Spectral Flows of Dilations of Fredholm Operators

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Abstract. Given an essentially unitary contraction and an arbitrary unitary dilation of it, there is a naturally associated spectral flow that is shown to be equal to the index of the operator. This result is interpreted in terms of the K-theory of an associated mapping cone. It is then extended to connect $\mathbb{Z}_2$ indices of odd symmetric Fredholm operators to a $\mathbb{Z}_2$-valued spectral flow.

The spectral flow of a one-parameter family of self-adjoint Fredholm operators was introduced by Atiyah, Patodi, and Singer [APS] and was under certain conditions shown to be connected to indices of Fredholm operators. A particularly convenient technical reformulation was given by Phillips [Phi1] that also extends to unbounded operators [CP]. As this version of spectral flow is also used here, it is reviewed in Section 1. The first part of the paper considers an essentially unitary contraction operator T on a Hilbert space $\mathcal{K}$ and an arbitrary unitary dilation U_T associated with it, namely a unitary operator on a larger Hilbert space $\mathcal{H}$ so that its compression to $\mathcal{K}$ is T. If P is the projection onto $\mathcal{K}$ and one sets $F = 2P - \mathbf{1}$, then Theorem 2.1 shows that the spectral flow of any path $s \in [0,1] \mapsto F_s$ from F to $U_T^* F U_T$ and with compact $F_s - F$ is equal to the index of T, up to a sign. Two proofs are provided, one by a homotopy argument and one using indices of pairs of projections [Kat, ASS]. This result appears in [Phi2], but for sake of completeness and as preparation for the second part of the paper, we provide the two proofs. Inspired by [Put, CPR], Section 3 then provides a K-theoretic interpretation of the theme *index = spectral flow* based on a mapping cone exact sequence and the pairing of its K-groups with adequate Fredholm modules. Indeed, the exposition makes explicit the tight connection between unitary dilations and Fredholm modules [HR].

More novel is an extension of the above result to essentially unitary operators and dilations having symmetry properties linked to a real or quaternionic structure on the Hilbert space. Examples are real, quaterionic, symmetric, and anti-symmetric operators. Anti-symmetric operators are in bijection with so-called odd symmetric operators, and such Fredholm operators have $\mathbb{Z}_2$ indices given by the dimension of their kernel modulo 2; see Section 4. This can also be deduced from [AS], and an explicit proof was recently given in [SB]. In Section 5 a $\mathbb{Z}_2$-valued spectral flow is defined for paths with an adequate symmetry property. The basic idea leading to this definition is already contained in the analysis of $\mathbb{Z}_2$ invariants of edge states in quantum systems with odd time-reversal symmetry [ASV]. In Section 6 this $\mathbb{Z}_2$-valued spectral flow of

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an odd symmetric dilation is again shown to be equal to the $\mathbb{Z}_2$ index of the dilated Fredholm operator. A mapping cone interpretation with the flavor of Real K-theory [Sch] is sketched in Section 7.

Our motivation to revisit the problem of spectral flow and to extend it to $\mathbb{Z}_2$ indices stems from applications to solid state physics. This is the subject of a companion paper, [DS].

1   Spectral Flow of a Pair of Unitary Equivalent Operators

In order to fix notations and terminology, this section reviews some standard facts from the theory of Fredholm operators on a separable Hilbert space $\mathcal{H}$ as well as the notion of spectral flow. Let $\mathbb{B}(\mathcal{H})$, $\mathbb{K}(\mathcal{H})$, and $\mathbb{F}(\mathcal{H})$ denote the sets of bounded, compact, and Fredholm operators on $\mathcal{H}$, respectively. Recall that $F \in \mathbb{B}(\mathcal{H})$ is Fredholm if and only if it has closed range $T\mathcal{H}$ and finite-dimensional kernel $\mathrm{Ker}(F)$ and cokernel $\mathrm{Ker}(F^*)$. Atkinson's theorem states that Fredholm operators are the invertibles modulo compact perturbation, and this provides the characterization

$$\mathbb{F}(\mathcal{H}) = \left\{ F \in \mathbb{B}(\mathcal{H}) \mid \exists\, S \in \mathbb{B}(\mathcal{H}),\ K_1, K_2 \in \mathbb{K}(\mathcal{H}) \text{ with } SF - K_1 = \mathbf{1} = FS - K_2 \right\}.$$

In particular, the property to be Fredholm is invariant under compact perturbations. Furthermore, $\mathbb{F}(\mathcal{H})$ is an open subset of $\mathbb{B}(\mathcal{H})$ that is closed under the adjoint involution. The Noether index of a Fredholm operator F is defined by $\mathrm{Ind}(F) = \dim(\mathrm{Ker}(F)) - \dim(\mathrm{Ker}(F^*)) \in \mathbb{Z}$. The index is a homotopy invariant that is stable under compact perturbations, namely $\mathrm{Ind}(F) = \mathrm{Ind}(F + K)$ for all $K \in \mathbb{K}(\mathcal{H})$. Moreover, $\mathbb{F}_n(\mathcal{H}) = \mathrm{Ind}^{-1}(n)$ is a path-connected component for any $n \in \mathbb{Z}$ so that the index map $\mathrm{Ind} \colon \mathbb{F}(\mathcal{H}) \to \mathbb{Z}$ establishes a bijection between $\mathbb{Z}$ and the path-connected components of $\mathbb{F}(\mathcal{H}) = \bigcup_{n \in \mathbb{Z}} \mathbb{F}_n(\mathcal{H})$. The set $\mathbb{F}(\mathcal{H})$ has a group structure under the multiplication of operators, and the index map is a group homomorphism onto the additive group $\mathbb{Z}$.

Let $\mathbb{SF}(\mathcal{H}) = \{F \in \mathbb{F}(\mathcal{H}) \mid F = F^*\}$ be the subset of self-adjoint Fredholm operators. Equivalently, $\mathbb{SF}(\mathcal{H}) = \{F \in \mathbb{B}(\mathcal{H}) \mid F = F^* \text{ and } 0 \notin \sigma_{\mathrm{ess}}(F)\}$, where $\sigma_{\mathrm{ess}}(F)$ denotes the essential spectrum. Although $\mathbb{SF}(\mathcal{H}) \subset \mathbb{F}_0(\mathcal{H})$, the space $\mathbb{SF}(\mathcal{H})$ has a non-trivial topology. First of all, it has three disjoint components:

$$\mathbb{SF}_\pm(\mathcal{H}) = \left\{ F \in \mathbb{SF}(\mathcal{H}) \mid \sigma_{\mathrm{ess}}(F) \subset \mathbb{R}_\pm \right\},$$
$$\mathbb{SF}_*(\mathcal{H}) = \left\{ F \in \mathbb{SF}(\mathcal{H}) \mid \sigma_{\mathrm{ess}}(F) \cap \mathbb{R}_\pm \neq \varnothing \right\},$$

where the notation $\mathbb{R}_\pm = \{x \in \mathbb{R} \mid \pm x > 0\}$ was used. The two components $\mathbb{SF}_\pm(\mathcal{H})$ are contractible [AS, Theorem B], but $\pi_1(\mathbb{SF}_*(\mathcal{H})) \cong \mathbb{Z}$ via the spectral flow isomorphism. The spectral flow was introduced in [APS] using the intuitive notion of intersection theory of spectral curves. Here we work instead with the versatile, but equivalent, notion of spectral flow proposed in [Phil].

