathbf{P}^3_k}(c_1, c_2)$, then $\bar{\mathcal{E}} = \mathcal{E} \otimes_k \bar{k}$ on $\mathbf{P}^3_{\bar{k}}$ gives a geometric point. The Zariski tangent space at this geometric point is given by $H^1(\mathbf{P}^3_{\bar{k}}, \underline{\operatorname{Hom}}(\bar{\mathcal{E}}, \bar{\mathcal{E}}))$ and if $H^2(\mathbf{P}^3_{\bar{k}}, \underline{\operatorname{Hom}}(\bar{\mathcal{E}}, \bar{\mathcal{E}})) = 0$, the moduli space is smooth at this point, with dimension equal to $h^1(\mathbf{P}^3_{\bar{k}}, \underline{\operatorname{Hom}}(\bar{\mathcal{E}}, \bar{\mathcal{E}}))$. In this case, we shall say that $\mathcal{E}$ is unobstructed. Of course, the dimensions of these vector spaces can be computed using $\mathcal{E}$ over k.

1.7. The spectrum of a semi-stable rank two bundle on $\mathbf{P}^3_k$ has been defined in characteristic zero in [B-E] and in arbitrary characteristic in [H-3]. Since the spectrum determines and is determined by dimensions of $h^1(\mathbf{P}^3_k, \mathcal{E}(l))$, we do not need to assume that k is algebraically closed. Let us recall the spectrum as found in [H-3]. Let $\mathcal{E}$ be a semi-stable rank two bundle on $\mathbf{P}^3_k$ with $c_1 = 0$ or -1 and $c_2 = n$. There is a unique set of n integers $k_1, k_2, \ldots k_n$ called the spectrum of $\mathcal{E}$ with the following properties: let $\mathcal{H}$ denote the sheaf $\oplus \mathcal{O}_{\mathbf{P}^1}(k_i)$ on $\mathbf{P}^1$.

(a) $h^1(\mathbf{P}^3_k, \mathcal{E}(l)) = h^0(\mathbf{P}^1, \mathcal{H}(l+1))$ for $l \leqslant -1$.
(b) $\{-k_i\} = \{k_i\}$ if $c_1 = 0$ and
$\{-k_i\} = \{k_i + 1\}$ if $c_1 = -1$.

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(c) The spectrum is connected, except possibly for a gap at 0. If $\mathcal{E}$ is stable, then the spectrum is connected.

(d) An integer l may appear more than once in the spectrum. If $l \leqslant -2$ and l appears exactly once in the spectrum, then any smaller integer can occur at most once in the spectrum. If $\mathcal{E}$ is stable and $c_1 = 0$, we can say the same for $l \leqslant -1$.

Property (d) was proved in [H-3 Prop. 5.1] using a characteristic zero hypotheses. However, this hypotheses was really needed only to prove a stronger statement about unstable planes and as was pointed out in [H-R], for the proof of (d), characteristic zero is not required.

1.8. When $c_1 = 0$, the condition that $H^1(\mathbf{P}^3_k, \mathcal{E}(-2)) = 0$ is equivalent to the condition that the spectrum of $\mathcal{E}$ (with $c_2 = n$) consists of n zeroes. Likewise, let $\mathcal{E}$ be a rank two bundle with $c_1 = -1, c_2 = n$ and with $H^1(\mathbf{P}^3_k, \mathcal{E}(-2)) = 0$. In our proof of Theorem 1.5, we showed that $H^1(\mathbf{P}^3_k, \mathcal{E}(-k)) = 0$ for all $k \geqslant 2$. It is easy to prove by the same techniques that, in the minimal monad for $\mathcal{E}$, the term L_1 has only $S(0)$'s and $S(-1)$'s, hence L_2 can contain only terms $S(a)$ with $a \leqslant -1$. Therefore, $\mathcal{E}$ is stable. $\mathcal{E}$ has spectrum consisting of $\frac{n}{2}$ 0's and $\frac{n}{2}$ -1's.

2. Let $\pi: X \to Y$ be a morphism of schemes defined over a field k of characteristic p different from zero. We can define Frobenius automorphisms F of X and Y induced by the Frobenius homomorphism $a \mapsto a^p$ on affine rings. Then the square

$$\begin{array}{ccc} X & \xrightarrow{F} & X \\ \pi \downarrow & & \downarrow \pi \\ Y & \xrightarrow{F} & Y \end{array}$$

commutes.

LEMMA 2.1. *Let $F: \mathbf{P}^n_k \to \mathbf{P}^n_k$ be the Frobenius morphism in characteristic p. Then (i) $F_*[\mathcal{O}_{\mathbf{P}^n_k}(l)] \cong \oplus_{i \geqslant -\frac{l}{p}} a_i \mathcal{O}_{\mathbf{P}^n_k}(-i)$ where a_i is the number of monomials $X_0^{b_0} X_1^{b_1} \ldots X_n^{b_n}$ of degree $l + pi$ with each exponent $b_j < p$. (ii) When k has characteristic two,*

$$F_*[\mathcal{O}_{\mathbf{P}^n_k}(l)] \cong \bigoplus_{\frac{n+1-l}{2} \geqslant i \geqslant -\frac{l}{2}} \binom{n+1}{l+2i} \mathcal{O}_{\mathbf{P}^n_k}(-i).$$

Proof. For $1 \leqslant i \leqslant n-1$,

$$\begin{aligned} H^i(\mathbf{P}^n_k, F_*[\mathcal{O}_{\mathbf{P}^n_k}(l)] \otimes \mathcal{O}_{\mathbf{P}^n_k}(m)) &= H^i(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}(l) \otimes F^*[\mathcal{O}_{\mathbf{P}^n_k}(m)]) \\ &= H^i(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}(l + pm)) = 0 \end{aligned}$$

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for all m. Hence by Horrock's theorem, the bundle $F_*[\mathcal{O}_{\mathbf{P}^n_k}(l)]$ is a sum of line bundles. The number of summands in this bundle of the form $\mathcal{O}_{\mathbf{P}^n_k}(-i)$ can be computed by finding the dimension of $H^1(\mathbf{P}^n_k, F_*[\mathcal{O}_{\mathbf{P}^n_k}(l)] \otimes \Omega^1_{\mathbf{P}^n_k}(i))$ which is just $h^1(\mathbf{P}^n_k, F^*[\Omega^1_{\mathbf{P}^n_k}](l+pi))$. We have the sequence

$$\begin{aligned} 0 \;\to\;& F^*[\Omega^1_{\mathbf{P}^n_k}](l+pi) \\ \to\;& (n+1)\mathcal{O}_{\mathbf{P}^n_k}(l+p(i-1)) \xrightarrow{[x_0^p,\dots,x_n^p]} \mathcal{O}_{\mathbf{P}^n_k}(l+pi) \to 0. \end{aligned}$$

The lemma follows now from this sequence. □

PROPOSITION 2.2. *Let $\mathcal{E}$ be a rank two bundle on $\mathbf{P}^n_k$ with first Chern class c_1, where k has characteristic p. Let F be the Frobenius mapping on $\mathbf{P}^n_k$. Then we have exact sequences*

$$0 \to F^*\mathcal{E} \to S_p(\mathcal{E}) \to S_{p-2}(\mathcal{E}) \otimes \mathcal{O}_{\mathbf{P}^n_k}(c_1) \to 0$$

and

$$0 \to \mathcal{O}_{\mathbf{P}^n_k} \to \underline{\mathrm{Hom}}(\mathcal{E},\mathcal{E}) \to S_2\mathcal{E} \otimes \mathcal{O}_{\mathbf{P}^n_k}(-c_1) \to 0.$$

Proof. Consider the commuting square (not Cartesian)

$$\begin{array}{ccc} \mathbf{P}(\mathcal{E}) & \xrightarrow{F} & \mathbf{P}(\mathcal{E}) \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \mathbf{P}^n_k & \xrightarrow{F} & \mathbf{P}^n_k. \end{array}$$

Let $\mathcal{O}_\pi(1)$ denote the tautological line quotient bundle on $\mathbf{P}(\mathcal{E})$. So we have

$$0 \to \wedge^2(\pi^*\mathcal{E}) \otimes \mathcal{O}_\pi(-1) \to \pi^*\mathcal{E} \to \mathcal{O}_\pi(1) \to 0.$$

Since $\wedge^2(\pi^*\mathcal{E}) \cong \pi^*\mathcal{O}_{\mathbf{P}^n_k}(c_1)$, after applying F^* we get

$$0 \to F^*\pi^*\mathcal{O}_{\mathbf{P}^n_k}(c_1) \otimes F^*\mathcal{O}_\pi(-1) \to F^*\pi^*\mathcal{E} \to F^*\mathcal{O}_\pi(1) \to 0,$$

hence

$$0 \to \pi^*\mathcal{O}_{\mathbf{P}^n_k}(pc_1) \otimes \mathcal{O}_\pi(-p) \to \pi^*F^*\mathcal{E} \to \mathcal{O}_\pi(p) \to 0.$$

Applying π_*, we get (since $\pi_*(\mathcal{O}_\pi(-p)) = 0$)

$$0 \to F^*\mathcal{E} \to \pi_*\mathcal{O}_\pi(p) \to \mathcal{O}_{\mathbf{P}^n_k}(pc_1) \otimes R^1\pi_*\mathcal{O}_\pi(-p) \to 0.$$

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Now $R^1\pi_*\mathcal{O}_\pi(-p) \cong [\pi_*\mathcal{O}_\pi(p-2)]^\vee \otimes (\wedge^2\mathcal{E})^\vee$ ([H-1], III, 8.4), hence we get

$$0 \to F^*\mathcal{E} \to S_p(\mathcal{E}) \to [S_{p-2}\mathcal{E}]^\vee \otimes \mathcal{O}_{\mathbf{P}^n_k}(pc_1 - c_1) \to 0.$$

Now in characteristic p, $[S_{p-2}\mathcal{E}]^\vee \cong S_{p-2}(\mathcal{E}^\vee) \cong S_{p-2}(\mathcal{E}) \otimes \mathcal{O}_{\mathbf{P}^n_k}(-(p-2)c_1)$ hence we end with

$$0 \to F^*\mathcal{E} \to S_p(\mathcal{E}) \to S_{p-2}(\mathcal{E}) \otimes \mathcal{O}_{\mathbf{P}^n_k}(c_1) \to 0.$$

For the other part, observe that $\underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{E}) \cong \mathcal{E}^* \otimes \mathcal{E} \cong \mathcal{E} \otimes \mathcal{E} \otimes \mathcal{O}_{\mathbf{P}^n_k}(-c_1)$, hence we have the sequence

$$0 \to \mathcal{O}_{\mathbf{P}^n_k} \to \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{E}) \to S_2\mathcal{E} \otimes \mathcal{O}_{\mathbf{P}^n_k}(-c_1) \to 0$$

(obtained for example by the push down of the tautological sequence on $\mathbf{P}(\mathcal{E})$ tensored by $\mathcal{O}_\pi(1)$). □