Let $s \in [0, 1] \mapsto F_s \in \mathbb{SF}_*(\mathcal{H})$ be a continuous path, not necessarily closed. For $a \in (-1, 0]$ and $b \in [0, +1)$ set $Q_{a,b}(s) = \chi_{(a,b]}(F_s)$, where χ_I denotes the characteristic function on $I \subset \mathbb{R}$. By compactness (see Figure 1 where $\sigma_{\mathrm{ess}}(F) = \{-1, 1\}$ is assumed), it is possible to choose a finite partition $0 = s_0 < s_1 < \cdots < s_{N-1} < s_N = 1$
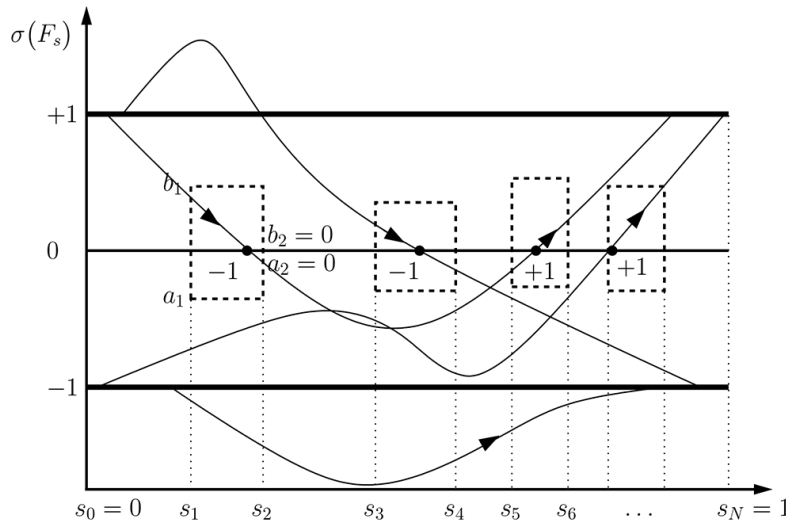
Figure 1: Schematic representation of the objects used in the definition (1.1) of the spectral flow as well as in the second proof of Theorem 2.1. Far from the crossings it is possible to set $a = b = 0$.

of $[0, 1]$ and $a_n < 0 < b_n$, $n = 1, \ldots, N$, such that $s \in [s_{n-1}, s_n] \mapsto Q_{a_n, b_n}(s)$ is continuous with constant (necessarily) finite rank. Then define the spectral flow by

$$(1.1) \qquad \mathrm{Sf}\big(s \in [0, 1] \mapsto F_s\big) = \sum_{n=1}^{N} \mathrm{Tr}_{\mathcal{H}}\big(Q_{a_n, 0}(s_{n-1}) - Q_{a_n, 0}(s_n)\big).$$

Note that $Q_{a_n, 0}(s_{n-1})$ and $Q_{a_n, 0}(s_n)$ are both finite dimensional projections so that the trace is finite. The basic result about the spectral flow is that it is well-defined by the above procedure and it is homotopy invariant. A detailed proof can be found in [Phil].

Theorem 1.1 *The definition of* $\mathrm{Sf}\big(s \in [0, 1] \mapsto F_s\big)$ *is independent of the choice of the partition* $0 = s_0 < s_1 < \cdots < s_{N-1} < s_N = 1$ *of* $[0, 1]$ *and values* $a_n < 0 < b_n$ *such that* $s \in [s_{n-1}, s_n] \mapsto Q_{a_n, b_n}(s)$ *is continuous. Moreover, let* $s \in [0, 1] \mapsto F_s$ *and* $s \in [0, 1] \mapsto G_s$ *be two continuous paths in* $\mathbb{SF}_*(\mathcal{H})$ *such that* $F_0 = G_0$ *and* $F_1 = G_1$. *Then* $\mathrm{Sf}\big(s \in [0, 1] \mapsto F_s\big) = \mathrm{Sf}\big(s \in [0, 1] \mapsto G_s\big)$ *if and only if there exists a norm continuous homotopy between the two paths leaving the endpoints fixed.*

Remark Let us briefly sketch the connection between definition (1.1) and the intuitive notion of spectral flow. By standard perturbation theory arguments [Kat], it is possible to label the spectral curves $\lambda_j(s)$ such that each varies continuously in s. When s increases, the spectral curves $s \mapsto \lambda_j(s)$ can cross the segment $[0, 1] \times \{0\}$. One has a spectral crossing of positive signature if there is a passage from a negative to a positive eigenvalue or a spectral crossing of negative signature if the passage is from a positive to a negative eigenvalue. If there is a finite number of crossings and

no crossings at the boundaries $s = 0$ and $s = 1$, the sum of these signatures over all crossings is equal to $\mathrm{Sf}(s \in [0,1] \mapsto F_s)$. The advantage of definition (1.1) is that the boundaries do not require special treatment and that there may well be an infinite number of crossings.

When Theorem 1.1 is applied to closed paths, it shows that $\mathrm{Sf} \colon \pi_1(\mathbb{SF}_*(\mathcal{H})) \to \mathbb{Z}$ is a well-defined group homomorphism. To verify that this map is bijective, one can use the fact that the space

$$\widehat{\mathbb{SF}}_*(\mathcal{H}) = \left\{ F \in \mathbb{SF}_*(\mathcal{H}) \mid \|F\| = 1 \quad \sigma_{\mathrm{ess}}(F) = \{-1,1\} \right\}$$

is a deformation retract of $\mathbb{SF}_*(\mathcal{H})$ and that the map $\varphi \colon \widehat{\mathbb{SF}}_*(\mathcal{H}) \to \mathbb{U}_*(\mathcal{H})$ defined by $\varphi(F) = e^{\imath \pi (F+\mathbf{1})}$ is an homotopy equivalence [AS]. Here, $\mathbb{U}_*(\mathcal{H})$ is the subgroup of those unitary operators $U \in \mathbb{U}(\mathcal{H})$ for which $U - \mathbf{1} \in \mathbb{K}(\mathcal{H})$. The sequence of isomorphisms $\pi_1(\mathbb{SF}_*(\mathcal{H})) \cong \pi_1(\widehat{\mathbb{SF}}_*(\mathcal{H})) \cong \pi_1(\mathbb{U}_*(\mathcal{H}))$ combined with the standard isomorphism $\pi_1(\mathbb{U}_*(\mathcal{H})) \to \mathbb{Z}$ given by the *winding number* concludes the description of the fundamental group of $\mathbb{SF}_*(\mathcal{H})$. Just for sake of completeness let us recall that the winding number of a differentiable loop $s \in [0,1] \mapsto U_s \in \mathbb{U}_*(\mathcal{H})$ is given by

$$\mathrm{Wind}(s \in [0,1] \mapsto U_s) = \frac{1}{2\pi \imath} \int_0^1 ds \, \mathrm{Tr}\left((U_s)^{-1} \partial_s U_s \right),$$

whenever $\partial_s U_s$ is traceclass. For adequate paths, the equality $\mathrm{Sf}(s \in [0,1] \mapsto F_s) = \mathrm{Wind}(s \in [0,1] \mapsto \varphi(F_s))$ provides an alternative formula for the computation of the spectral flow.

Focus here will be on the spectral flow of certain special paths in $\widehat{\mathbb{SF}}_*(\mathcal{H})$. Given $F \in \widehat{\mathbb{SF}}_*(\mathcal{H})$ and a unitary $U \in \mathbb{U}(\mathcal{H})$, let $\Theta(F,U)$ denote the set of paths $s \in [0,1] \mapsto F_s$ such that

(i) $\qquad\qquad\qquad\qquad F_0 = F,$

(ii) $\qquad\qquad\qquad\qquad F_s - F \in \mathbb{K}(\mathcal{H})$ for all $s \in [0,1],$

(iii) $\qquad\qquad\qquad\qquad F_1 = U^* F U.$

Note that $\sigma(F_1) = \sigma(F_0)$. The spectral flow $\mathrm{Sf}(s \in [0,1] \mapsto F_s)$ of such a path is well defined and equal to the spectral flow of the path obtained by concatenation with the isospectral path $r \in [0,1] \mapsto (U^r)^* F U^r$.

Proposition 1.2   *The spectral flow* $\mathrm{Sf} \colon \Theta(F,U) \to \mathbb{Z}$ *is equal to a constant denoted by* $\mathrm{Sf}(F,U)$. *In particular,*

(1.2) $$\mathrm{Sf}(F,U) = \mathrm{Sf}\left(s \in [0,1] \mapsto F + s\, U^*[F,U] \right).$$

Proof   Let $s \in [0,1] \mapsto F_s$ and $s \in [0,1] \mapsto F_s'$ be two paths in $\Theta(F,U)$. Then $r \in [0,1] \mapsto r F_s + (1-r) F_s'$ is a homotopy in $\Theta(F,U)$ keeping the initial and final point fixed, so that Theorem 1.1 implies the first claim. The second follows because $s \in [0,1] \mapsto F + s\, U^*[F,U]$ is indeed a path in $\Theta(F,U)$, since $U^*[F,U] = F_1 - F_0 \in \mathbb{K}(\mathcal{H})$.   ∎

Let us introduce the set of operator pairs for which a spectral flow can be defined:

$$\mathbb{P}(\mathcal{H}) = \left\{ (F,U) \in \widehat{\mathbb{SF}}_*(\mathcal{H}) \times \mathbb{U}(\mathcal{H}) \mid [F,U] \text{ compact} \right\}.$$

It carries the subspace topology induced from the norm topology on $\mathbb{B}(\mathcal{H}) \times \mathbb{B}(\mathcal{H})$. Note that indeed each point $(F, U) \in \mathbb{P}(\mathcal{H})$ defines a class of paths $\Theta(F, U)$ so that it is possible to view the spectral flow as a map $\mathrm{Sf} \colon \mathbb{P}(\mathcal{H}) \to \mathbb{Z}$ (strictly speaking this is $\mathrm{Sf} \circ \Theta$).

Proposition 1.3 *The spectral flow* $\mathrm{Sf} \colon \mathbb{P}(\mathcal{H}) \to \mathbb{Z}$ *is locally constant.*

Proof Let $(F(0), U(0))$ and $(F(1), U(1))$ be two points connected by a continuous path $r \in [0, 1] \mapsto (F(r), U(r))$ in $\mathbb{P}(\mathcal{H})$. This means that there is a path α in $\widehat{\mathbb{SF}}_*(\mathcal{H})$, given by $r \mapsto F(r)$, which connects $F(0)$ with $F(1)$ and a second path β in $\widehat{\mathbb{SF}}_*(\mathcal{H})$, given by $r \mapsto U(r)^* F(r) U(r)$, which connects $U(0)^* F(0) U(0)$ with $U(1)^* F(1) U(1)$. Let $\gamma_r \in \Theta(F(r), U(r))$ be any path connecting $F(r)$ to $U(r)^* F(r) U(r)$ by compact perturbations. For each $r \in [0, 1]$ let us denote by α_r the reduced path that connects $F(0)$ with $F(r)$ along the path α. In similar way let us introduce also the reduced path β_r. The composed paths $\theta_r = \beta_r^{-1} \circ \gamma_r \circ \alpha_r$ produce a homotopy between $\theta_0 = \gamma_0$ and θ_1 within the class of paths having same extreme points $F(0)$ and $U(0)^* F(0) U(0)$. Theorem 1.1 applies so that $\mathrm{Sf}(\theta_1) = \mathrm{Sf}(\theta_0) = \mathrm{Sf}(F(0), U(0))$. Now, by applying the composition rule of the spectral flow, $\mathrm{Sf}(\theta_1) = \mathrm{Sf}(\alpha) + \mathrm{Sf}(\gamma_1) - \mathrm{Sf}(\beta)$, and observing that $\mathrm{Sf}(\alpha) = \mathrm{Sf}(\beta)$ one gets $\mathrm{Sf}(F(0), U(0)) = \mathrm{Sf}(\gamma_1) = \mathrm{Sf}(F(1), U(1))$. ∎

2 Dilation of a Fredholm Operator and its Spectral Flow

In this section, the index of an arbitrary Fredholm operator is being calculated as a spectral flow, in fact, of a whole family of spectral flows associated with arbitrary dilations. Let $\mathcal{K}$ be a separable Hilbert space. Let $T \in \mathbb{B}(\mathcal{K})$ be a contraction, namely $\|T\| \le 1$. A unitary dilation of T is a unitary operator $U_T \in \mathbb{B}(\mathcal{H})$ on some Hilbert space $\mathcal{H}$ in which $\mathcal{K}$ is isometrically embedded by an injective partial isometry $\Pi \colon \mathcal{K} \hookrightarrow \mathcal{H}$ such that

$$(2.1) \qquad\qquad T = \Pi^* U_T \Pi.$$

To exclude a trivial case, it will always be assumed that $\mathcal{H} \ominus \mathcal{K}$ is infinite dimensional. Recall that an operator T is called essentially unitary if $T^* T - \mathbf{1}$ and $T T^* - \mathbf{1}$ are compact operators. The set $\mathbb{EU}(\mathcal{K})$ of essentially unitary operators is a subset of the Fredholm operators $\mathbb{F}(\mathcal{K})$.