COROLLARY 2.3. *Let $\mathcal{E}$ be a rank two bundle on $\mathbf{P}^3_k$, where k has characteristic two.*

(i) *Let $c_1 = 0$ and let $m \geqslant -4$.*
If m is even, then $h^2(\mathbf{P}^3_k, \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{E})(m)) = h^1(\mathbf{P}^3_k, \mathcal{E}(-2 - \frac{m}{2})) + 6h^1(\mathbf{P}^3_k, \mathcal{E}(-3 - \frac{m}{2})) + h^1(\mathbf{P}^3_k, \mathcal{E}(-4 - \frac{m}{2}))$.
If m is odd, then $h^2(\mathbf{P}^3_k, \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{E})(m)) = 4h^1(\mathbf{P}^3_k, \mathcal{E}(-\frac{3+m}{2} - 1) + 4h^1(\mathbf{P}^3_k, \mathcal{E}(-\frac{3+m}{2} - 2))$.

(ii) *Let $c_1 = -1$ and let $m \geqslant -4$.*
If m is even, then $h^2(\mathbf{P}^3_k, \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{E})(m)) = 4h^1(\mathbf{P}^3_k, \mathcal{E}(-2 - \frac{m}{2})) + 4h^1(\mathbf{P}^3_k, \mathcal{E}(-3 - \frac{m}{2}))$.
If m is odd, then $h^2(\mathbf{P}^3_k, \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{E})(m)) = h^1(\mathbf{P}^3_k, \mathcal{E}(-\frac{m+3}{2}) + 6h^1(\mathbf{P}^3_k, \mathcal{E}(-\frac{m+3}{2} - 1)) + h^1(\mathbf{P}^3_k, \mathcal{E}(-\frac{m+3}{2} - 2))$.

Proof. When $m \geqslant -4$,

$$\begin{aligned} h^2(\mathbf{P}^3_k, \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{E})(m)) &= h^2(\mathbf{P}^3_k, S_2\mathcal{E} \otimes \mathcal{O}_{\mathbf{P}^3_k}(m - c_1)) \\ &= h^2(\mathbf{P}^3_k, F^*[\mathcal{E}] \otimes \mathcal{O}_{\mathbf{P}^3_k}(m - c_1)) \\ &= h^1(\mathbf{P}^3_k, F^*(\mathcal{E}^\vee) \otimes \mathcal{O}_{\mathbf{P}^3_k}(c_1 - m - 4)) \\ &= h^1(\mathbf{P}^3_k, F^*(\mathcal{E}) \otimes \mathcal{O}_{\mathbf{P}^3_k}(-2c_1 + c_1 - m - 4)) \\ &= h^1(\mathbf{P}^3_k, \mathcal{E} \otimes F_*[\mathcal{O}_{\mathbf{P}^3_k}(-c_1 - m - 4)]). \end{aligned}$$

Now if $-c_1 - m - 4 = 2t$, then $F_*[\mathcal{O}_{\mathbf{P}^3_k}(-c_1 - m - 4)] = F_*F^*[\mathcal{O}_{\mathbf{P}^3_k}(t)] = \mathcal{O}_{\mathbf{P}^3_k}(t) \otimes F_*[\mathcal{O}_{\mathbf{P}^3_k}] = \mathcal{O}_{\mathbf{P}^3_k}(t) \otimes [\mathcal{O}_{\mathbf{P}^3_k} \oplus 6\mathcal{O}_{\mathbf{P}^3_k}(-1) \oplus \mathcal{O}_{\mathbf{P}^3_k}(-2)]$. If $-c_1 - m - 4 = 2t - 1$, then $F_*[\mathcal{O}_{\mathbf{P}^3_k}(-c_1 - m - 4)] = F_*[F^*[\mathcal{O}_{\mathbf{P}^3_k}(t)] \otimes \mathcal{O}_{\mathbf{P}^3_k}(-1)] = \mathcal{O}_{\mathbf{P}^3_k}(t) \otimes F_*[\mathcal{O}_{\mathbf{P}^3_k}(-1)] = \mathcal{O}_{\mathbf{P}^3_k}(t) \otimes [4\mathcal{O}_{\mathbf{P}^3_k}(-1) \oplus 4\mathcal{O}_{\mathbf{P}^3_k}(-2)]$. □

THEOREM 2.4. *If k is a field of characteristic two, then a stable rank two bundle $\mathcal{E}$ with $c_1 = 0$ is unobstructed if and only if $\mathcal{E}$ is a mathematical instanton on $\mathbf{P}^3_k$.*

Proof. $h^2(\mathbf{P}^3_k, \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{E})) = h^1(\mathcal{E}(-2)) + 6h^1(\mathcal{E}(-3)) + h^1(\mathcal{E}(-4))$ by the Corollary above. We saw earlier (Theorem 1.5 (a)) that $h^1(\mathcal{E}(-2)) = 0$ implies that the other terms are also zero. □

THEOREM 2.5. *If k is a field of characteristic two, then a stable rank two bundle $\mathcal{E}$ with $c_1 = -1$ is unobstructed if and only if $h^1(\mathbf{P}^3_k, \mathcal{E}(-2)) = 0$.*

Proof. In this case, $h^2(\mathbf{P}^3_k, \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{E})) = 4h^1(\mathcal{E}(-2)) + 4h^1(\mathcal{E}(-3))$. Again, if $h^1(\mathcal{E}(-2)) = 0$ then so is $h^1(\mathcal{E}(-3))$ (see 1.8). □

With this situation, one expects that the moduli schemes in characteristic two will be highly singular. Indeed, in the first example of a bundle family not of mathematical instanton type, we find that the moduli scheme has a nonreduced component. In contrast, at this time, very few examples of singular components of the moduli scheme are known in characteristic zero ([Ma], [A-O]).