Theorem 2.1 ([Phi2]) *Let $U_T \in \mathbb{U}(\mathcal{H})$ be a unitary dilation of an essentially unitary contraction $T \in \mathbb{EU}(\mathcal{K})$ and let $\Pi \colon \mathcal{K} \hookrightarrow \mathcal{H}$ be the associated injective partial isometry. Then, with $F = 2\, \Pi \Pi^* - \mathbf{1} \in \widehat{\mathbb{SF}}_*(\mathcal{H})$,*

$$(2.2) \qquad \mathrm{Ind}(T) = -\mathrm{Sf}(F, U_T) = -\mathrm{Sf}\big(s \in [0, 1] \mapsto F + s\, U_T^* [F, U_T] \big) .$$

Two proofs will be provided, one by constructing a homotopy to a special dilation for which the identity (2.2) can be checked by direct computation, and one that uses the index of Fredholm pairs of projections introduced by Kato [Kat] and studied in detail in [ASS].

Proof of Theorem 2.1 by Homotopy   Let $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}'$ for some Hilbert space $\mathcal{K}'$. In this grading,

$$(2.3) \qquad U_T = \begin{pmatrix} T & B \\ C & D \end{pmatrix}, \qquad F = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix},$$

with adequate operators B, C, and D. By unitarity, $BB^* = \mathbf{1} - TT^*$ and $C^*C = \mathbf{1} - T^*T$. As T is essentially unitary, it follows that BB^* and C^*C are compact, and thus B and C are also compact (*e.g.*, by the polar decomposition). This implies that $[F, U_T] \in \mathbb{K}(\mathcal{H})$. Furthermore, $F \in \widehat{\mathbb{SF}}_*(\mathcal{H})$ and thus by Proposition 1.2, the spectral flow $\mathrm{Sf}(F, U_T)$ is well defined and the second equality in (2.2) holds. Because it was assumed that $\mathcal{K}' = \mathcal{H} \ominus \mathcal{K}$ is infinite dimensional, there exists a unitary map from $\mathcal{K}$ to $\mathcal{K}'$ and, as a basis change does not change the spectral flow, one may suppose $\mathcal{K} = \mathcal{K}'$ from now on. Actually, the infinite dimensionality of $\mathcal{K}'$ follows if $\mathrm{Ind}(T) \neq 0$. Indeed, by unitarity of U_T, $D^*D - \mathbf{1} = B^*B$ and $DD^* - \mathbf{1} = CC^*$ are also compact, so that D is also an essential unitary operator. Since the index of U_T vanishes and the index is stable under compact perturbations, it follows that $0 = \mathrm{Ind}(T \oplus D) = \mathrm{Ind}(T) + \mathrm{Ind}(D)$. But a non-vanishing index only exists in infinite dimension.

The basic idea of the proof is to verify (2.2) for one particular unitary dilation U_T^{H} and then to show that any other unitary dilation U_T is connected with U_T^{H} by a continuous path $r \in [0,1] \mapsto U(r)$, which is such that $(F, U(r)) \in \mathbb{P}(\mathcal{H})$ holds (which is equivalent to the off-diagonal entry of $U(r)$ being compact). By Proposition 1.2 one then has that $\mathrm{Sf}(F, U_T) = \mathrm{Sf}(F, U_T^{\mathrm{H}})$ and one can conclude that (2.2) holds for any unitary dilation if only it can be checked for the special dilation U_T^{H}. The latter is chosen to be the Halmos dilation [Hal]:

$$(2.4) \qquad U_T^{\mathrm{H}} = \begin{pmatrix} T & (\mathbf{1} - TT^*)^{\frac{1}{2}} \\ (\mathbf{1} - T^*T)^{\frac{1}{2}} & -T^* \end{pmatrix}.$$

Let $T = V|T|$ be the polar decomposition with partial isometry V such that $\mathrm{Ker}(V) = \mathrm{Ker}(T)$. Due to the continuous path $r \in [0,1] \mapsto V|T|^r$ and the stability of the index, one has $\mathrm{Ind}(T) = \mathrm{Ind}(V)$. Let us consider the path $r \in [0,1] \mapsto U(r)$ with

$$U(r) = \begin{pmatrix} V|T|^r & (\mathbf{1} - V|T|^{2r}V^*)^{\frac{1}{2}} \\ (\mathbf{1} - |T|^r V^*V|T|^r)^{\frac{1}{2}} & -|T|^r V^* \end{pmatrix}.$$

Indeed, $(F, U(r)) \in \mathbb{P}(\mathcal{H})$ and the path connects $U(1) = U_T^{\mathrm{H}}$ to

$$U(0) = \begin{pmatrix} V & (\mathbf{1} - VV^*)^{\frac{1}{2}} \\ (\mathbf{1} - V^*V)^{\frac{1}{2}} & -V^* \end{pmatrix}.$$

Now

$$F + sU(0)^*[F, U(0)] = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} + 2s \begin{pmatrix} -(\mathbf{1} - V^*V) & 0 \\ 0 & \mathbf{1} - VV^* \end{pmatrix},$$

where one uses that $\mathbf{1} - V^*V$ and $\mathbf{1} - VV^*$ are the orthogonal projections on $\mathrm{Ker}(V) = \mathrm{Ker}(T)$ and $\mathrm{Ker}(V^*) = \mathrm{Ker}(T^*)$, respectively. These relations also imply that the spectral flow of $s \in [0,1] \mapsto F + sU(0)^*[F, U(0)]$ is equal to $-\mathrm{Ind}(T)$, and therefore the identity (2.2) for the Halmos dilation follows.

It now remains to construct a continuous map from an arbitrary dilation U_T to U_T^{H} such that, when combined with F, one stays in $\mathbb{P}(\mathcal{H})$. For that purpose, let us consider $W = (U_T^{\mathrm{H}})^* U_T$ and show that it is homotopic to the identity with a path having compact off-diagonal. The matrix entries of $W = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ satisfy $\mathrm{Ind}(A) = \mathrm{Ind}(D) = 0$ (actually, one also has $A = T^*T$, but this is irrelevant in the sequel), and B and C are compact. Consequently, by standard Fredholm theory there are partial isometries V_A and V_D such that $A + \epsilon V_A$ and $D + \epsilon V_D$ are invertible for $\epsilon > 0$. Now choose $\epsilon \in (0, 1)$ sufficiently small such that the path

$$r \in [0, \epsilon] \mapsto W(r) = \begin{pmatrix} A + r\,V_A & B \\ C & D + r\,V_D \end{pmatrix}$$

remains in the invertibles (the spectrum of $W(r)$ does not touch 0 if $r < 1$). Next decompose

$$W(r) = \begin{pmatrix} A + r\,V_A & 0 \\ 0 & D + r\,V_D \end{pmatrix} \begin{pmatrix} \mathbf{1} & (A + r\,V_A)^{-1} B \\ (D + r\,V_D)^{-1} C & \mathbf{1} \end{pmatrix}.$$

The second factor is a compact perturbation of the identity and can therefore be continuously deformed within the compact operators to the identity, *e.g.,* using spectral theory for compact operators. Thus one obtains a path $r \in [\epsilon, 1] \mapsto W(r)$ in the invertibles on $\mathcal{K} \oplus \mathcal{K}$ with compact off-diagonal entries connecting $W(\epsilon)$ to the identity (as the invertibles on $\mathcal{K}$ are also path connected by the polar decomposition). From this path of invertibles one obtains the desired path of unitaries $r \in [0, 1] \mapsto W(r)|W(r)|^{-1}$. Let us note that the homotopy is not a path of unitary dilations of T, but this is not needed in order to connect the spectral flows. $\blacksquare$

Proof of Theorem 2.1 using Fredholm Pairs of Projections (This proof looks shorter than the above, but appeals to several results from [ASS].) Let U_T and F be as in (2.3) and consider any path $[0, 1] \in s \to F_s$ in $\Theta(F, U_T)$, namely such that $F_0 = F$, $F_1 = U_T^* F U_T$ and $F_s - F \in \mathbb{K}(\mathcal{H})$ for all $s \in [0, 1]$. Let us start from (1.1). As $Q_{a_n,0}(s_{n-1})$ and $Q_{a_n,0}(s_n)$ are both finite dimensional projections, the difference of their races can be expressed in terms of the index of a pair of projections. One of the equivalent definitions in [ASS] is $\mathrm{Ind}(P, Q) = \mathrm{Ind}(QPQ)$ whenever QPQ is a Fredholm operator on $Q\mathcal{H}$. With this,

$$\mathrm{Sf}\big(s \in [0, 1] \mapsto F_s \big) = \sum_{n=1}^{N} \mathrm{Ind}\big(Q_{a_n,0}(s_{n-1}), Q_{a_n,0}(s_n) \big).$$

Let us consider the orthogonal decomposition $P(s) = Q_a(s) \oplus Q_{a,0}(s)$ for $a < 0$. The map $[s_{n-1}, s_n] \ni s \mapsto Q_{a_n}(s)$ is continuous, and for any pair $s_{n-1} \leqslant s_1' < s_2' \leqslant s_n$ an application of the Riesz integral and of the resolvent identity shows that the difference $Q_{a_n}(s_1') - Q_{a_n}(s_2')$ is compact and $\|Q_{a_n}(s_1') - Q_{a_n}(s_2')\| \leqslant C \|F_{s_1'} - F_{s_2'}\|$. It follows from these two facts together that $Q_{a_n}(s_{n-1})$ and $Q_{a_n}(s_n)$ are a Fredholm pair with $\mathrm{Ind}(Q_{a_n}(s_{n-1}), Q_{a_n}(s_n)) = 0$ [ASS, Proposition 3.2 and Theorem 3.4(c)]; see also [BCP, Lemma 4.1] where a characterization for being a Fredholm pair is given. Since the computation of the index is linear with respect to the orthogonal sum of projections [ASS, Lemma A.9], one has $\mathrm{Ind}(Q_{a_n,0}(s_{n-1}), Q_{a_n,0}(s_n)) =$

$\mathrm{Ind}(P(s_{n-1}), P(s_n))$. Observing that the difference $P(s_{n-1}) - P(s_n)$ is compact, one uses again [ASS, Theorem 3.4(c)] for the sum of the telescopic series:

$$\mathrm{Sf}\big(s \in [0,1] \mapsto F_s \big) = \mathrm{Ind}\big(P(0), P(1) \big) = \mathrm{Ind}\big(P(0), U_T^* P(0) U_T \big)$$
$$= \mathrm{Ind}\big(P(0) U_T P(0) \big),$$

where the last step is provided by [ASS, Theorem 5.2] and $P(0)U_T P(0)$ has to be considered as an operator on $P(0)\mathcal{H}$. Observing that $P(0) \oplus P = \mathbf{1}$, with $P = \chi_{[0,+\infty)}(F)$ the spectral projection of F on the positive spectrum, one concludes that $\mathrm{Ind}(P(0)U_T P(0)) = -\mathrm{Ind}(PU_T P) = -\mathrm{Ind}(T)$, since $PU_T P|_{P\mathcal{H}} = \Pi^* U_T \Pi = T$. ∎

3   Mapping Cone of a Fredholm Module

Let $\mathcal{A}$ be a C*-algebra that is realized as a subalgebra of the bounded operators $\mathbb{B}(\mathcal{H})$ on a separable Hilbert space $\mathcal{H}$. Associated are the K-groups $K_0(\mathcal{A})$ and $K_1(\mathcal{A})$ of homotopy classes of projections and unitaries in $\mathcal{A}$. Topological content can be extracted from these groups via pairing with Fredholm modules. The ungraded version of an even bounded Fredholm module $(\mathcal{H}, F)$ for $\mathcal{A}$ is a unitary operator F on $\mathcal{H}$ such that for all $A \in \mathcal{A}$ the commutators $[F, A] \in \mathbb{K}(\mathcal{H})$. An odd bounded Fredholm module is an even bounded Fredholm module $(\mathcal{H}, F)$ for which, moreover, $F^2 = \mathbf{1}$ and $\sigma_{\mathrm{ess}}(F) = \{-1, 1\}$. Equivalently, the unitary F lies in $\widehat{\mathbb{SF}}_*(\mathcal{H})$. In the literature ([Con]), even Fredholm modules are always graded, so let us briefly explain the connection as well as the relation between unitary dilations and odd Fredholm modules.

Remark 1   By definition [Con], a (graded version of an) even Fredholm module $(\widehat{\mathcal{H}}, \widehat{F}, \Gamma)$ for a C*-algebra $\widehat{\mathcal{A}}$ is an odd Fredholm module $(\widehat{\mathcal{H}}, \widehat{F})$ together with a (unitary) grading operator Γ satisfying $\Gamma^2 = \mathbf{1}$, $\Gamma A \Gamma = A$ for all $A \in \widehat{\mathcal{A}}$, and $\Gamma \widehat{F} \Gamma = -\widehat{F}$. Given an ungraded version of an even bounded Fredholm module $(\mathcal{H}, F)$ for $\mathcal{A}$, one obtains a graded version by setting $\widehat{\mathcal{H}} = \mathcal{H} \otimes \mathbb{C}^2$ and $\widehat{\mathcal{A}} = \mathcal{A} \otimes \mathbf{1}_2$, furthermore setting $\widehat{F} = \mathfrak{Re}(F)\,\sigma^1 + \mathfrak{Im}(F)\,\sigma^2$ and $\Gamma = \sigma^3$, where σ^1, σ^2, and σ^3 are the Pauli matrices and $\mathfrak{Re}(F) = \frac{1}{2}(F + F^*)$ as well as $\mathfrak{Im}(F) = \frac{1}{2\imath}(F - F^*)$. Inversely, given a graded version of an even Fredholm module, the reduction to the spectral subspaces of Γ allows the extraction of two ungraded versions of an even Fredholm module.

Remark 2   The operator F of an odd Fredholm module is also called the Dirac phase because it can often be obtained from a so-called unbounded Fredholm module $(\mathcal{H}, D)$ with Dirac operator D. Here, D is an invertible self-adjoint operator with compact resolvent and a domain $\mathcal{D}(D)$ that is left invariant under $\mathcal{A}$, and such that the commutators $[A, D]$ are compact operators for all $A \in \mathcal{A}$. From this data one constructs a bounded Fredholm module by setting $F = D|D|^{-1}$; see [Con, CP]. Similarly, the operator $\widehat{F}$ of a graded version of an even Fredholm module is often obtained from a Dirac operator.