EXAMPLE 2.6. A nonreduced component of $\mathcal{M}(0, 3)$ in characteristic two.

Consider stable bundles on $\mathbf{P}^3_k$ with $c_1 = 0$, $c_2 = 3$ and spectrum $-1, 0, 1$. The monad of such a bundle $\mathcal{E}$ (regardless of characteristic) has the form

$$0 \to \mathcal{O}_{\mathbf{P}^3_k}(-2) \xrightarrow{\alpha} \mathcal{O}_{\mathbf{P}^3_k}(-1) \oplus 2\mathcal{O}_{\mathbf{P}^3_k} \oplus \mathcal{O}_{\mathbf{P}^3_k}(1) \xrightarrow{\beta} \mathcal{O}_{\mathbf{P}^3_k}(2) \to 0.$$

Calling it $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$, let $\mathcal{G}$ be the kernel of the right hand map. $\mathcal{G}$ has a ten dimensional family of sections in degree 2, hence the set of all such monads is parametrized by a quasi-projective variety V of dimension $10 + \dim \mathrm{Hom}(B, C) = 54$. This space maps onto the moduli space of all stable bundles of the type being considered. The group $\mathrm{Aut}(A) \times \mathrm{Aut}(B) \times \mathrm{Aut}(C)$ acts on V and the orbit of a monad consists of monads for isomorphic stable bundles. The subgroup k^*, embedded diagonally, stabilizes a monad. On the other hand, since $\mathrm{Hom}(C, B) = \mathrm{Hom}(B, A) = 0$, results of Barth and Hulek [B-H] tell us that any automorphism of a bundle $\mathcal{E}$ is uniquely lifted to an automorphism of monads. Since $\mathcal{E}$ is stable, $\mathrm{Aut}(\mathcal{E})$ consists of elements of k^*, hence the stabilizer of a monad is exactly k^*. Thus we get a dimension of 54 less $(1 + 32 + 1 - 1)$ or a dimension of 21 for this component of the moduli space.

The mathematical instantons form smooth components of the moduli space, of dimension 21, proved above in characteristic two. (See also, for example, [L], for a proof valid in any characteristic.) There are just two possible spectra for these Chern classes. Hence, by reasons of dimension, the bundles with spectrum $-1, 0, 1$ give a distinct irreducible component of the moduli scheme.

Now these bundles $\mathcal{E}$ (with spectrum $-1, 0, 1$) have $h^1(\mathbf{P}^3_k, \mathcal{E}(-2)) = 1$, Hence, in characteristic two, by Corollary 2.3, $h^2(\mathbf{P}^3_k, \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{E})) = 1$. Since $h^1(\mathbf{P}^3_k, \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{E})) - h^2(\mathbf{P}^3_k, \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{E})) = 8c_2 - 3 = 21$, it follows that the Zariski tangent space to each such point $\mathcal{E}$ on this component of the moduli scheme is 22-

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dimensional and hence larger than the dimension of the component. Thus we get a non-reduced component. □

THEOREM 2.7. *Let $\mathcal{E}$ be a semistable bundle of rank two with $c_1 = 0$ or -1 and $c_2 = n$ on $\mathbf{P}^3_k$, in characteristic* 2.

(i) *If $c_1 = 0$, then $h^2(\mathbf{P}^3_k, \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{E})) \leqslant (n-1)^2$.*
(ii) *If $c_1 = 0$ and $\mathcal{E}$ is stable, then $h^2(\mathbf{P}^3_k, \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{E})) \leqslant (n-2)^2$.*
(iii) *If $c_1 = -1$ (so $\mathcal{E}$ is stable), $h^2(\mathbf{P}^3_k, \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{E})) \leqslant (n-2)^2$.*

Proof. By the earlier discussion, we just need to bound $h^1(\mathbf{P}^3_k, (F^*\mathcal{E})(-c_1-4))$ above.

First, let $c_1 = 0$. Using Lemma 2.1, we see that

$$\begin{aligned} h^1(\mathbf{P}^3_k, (F^*\mathcal{E})(-4)) &= h^1(\mathbf{P}^3_k, F^*(\mathcal{E}(-2))) \\ &= h^1(\mathbf{P}^3_k, \mathcal{E}(-2) \otimes F_*(\mathcal{O}_{\mathbf{P}^3_k}) \\ &= h^1(\mathbf{P}^3_k, \mathcal{E}(-2)) + 6h^1(\mathbf{P}^3_k, \mathcal{E}(-3)) + h^1(\mathbf{P}^3_k, \mathcal{E}(-4)). \end{aligned}$$

Now $\mathcal{E}$ has a spectrum of n integers and let the positive integers in the spectrum consist of a_1 ones, a_2 twos, $\ldots, a_r$ r's with no a_i equal to zero by the connectedness of the spectrum. Let $\Sigma a_i = b$. Let $\mathcal{K} = \oplus_{i>0} a_i \mathcal{O}_{\mathbf{P}^1}(i)$. Then $h^1(\mathbf{P}^3_k, \mathcal{E}(-m)) = h^0(\mathbf{P}^1, \mathcal{K}(-m+1))$ for $m \geqslant 1$.