Remark 3   Let $T \in \mathbb{EU}(\mathcal{K})$ be an essentially unitary Fredholm operator on a Hilbert space $\mathcal{K}$ and let $U_T \in \mathbb{U}(\mathcal{H})$ be a unitary dilation on a Hilbert space $\mathcal{H}$ in which

there is an isometric embedding $\Pi\colon \mathcal{K} \mapsto \mathcal{H}$; see (2.1). Then setting $\mathcal{A} = C^*(U_T)$ and $F = 2\,\Pi\Pi^* - \mathbf{1}$, one has a odd Fredholm module $(\mathcal{H}, F)$ for $\mathcal{A}$. Indeed, $F \in \widehat{\mathbb{SF}}_*(\mathcal{H})$ and the off-diagonal entries of U_T are compact by the argument of the proof of Theorem 2.1, which assures the required compactness. Inversely, if $(\mathcal{H}, F)$ is an odd Fredholm modul for $\mathcal{A} = C^*(U)$ for some unitary $U \in \mathbb{B}(\mathcal{H})$, then U is a unitary dilation of PUP where $P = \frac{1}{2}(F + \mathbf{1})$.

It is well known [Con, Chapter 4] that odd Fredholm modules pair integrally with the K-group $K_1(\mathcal{A})$ via

$$\big\langle (\mathcal{H}, F), [U]_1 \big\rangle_1 = \mathrm{Ind}(PUP), \quad P = \tfrac{1}{2}(F + \mathbf{1}),$$

where $U \in \mathcal{A}$ and PUP is understood as an operator on $\mathcal{K} = P\mathcal{H} = \mathrm{Ran}(P)$, and a natural generalization holds if U only lies in a matrix algebra over $\mathcal{A}$. This makes sense because PUP is indeed a Fredholm operator (with pseudo-inverse PU^*P). Similarly, ungraded versions of even Fredholm modules pair integrally with $K_0(\mathcal{A})$. Explicitly, again only for a projection $P \in \mathcal{A}$,

(3.1) $$\big\langle (\mathcal{H}, F), [P]_0 \big\rangle_0 = \mathrm{Ind}(PFP),$$

where PFP is a Fredholm operator again on $\mathcal{K} = P\mathcal{H} = \mathrm{Ran}(P)$. The presentation already makes it clear that the two pairings are quite similar and connected via

$$\big\langle (\mathcal{H}, F), [P]_0 \big\rangle_0 = \big\langle (\mathcal{H}, 2P - \mathbf{1}), [F]_1 \big\rangle_1,$$

where on the right-hand side one has a pairing over the algebra $C^*(F)$.

Remark 4 If one works with the graded version of an even Fredholm module $(\widehat{\mathcal{H}}, \widehat{F}, \Gamma)$ for a C*-algebra $\widehat{\mathcal{A}}$, the even pairing takes the familiar form [Con]

$$\big\langle (\widehat{\mathcal{H}}, \widehat{F}, \Gamma), [\widehat{P}]_0 \big\rangle_0 = \mathrm{Ind}(P_- F P_+), \quad \widehat{F} = \begin{pmatrix} 0 & F^* \\ F & 0 \end{pmatrix}, \quad \widehat{P} = \begin{pmatrix} P_+ & 0 \\ 0 & P_- \end{pmatrix},$$

where $\widehat{F}$ and $\widehat{P}$ are written in the grading of Γ and $P_- F P_+$ is understood to be an operator from $\mathrm{Ran}(P_+)$ to $\mathrm{Ran}(P_-)$. If $P_+ = P_-$, this reduces to formula (3.1).

It is possible to calculate these two indices from the Chern characters constructed from the corresponding Fredholm modules [Con], but here the aim will be to link these indices to adequate spectral flows. For this purpose, let us define the F-twisted mapping cone associated with $(\mathcal{H}, F)$ by

$$\mathcal{M} = \Big\{ (A_s)_{s \in [0,1]} \in C([0,1], \mathcal{A} + \mathbb{K}(\mathcal{H})) \mid A_0 = F^* A_1 F \in \mathcal{A}$$
$$\text{and } K_s = A_s - A_0 \in \mathbb{K}(\mathcal{H}) \Big\}.$$

This is a C*-algebra, and the evaluation application $\mathrm{ev}\colon \mathcal{M} \to \mathcal{A}$ defined by $\mathrm{ev}((A_s)_{s \in [0,1]}) = A_0$ is an algebra homomorphism. Furthermore, the suspension $S\mathbb{K}(\mathcal{H}) = C_0((0,1), \mathbb{K}(\mathcal{H}))$ is a subalgebra embedded in $\mathcal{M}$ so that $S\mathbb{K}(\mathcal{H}) \hookrightarrow \mathcal{M} \to \mathcal{A}$ is exact. Finally, an explicit lift of $A \in \mathcal{A}$ into $\mathcal{M}$ is given by $(1 - s)A + s\,F^*AF = A + s\,F^*[A, F]$. An arbitrary lift of $A \in \mathcal{A}$ is denoted by $\mathrm{Lift}(A) = (\mathrm{Lift}(A)_s)_{s \in [0,1]}$. If A is selfadjoint, one can always assume the lift to be selfajoint as

well (just work with $\frac{1}{2}(\mathrm{Lift}(A)_s^* + \mathrm{Lift}(A)_s)$ if necessary). Resuming, one has the following proposition.

Proposition 3.1 *The sequence*

$$0 \longrightarrow S\mathbb{K}(\mathcal{H}) \hookrightarrow \mathcal{M} \xrightarrow{\mathrm{ev}} \mathcal{A} \longrightarrow 0$$

is exact.

This sequence is splitting if and only if F commutes with $\mathcal{A}$. The 6-term exact sequence of K-theory associated with the exact sequence is

$$
\begin{array}{ccccc}
K_0(S\mathbb{K}(\mathcal{H})) & \xrightarrow{\imath_*} & K_0(\mathcal{M})) & \xrightarrow{\mathrm{ev}_*} & K_0(\mathcal{A}) \\
{\scriptstyle \mathrm{Ind}} \big\uparrow & & & & \big\downarrow {\scriptstyle \exp} \\
K_1(\mathcal{A}) & \xleftarrow{\mathrm{ev}_*} & K_1(\mathcal{M}) & \xleftarrow{\imath_*} & K_1(S\mathbb{K}(\mathcal{H}))
\end{array}
$$

Here, $K_0(S\mathbb{K}(\mathcal{H})) = K_1(\mathbb{K}(\mathcal{H})) = 0$ and $K_1(S\mathbb{K}(\mathcal{H})) = K_0(\mathbb{K}(\mathcal{H})) = \mathbb{Z}$. The connecting map $\exp\colon K_0(\mathcal{A}) \to K_1(S\mathbb{K}(\mathcal{H}))$ on the right-hand side is, in terms of a (self-adjoint) lift, explicitly given by

$$(3.2) \qquad \exp([P]_0) = \big[\exp\big(2\pi\imath \, \mathrm{Lift}(P)_s \big) \big]_1 = \big[\exp\big(2\pi\imath(P + s\,F^*[P,F]) \big) \big]_1,$$

where in the second equality an explicit lift was chosen. Furthermore, the index map from $K_1(\mathcal{A})$ to $K_0(S\mathbb{K}(\mathcal{H})) = 0$ is trivial. Thus, the other map of interest to us, $\mathrm{ev}_*\colon K_1(\mathcal{M}) \to K_1(\mathcal{A})$, is surjective and has an inverse $\mathrm{ev}^*\colon K_1(\mathcal{A}) \to K_1(\mathcal{M})$. Therefore, every unitary $U \in \mathcal{A}$ has a unitary lift $\mathrm{Lift}(U) \in \mathcal{M}$. In general, it may be difficult to find an explicit formula for such a unitary lift, but if the Fredholm module is odd, then an explicit choice is given by

$$\mathrm{ev}^*([U]_1) = \big[U \, \exp\big(\tfrac{\imath\pi}{2}\big(-\mathbf{1} + (1-s)F + s\,U^*FU \big) \big) \, F \big]_1$$
$$= \big[U \, \exp\big(\tfrac{\imath\pi}{2}\big(F - \mathbf{1} + s\,U^*[F,U] \big) \big) \, F \big]_1.$$

A further difference between odd and ungraded even Fredholm modules is pointed out in the following result.

Proposition 3.2 *If $(\mathcal{H}, F)$ is odd, then $\exp = 0$; that is, every class of projections in $\mathcal{A}$ is the image of a class of projections in $\mathcal{M}$.*

Proof If the Fredholm module is odd, then it is possible to choose a lift given by a path of projections. By (3.2) this implies the result. Let $Q = \frac{1}{2}(F + \mathbf{1})$ be the positive spectral projection of F. Then set

$$\mathrm{Lift}(P)_s = QPQ + e^{\imath\pi s}QP(\mathbf{1} - Q) + e^{-\imath\pi s}(\mathbf{1} - Q)PQ + (\mathbf{1} - Q)P(\mathbf{1} - Q),$$

which can indeed be checked to be self-adjoint and idempotent. ∎

Resuming, for an odd Fredholm module and a unitary U one has $\mathrm{ev}^*([U]_1) \in K_1(\mathcal{M})$, while for an even Fredholm module and a projection P one has $\exp([P]_0) \in K_1(S\mathbb{K}(\mathcal{H}))$. In both cases, there is an associated spectral flow. Let us begin with the odd case. The path of unitaries $s \in [0, 1] \mapsto V_s = U^* \mathrm{Lift}(U)_s F$ satisfies $V_0 = F$ and $V_1 = U^*FU$. Hence the spectrum of V_0 and V_1 consists of two infinitely degenerate

eigenvalues $\{-1, 1\}$ so that $V_0, V_1 \in \widehat{\mathbb{SF}}_*(\mathcal{H})$. As V_s is a compact perturbation of V_0, there are a finite number of eigenvalues (on the unit circle) moving from -1 to 1 along the path $s \in [0, 1] \mapsto V_s$, and this allows one to define a spectral flow. In order to formulate this spectral flow again in the picture of paths of self-adjoint operators presented in Section 1, one can consider $\mathfrak{Re}(V_s) = \frac{1}{2}(V_s - V_s^*)$ the spectrum of which consists of the spectrum of V_s projected onto the real line (along the imaginary axis). Thus,

$$\mathrm{Sf}\big(s \in [0, 1] \mapsto U^* \mathrm{Lift}(U)_s F\big) = \mathrm{Sf}\big(s \in [0, 1] \mapsto \mathfrak{Re}(V_s)\big) = \mathrm{Sf}(F, U),$$

where the last equality reflects $\mathfrak{Re}(V_0) = F$ and $\mathfrak{Re}(V_1) = U^* F U$. Theorem 2.1 combined with the above discussion then implies the following theorem.

Theorem 3.3 *Let $U \in \mathcal{A}$ and $(\mathcal{H}, F)$ be an odd Fredholm module. Then*

$$\big\langle (\mathcal{H}, F), [U]_1 \big\rangle_1 = \mathrm{Sf}(F, U).$$

Next let us consider the even case. First of all, $\exp(2\pi\imath \mathrm{Lift}(P)_s) - \mathbf{1} \in \mathbb{K}(\mathcal{H})$ so that the essential spectrum of $\exp(2\pi\imath \mathrm{Lift}(P)_s)$ is $\{1\}$ for all s independent of the choice of the self-adjoint lift, namely $\exp(2\pi\imath \mathrm{Lift}(P)_s) \in \mathbb{U}_*(\mathcal{H})$. Therefore, the path has a winding number

$$\mathrm{Wind}\big(s \in [0, 1] \mapsto \exp(2\pi\imath \mathrm{Lift}(P)_s)\big) = \mathrm{Sf}(2P - \mathbf{1}, F).$$

Theorem 3.4 *Let $P \in \mathcal{A}$ and let $(\mathcal{H}, F)$ be an even Fredholm module for $\mathcal{A}$. Then*

$$\langle (\mathcal{H}, F), [P]_0 \rangle_0 = \mathrm{Sf}(2P - \mathbf{1}, F).$$

4   Real and Quaternionic Structures

Let $\mathcal{H}$ be a complex separable Hilbert space with complex conjugation $\mathcal{C}$ (also called real structure), namely $\mathcal{C}$ is anti-unitary and $\mathcal{C}^2 = \mathbf{1}$. This allows to define the complex conjugate $\overline{T} = \mathcal{C}T\mathcal{C} \in \mathbb{B}(\mathcal{H})$ for any operator $T \in \mathbb{B}(\mathcal{H})$ as well as its transpose $T^t = (\overline{T})^*$. Let a further structure on $\mathcal{H}$ be given unitary operators $J, I \in \mathbb{B}(\mathcal{H})$ which are real $J = \overline{J}$ and $I = \overline{I}$, and satisfy $J^2 = \mathbf{1}$ as well as $I^2 = -\mathbf{1}$. Note that $J = \mathbf{1}$ is a possible choice, and that the existence of I forces a finite dimensional $\mathcal{H}$ to be of even dimension. Then $J\mathcal{C}$ and $I\mathcal{C}$ are called real and quaternionic structures on $\mathcal{H}$, respectively.

Definition 4.1   Let the above data and an operator $T \in \mathbb{B}(\mathcal{H})$ be given.

(i) T is even real (or simply real) if and only if $J^* \overline{T} J = T$.
(ii) T is odd real (or quaternionic) if and only if $I^* \overline{T} I = T$.
(iii) T is even symmetric if and only if $J^* T^t J = T$.
(iv) T is odd symmetric if and only if $I^* T^t I = T$.

The sets of even and odd real operators are denoted by $\mathbb{BR}(\mathcal{H}, J)$ and $\mathbb{BR}(\mathcal{H}, I)$, and those of even and odd symmetric operators $\mathbb{BS}(\mathcal{H}, J)$ and $\mathbb{BS}(\mathcal{H}, I)$. Similarly, the even and odd real Fredholm operators are $\mathbb{FR}(\mathcal{H}, J)$ and $\mathbb{FR}(\mathcal{H}, I)$, and the even and odd symmetric Fredholm operators are $\mathbb{FS}(\mathcal{H}, J)$ and $\mathbb{FS}(\mathcal{H}, I)$.