Hence

$$h^1(\mathbf{P}^3_k, \mathcal{E}(-2)) = a_1 + 2a_2 + 3a_3 \cdots + ra_r,$$
$$6h^1(\mathbf{P}^3_k, \mathcal{E}(-3)) = 6(a_2 + 2a_3 \cdots + (r-1)a_r),$$
$$h^1(\mathbf{P}^3_k, \mathcal{E}(-4)) = a_3 + 2a_4 \cdots + (r-2)a_r,$$

and

$$h^1(\mathbf{P}^3_k, (F^*\mathcal{E})(-4)) = a_1 + 8(a_2 + 2a_3 + \cdots + (r-1)a_r).$$

If b is fixed, this sum will be maximized when each $a_i = 1$, giving

$$h^1(\mathbf{P}^3_k, (F^*\mathcal{E})(-4)) \leqslant 1 + 4(b-1)b = (2b-1)^2.$$

Now when $\mathcal{E}$ is semistable, we know by symmetry of the spectrum that $b \leqslant n/2$. If in addition $\mathcal{E}$ is stable, then the spectrum is connected, hence $b \leqslant (n-1)/2$. This gives (i) and (ii).

If $c_1 = -1$, we need to bound $4[h^1(\mathbf{P}^3_k, \mathcal{E}(-2)) + h^1(\mathbf{P}^3_k, \mathcal{E}(-3))]$ which is equal to $4[a_1 + 3a_2 + 5a_3 + \cdots + (2r-1)a_r]$. For fixed $b = \Sigma a_i$, this sum is maximized when all a_i's are 1, hence by $4b^2$. Now $b \leqslant (n/2) - 1$ since the spectrum is connected and symmetric about $-1/2$. Hence the bound of (iii). □

COROLLARY 2.8. *In characteristic two, the moduli spaces of stable rank two bundles on $\mathbf{P}^3_k$ have each component bounded above in dimension: if n is the*

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normalized second Chern class, the dimension is less than or equal to $n^2 + 4n + 1$ *for* $c_1 = 0$ *and less than or equal to* $n^2 + 4n - 1$ *for* $c_1 = -1$.

Proof: If $c_1 = 0$ (respectively -1), then $h^1(\underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{E})) - h^2(\underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{E}))$ equals $8n - 3$ (respectively $8n - 5$), for such a stable bundle $\mathcal{E}$. Now use the last theorem to bound $h^1(\underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{E}))$. □

Remarks 2.9. It is quite likely that this upper bound is too coarse. In fact, one expects that nonreducedness contributes a part to this bound for $h^1(\underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{E}))$. Known examples of large families of stable bundles have dimension much below this bound. Examples of Ellingsrud and Strømme give components of $\mathcal{M}(0, 2k-1)$ of dimension $3k^2 + 4k + 1$, while Ein gives examples in $\mathcal{M}(-1, 2k)$ of dimension $3k^2 + 7k + 2$ (See [E] for both. $k \geqslant 2$ in these examples.) These dimensions are of the order $\frac{3}{4}n^2$.

3. We will try to draw some conclusions about bundles in characteristic zero from these results in characteristic two. So let $\mathcal{E}$ be a rank two bundle on $\mathbf{P}^3_k$, defined over a field k of characteristic zero. By the discussion in the first section, $\mathcal{E}$ is the homology of a minimal monad

$$0 \to \tilde{L}_0^\vee(c_1) \xrightarrow{\beta} \tilde{L}_1 \xrightarrow{\alpha} \tilde{L}_0 \to 0$$

where α, β are matrices of homogeneous polynomials in $S = k[X_0, X_1, X_2, X_3]$. Let $A \subset k$ be a sub-integral domain of k whose field of fractions is k. Since the monad gives an equivalent monad if α, β are multiplied by nonzero elements of k, we may assume that α, β are matrices of homogeneous polynomials in $A[X_0, X_1, X_2, X_3]$. We will call this 'a lift of the monad (and of the bundle $\mathcal{E}$) to A'. This lift is of course by no means unique. Now let $\mathfrak{p}$ be a prime ideal of A such that the residue field $k(\mathfrak{p})$ has characteristic two. For a given A, such an ideal may not exist in A, but there will always be an A in k for which such an ideal exists. Then, taking the lifted monad for $\mathcal{E}$, we may reduce it modulo this ideal, that is to say, we may apply $\otimes_A k(\mathfrak{p})$. In general, we do not expect this to be a monad over the field $k(\mathfrak{p})$; for example, the reduced β may not be an inclusion of bundles.

DEFINITION 3.1. We will say that $\mathcal{E}$ has a good monad reduction to characteristic two if there is a minimal monad for $\mathcal{E}$, as described above for some choice of A and $\mathfrak{p}$, such that the reduction modulo $\mathfrak{p}$ is still a monad over the residue field.

Remarks. Note that in this definition, we are ending up with a bundle in characteristic two which has the same monad type as the bundle we started with. So, for example, if a mathematical instanton bundle has a good monad reduction to characteristic two, it's lift to A specializes to a mathematical instanton bundle in characteristic two.

We can also assume that $(A, \mathfrak{p})$ is a discrete valuation ring. Indeed, by the 'Lefschetz Principle', we may assume that k (the field of definition of $\mathcal{E}$) is finitely generated over $\mathbb{Q}$. If an A has been found in k with a $\mathfrak{p}$ giving good monad

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reduction, we can replace A with its integral closure A' in k and replace $\mathfrak{p}$ with an over-ideal $\mathfrak{p}'$ in A', since reduction to $k(\mathfrak{p}')$ is obtained by base change after first reducing to $k(\mathfrak{p})$. Next, replacing A' with its localization at $\mathfrak{p}'$, we may assume that $(A, \mathfrak{p})$ is a local normal domain with fraction field k. The condition of good reduction modulo $\mathfrak{p}$ and Nakayama's Lemma tell us that the lift of the monad to A defines a monad for a vector bundle on $\mathbf{P}^3_A$, hence also at each localization $A_\mathfrak{q}$ of A. Since $2 \in \mathfrak{p}$, we can choose a height one prime sub-ideal $\mathfrak{q}$ containing 2, getting a discrete valuation ring $A_\mathfrak{q}$ with the required properties.

THEOREM 3.2. *Let $\mathcal{E}$ be a mathematical instanton bundle on $\mathbf{P}^3_k$, where k has characteristic zero.*

(1) *If $\mathcal{E}$ has a good monad reduction to characteristic two, then $\mathcal{E}$ is unobstructed.*
(2) *Suppose there is an element $\varphi \in \mathrm{GL}(4, k)$ such that $\varphi^*\mathcal{E}$ has a good monad reduction to characteristic two. Then $\mathcal{E}$ is unobstructed.*

Proof. This is a standard upper semi-continuity argument. Let A, $\mathfrak{p}$ be as in the definition above. Then since modulo $\mathfrak{p}$, the lift of the monad gives a monad over the residue field, there is an open set U in Spec A which contains $\mathfrak{p}$, and such that on $\mathbf{P}^3_U$, the lift of the monad is a monad, ie. $\mathcal{E}$ lifts to a bundle $\mathcal{E}_U$ on $\mathbf{P}^3_U$. Since at the prime $\mathfrak{p}$, the restriction of $\mathcal{E}_U$ is a mathematical instanton, hence unobstructed (in characteristic two), there is a perhaps smaller open set V in U, where we may take $V = \mathrm{Spec}\, A_f$ for some $f \in A$, over which $H^2(\mathbf{P}^3_V, \underline{\mathrm{Hom}}(\mathcal{E}_V, \mathcal{E}_V)) = 0$, hence also $H^2(\mathbf{P}^3_k, \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{E})) = 0$.

The second result follows since $\mathcal{E}$ and $\varphi^*\mathcal{E}$ have appropriate cohomology groups of the same dimension. □

EXAMPLE 3.3. On $\mathbf{P}^3_\mathbb{Q}$, consider the mathematical instanton bundle $\mathcal{E}$ with $c_1 = 0$, $c_2 = 1$ and minimal monad

$$0 \to \mathcal{O}_{\mathbf{P}^3}(-1) \xrightarrow{\beta} 4\mathcal{O}_{\mathbf{P}^3} \xrightarrow{\alpha} \mathcal{O}_{\mathbf{P}^3}(1) \to 0,$$

where

$$\beta = \begin{bmatrix} -2X_1 \\ X_0 \\ -2X_3 \\ X_2 \end{bmatrix}, \quad \alpha = [X_0 \quad 2X_1 \quad X_2 \quad 2X_3].$$

Certainly, this monad (defined over $\mathbb{Z}$) has bad reduction modulo 2. However, $\mathcal{E}$ has an equivalent monad which has good reduction, for we see that there is an isomorphism of monads over $\mathbb{Q}$ given by

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbf{P}^3}(-1) & \xrightarrow{\beta} & 4\mathcal{O}_{\mathbf{P}^3} & \xrightarrow{\alpha} & \mathcal{O}_{\mathbf{P}^3}(1) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle \gamma} & & \| & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathbf{P}^3}(-1) & \xrightarrow{\beta'} & 4\mathcal{O}_{\mathbf{P}^3} & \xrightarrow{\alpha'} & \mathcal{O}_{\mathbf{P}^3}(1) & \longrightarrow & 0, \end{array}$$

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where

$$\gamma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \qquad \beta' = \begin{bmatrix} -X_1 \\ X_0 \\ -X_3 \\ X_2 \end{bmatrix}, \qquad \alpha' = [X_0 \quad X_1 \quad X_2 \quad X_3].$$

This new monad has good reduction modulo 2, and thus $\mathcal{E}$ has a good monad reduction modulo 2.

EXAMPLE 3.4. Consider the bundle $\mathcal{E}$ defined over $\mathbb{Q}$ by the monad

$$0 \to \mathcal{O}_{\mathbf{P}^3}(-1) \xrightarrow{(-2X_1, 2X_0, -X_3, X_2)^\vee} 4\mathcal{O}_{\mathbf{P}^3} \xrightarrow{(X_0, X_1, X_2, X_3)} \mathcal{O}_{\mathbf{P}^3}(1) \to 0.$$

I claim that this $\mathcal{E}$ does not have a good monad reduction modulo 2.

Proof. Of course, this particular monad is also a lift to $\mathbb{Z}$ which does not reduce well modulo 2. However, it may be that some lift of an equivalent monad may exist, that reduces well. So suppose that A is some integral domain between $\mathbb{Z}$ and $\mathbb{Q}$ and suppose there is a lift to $\mathbf{P}^3_A$ of a monad for $\mathcal{E}$ which reduces well modulo 2. This lift will have matrices β_A and α_A. The two monads are equivalent over $\mathbb{Q}$, hence $\alpha_A = [X_0, X_1, X_2, X_3]\psi$ for some matrix ψ in $\mathrm{GL}(4, \mathbb{Q})$. Clearly ψ can be found with entries in A. Since the right hand maps of both monads are surjective modulo 2, the determinant of ψ is nonzero modulo 2. By localizing at the multiplicative set obtained from the determinant of ψ, we may assume that ψ is invertible over A and 2 is still in $\mathrm{Spec}(A)$. Then we see that $c\beta_A = \psi^{-1}[-2X_1, 2X_0, -X_3, X_2]^\vee$ where $c \in \mathbb{Q}$. Write c as e/f, a ratio of integers in lowest terms. If 2 divides e, then $[-2X_1, 2X_0, -X_3, X_2]^\vee$ is identically zero modulo 2, which is not true. So e is invertible at 2. Hence modulo 2, $\beta_A = \frac{f}{e}\psi^{-1}[0, 0, -X_3, X_2]^\vee$. This contradicts our assumption that β_A modulo 2 is an injection of bundles. □

EXAMPLE 3.5. In Example (3.3), we could also have proceeded as follows: Consider the automorphism of $\mathbf{P}^3_{\mathbb{Q}}$ given by $X_1 \to X_0$, $X_1 \to X_1/2$, $X_2 \to X_2$, $X_3 \to X_3/2$. If φ is this automorphism, $\varphi^*\mathcal{E}$ has the monad with β', α' as described there (so that in this case, φ fixes $\mathcal{E}$).

EXAMPLE 3.6. Let $\mathcal{E}$ be any mathematical instanton bundle with $c_1 = 0$, $c_2 = 1$ defined over a field k. Then $\mathcal{E}$ has a minimal monad

$$0 \to \mathcal{O}_{\mathbf{P}^3}(-1) \xrightarrow{\beta} 4\mathcal{O}_{\mathbf{P}^3} \xrightarrow{\alpha} \mathcal{O}_{\mathbf{P}^3}(1) \to 0,$$

where after a change of basis, we may assume that $\alpha = [X_0 \quad X_1 \quad X_2 \quad X_3]$. Then the kernel of α is $\Omega^1_{\mathbf{P}^3_k}(1)$ and the map β can be understood as picking out a section of $H^0(\mathbf{P}^3_k, \Omega^1_{\mathbf{P}^3_k}(2))$. If V is the vector space $H^0(\mathbf{P}^3_k, \mathcal{O}_{\mathbf{P}^3}(1)$ (defined over k), β can be viewed as picking an element in $\wedge^2 V$. The fact that β is an inclusion of bundles

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means that β picks out an indecomposable vector in $\wedge^2 V$ (up to scalar multiples). Now, the action of $\mathrm{GL}(4, k)$ on $\wedge^2 V$ has one orbit consisting of the indecomposable vectors. Hence, given any $\mathcal{E}$, there is an automorphism φ of $\mathbf{P}^3_k$ such that $\varphi^*\mathcal{E}$ is the standard bundle with the monad described in (3.3)

$$\left(\text{i.e. with matrix } \beta' = \begin{bmatrix} -X_1 \\ X_0 \\ -X_3 \\ X_2 \end{bmatrix}\right).$$

Thus any $\mathcal{E}$ with $c_1 = 0, c_2 = 1$ defined over a field of characteristic zero satisfies the second condition of the Theorem 3.2.

EXAMPLE 3.7. A mathematical instanton which satisfies neither of the assumptions of Theorem 3.2.

Let $k = \mathbb{Q}(\sqrt{-3})$ and let A be the ring of elements integral over $\mathbb{Z}$. The prime number 2 of $\mathbb{Z}$ is undecomposed in A, hence at the prime 2, the extension of residue fields has degree 2. Let l, m be the skew lines in $\mathbf{P}^3_k$, with ideals (X_0, X_1), (X_2, X_3) respectively. Let P_1, P_2, P_3, P_4 be the four points on l with coordinates $(0, 0, 1, 2)$,$(0, 0, 1, 1)$,$(0, 0, 1, 0)$, $(0, 0, 0, 1)$ and Q_1, Q_2, Q_3, Q_4 the four points on m with coordinates $(1, a, 0, 0)$,$(1, 1, 0, 0)$, $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, where $a \in A$ is an element which reduces modulo 2 to an element $\bar{a}$ which is not in $\mathbb{Z}_2$. The cross-ratio on l of the four points in this order is 2 and that of the four points on m is a. Recall that the cross-ratio of four distinct points on a line is an element of the field which is not 0 or 1 and is invariant under automorphisms of the line.

Now let ϕ be an automorphism of $\mathbf{P}^3$ defined over an extension field k' of k. Let l', P'_i, m', Q'_i be the inverse images under ϕ of l, P_i, m, Q_i. The cross-ratio of the appropriate inverse image points is unchanged from the cross-ratio of the original points. Furthermore, suppose A' is a ring in k' whose field of fractions is k' and let $\mathfrak{p}$ be a prime ideal with residue field of characteristic two. In the light of the remarks following (3.1), we will assume that $(A', \mathfrak{p})$ is a discrete valuation ring. Then A' contains A and we can find equations and coordinates for l', P'_i, m', Q'_i defined over A'. The Hilbert schemes of lines and points in $\mathbf{P}^3_{A'}$ are proper over A', hence we can find equations and coordinates which reduce well modulo $\mathfrak{p}$ (more concretely, we can divide the equations and coordinates by a power of the uniformizing parameter of $\mathfrak{p}$ after which good reduction is possible.) Let l'_0, m'_0 be the reductions of l', m' modulo $\mathfrak{p}$. On l'_0, since the reduced cross-ratio is now zero, the four points are no longer distinct. On the other hand, the four points on m' do reduce to distinct points of m'_0 by our choice of a.

The union $Y = P_1Q_1 \cup P_2Q_2 \cup P_3Q_3 \cup P_4Q_4$ is the zero-scheme of a section $s \in H^0(\mathbf{P}^3_k, \mathcal{E}(1))$ for a mathematical instanton bundle $\mathcal{E}$ with $c_2 = 3$, defined over k. In fact, since Y does not lie on a quadric, $\mathcal{E}$ has a unique section up to scalar multiples. We will use three facts in the following:

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(i) The line m is a jumping line for $\mathcal{E}$ of order 3 since it is a quadri-secant for Y. (Indeed, consider the restriction of the sequence for $\mathcal{I}_Y$ to m: $\mathcal{O}_m(-1) \xrightarrow{s} \mathcal{E}_m \to \mathcal{I}_Y(1) \otimes \mathcal{O}_m \to 0$). Hence the same is true for m' as a jumping line of $\mathcal{E}' = \phi^*\mathcal{E}$. Furthermore, if $\mathcal{E}'_{A'}$ is a vector bundle on $\mathbf{P}^3_{A'}$, then the specialization m'_0 of m' after reducing modulo $\mathfrak{p}$ will be a jumping line for $\mathcal{E}'_0$ of order at least 3.

(ii) Let s be a section in degree one for a mathematical instanton $\mathcal{F}$ and let Y be the zero-scheme. Suppose C is a reduced subscheme of Y which lies on a plane. Then C is a line. For if H is the plane so obtained, when s is restricted to a section of $\mathcal{F}_H$, it vanishes along the curve C, hence is divisible by the equation of C. So $\mathcal{F}_H(-d)$ has a section where d is the degree of C. The result now follows from the restriction sequence of $\mathcal{F}$ to H.

(iii) A result of Nüßler and Trautmann (true in any characteristic) states that if $\mathcal{F}$ is a mathematical instanton with a section s in degree one and if m is any line contained in the support of the zero scheme of s, then $\mathcal{F}_m = \mathcal{O}_m(-1) \oplus \mathcal{O}_m(1)$ [N-T].

CLAIM. *Let k' be an extension field of k, and let ϕ be any automorphism of $\mathbf{P}^3_{k'}$. Then $\mathcal{E}' = \phi^*\mathcal{E}$ does not have a good monad reduction modulo two.*

Proof. Assume the contrary. By this assumption, using the same notation as above, there is a vector bundle $\mathcal{E}'_{A'}$ on $\mathbf{P}^3_{A'}$ which specializes to a mathematical instanton $\mathcal{E}'_0$ on $\mathbf{P}^3_{k(\mathfrak{p})}$. Along with a lift of the monad to A', we can lift the section s' of $\mathcal{E}'(1)$ to A'. Hence the specialization of s' will give a section s'_0 of $\mathcal{E}'_0(1)$ which defines a codimension two zero-scheme Y'_0 in $\mathbf{P}^3_{k(\mathfrak{p})}$. By our choice of cross-ratios, the lines in $Y_{A'}$ (the zero-scheme of s') cannot specialize to four disjoint lines in $\mathbf{P}^3_{k(\mathfrak{p})}$. In fact, we claim that m'_0, the specialization of m' is contained in Y'_0.

Indeed, the four points Q'_1, Q'_2, Q'_3, Q'_4 on m' reduce to four distinct points on m'_0, while the four points on l' don't. Since Y'_0 cannot have a component whose reduced subscheme is planar of degree $\geqslant 2$, in particular Y'_0 cannot contain two distinct lines meeting at a point. So the only conclusion is that m'_0 lies inside Y'_0. But this is a contradiction, since on the one hand the restriction of $\mathcal{E}'_0$ to m'_0 splits as $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ using (iii) above and on the other hand as $\mathcal{O}(-3) \oplus \mathcal{O}(3)$ since the limit of a jumping line of order 3 is a jumping line of order at least three (and hence equal to 3 as the maximal possible order of a jumping line is 3).

Thus the claim. □

Lastly, we get the following consequence of Corollary 2.8.

THEOREM 3.8. *Let k be a field of characteristic zero. Suppose N is a component of $\mathcal{M}_{\mathbf{P}^3_k}(c_1, n)$ which contains one bundle $\mathcal{E}$ which reduces modulo two to a stable bundle. Then N has dimension bounded above by $n^2 + 4n + 1$ for $c_1 = 0$ and by $n^2 + 4n - 1$ for $c_1 = -1$.*

Proof. Our definition of good reduction here is more general than the definition in (3.1). It means merely that there is a vector bundle $\mathcal{E}_A$ on $\mathbf{P}^3_A$ such that $\mathcal{E}_A \otimes_A k$

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equals $\mathcal{E}$, and such that $\mathcal{E}_A \otimes_A k(\mathfrak{p})$ is stable. The bounds for $h^1(\mathrm{Hom}(\mathcal{E}, \mathcal{E}))$ of (2.8) are valid over k as well, by upper semi-continuity. □

Remarks 3.9. We do not know if there are components N which violate the condition of Theorem 3.8. The condition of degenerating to a stable bundle can be relaxed to one of degenerating to a semi-stable bundle without any great change in the dimension bound, since Theorem 2.7 still applies. A remark similar to the above theorem can be made about mathematical instantons. Let N be a component of the moduli space of mathematical instantons with $c_1 = 0, c_2 = n$ in characteristic zero. Suppose that N contains one bundle $\mathcal{E}$ which has a good monad reduction modulo two. Then N is generically smooth of dimension $8n - 3$. Of course this is well known for the usual component of instantons, i.e. the component containing the bundles corresponding to skew lines. Even in the example we gave in (3.7), if the example is deformed in moduli by changing the cross-ratio of 2 to a value not in $\mathbb{Z}_2$, we end up with a bundle with good monad reduction. It may well be (as seems to be generally expected) that there is only one component for this moduli space, in which case this remark gives nothing new.

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