Via $T \in \mathbb{BS}(\mathcal{H}, I) \mapsto IT$, the odd symmetric operators are in bijection with the anti-symmetric operators, and similarly the odd symmetric Fredholm operators are in bijection with the anti-symmetric Fredholm operators (which are not necessarily real as in [AS]). The reason to put forward the odd symmetric operators is that they have even degeneracies at every level of the Jordan hierarchy [SB], a property that does not hold for antisymmetric operators. The following theorem can be deduced from [AS]. A detailed, purely functional analytic proof is given in [SB].

Theorem 4.2 *Let the above data be given.*

(i) *The connected components of $\mathbb{FR}(\mathcal{H}, J)$ are labelled by*

$$T \in \mathbb{FR}(\mathcal{H}, J) \mapsto \operatorname{Ind}(T) \in \mathbb{Z},$$

   which is a homotopy invariant with $\operatorname{Ind}(T + K) = \operatorname{Ind}(T)$ for compact $K \in \mathbb{BR}(\mathcal{H}, J)$.

(ii) *The connected components of $\mathbb{FR}(\mathcal{H}, I)$ are labelled $T \in \mathbb{FR}(\mathcal{H}, I) \mapsto \operatorname{Ind}(T) \in 2\mathbb{Z}$, which is a homotopy invariant with $\operatorname{Ind}(T + K) = \operatorname{Ind}(T)$ for compact $K \in \mathbb{BR}(\mathcal{H}, I)$.*

(iii) *$\mathbb{FS}(\mathcal{H}, J)$ has one connected component.*

(iv) *The $\mathbb{Z}_2$ index $T \in \mathbb{FS}(\mathcal{H}, I) \mapsto \operatorname{Ind}_2(T) = \dim(\operatorname{Ker}(T)) \bmod 2 \in \mathbb{Z}_2$ is a homotopy invariant that labels the two connected components of $\mathbb{FS}(\mathcal{H}, I)$ and satisfies $\operatorname{Ind}(T + K) = \operatorname{Ind}(T)$ for compact $K \in \mathbb{BS}(\mathcal{H}, I)$.*

Items (i) and (ii) are relatively straightforward, with (ii) following from a Kramers' degeneracy argument. The proofs of (iii) and (iv) are more involved, with (iv) pending on the above-mentioned even spectral degeneracy of odd symmetric matrices. Furthermore, the following representation theorem for odd symmetric operators of [SB] (generalizing a result of C. L. Siegel) will be used below. Note that it also implies that the sets of odd symmetric invertibles and odd symmetric unitaries are path connected.

Theorem 4.3 *$T \in \mathbb{BS}(\mathcal{H}, I)$ if and only if there exists $A \in \mathbb{B}(\mathcal{H})$ with $T = I^* A^t I A$.*

Example Let us provide examples of Fredholm operators in the above classes with non-trivial indices. Let S be the left shift on the Hilbert space $\mathcal{H} = \ell^2(\mathbb{N})$ furnished with the natural fiberwise complex conjugation $\mathcal{C}$ and $J = \mathbf{1}$. Then $S^n \in \mathbb{FR}(\mathcal{H}, J)$ and $\operatorname{Ind}(S^n) = n$. Next let us consider $\mathcal{H} = \ell^2(\mathbb{N}) \otimes \mathbb{C}^2$ with $I = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right)$. Then $T = \left(\begin{smallmatrix} S^n & 0 \\ 0 & S^n \end{smallmatrix} \right) \in \mathbb{FR}(\mathcal{H}, I)$ and $T' = \left(\begin{smallmatrix} S & 0 \\ 0 & S^* \end{smallmatrix} \right) \in \mathbb{FS}(\mathcal{H}, I)$ with $\operatorname{Ind}(T) = 2n$ and $\operatorname{Ind}_2(T') = 1$.

In view of Section 2, it is now natural to consider dilations of Fredholm operators in $\mathbb{FR}(\mathcal{H}, J)$, $\mathbb{FR}(\mathcal{H}, I)$, $\mathbb{FS}(\mathcal{H}, J)$, and $\mathbb{FS}(\mathcal{H}, I)$, which have the same symmetry properties, and then to study the associated spectral flow as in Theorem 2.1. For $\mathbb{FR}(\mathcal{H}, J)$ and $\mathbb{FR}(\mathcal{H}, I)$ there is little new to discover, except that the spectral flows associated with operators $\mathbb{FR}(\mathcal{H}, I)$ all have Kramers degeneracy. More interesting is the case of odd symmetric Fredholm operators for which the concept of $\mathbb{Z}_2$-valued

spectral flow will be introduced in the next section. This is then connected to $\mathbb{Z}_2$ indices in Section 6.

5 $\mathbb{Z}_2$-valued Spectral Flow

The aim of this section is to introduce the new concept of a $\mathbb{Z}_2$-valued spectral flow $\mathrm{Sf}_2(F, U)$ similar to in (1.2), but under the supplementary assumption that F and U are odd symmetric. Hence, let $F \in \widehat{\mathbb{SF}}_*(\mathcal{H}, I) = \widehat{\mathbb{SF}}_*(\mathcal{H}) \cap \mathbb{BS}(\mathcal{H}, I)$ and $U \in \mathbb{US}(\mathcal{H}, I) = \mathbb{U}(\mathcal{H}) \cap \mathbb{BS}(\mathcal{H}, I)$. Then let $\Theta(F, U, I)$ be the set of paths in $s \in [0, 1] \mapsto F_s \in \widehat{\mathbb{SF}}_*(\mathcal{H})$ such that

(i) $$F_0 = F,$$

(ii) $$F_s - F \in \mathbb{K}(\mathcal{H}) \text{ for all } s \in [0, 1],$$

(iii′) $$F_{1-s} = (IU)^*(F_s)^t(IU).$$

Let us note that conditions (i) and (ii) are the same as in Section 1, but (iii′) is strictly stronger because (iii′) indeed implies $F_1 = U^*FU$. Furthermore, it implies $\sigma(F_{1-s}) = \sigma(F_s)$. Hence the spectral curves are already determined by the first half $s \in [0, \frac{1}{2}] \mapsto F_s$ of the path. A particular path realizing (iii′) is given by $s \in [0, 1] \mapsto F + s\, U^*[F, U]$, and can readily be checked. The odd symmetry assumptions imply Kramers' degeneracy of F_s for $s = 0, \frac{1}{2}, 1$ by the following result.

Lemma 5.1 F_0, $F_{\frac{1}{2}}$, and F_1 have even degeneracy for every finitely degenerate eigenvalue.

Proof The arguments being similar, let us focus on the case $s = \frac{1}{2}$. One has $F_{\frac{1}{2}} V = V \overline{F_{\frac{1}{2}}}$, where $V = U^*I^*$. If $F_{\frac{1}{2}} \psi = \lambda \psi$ for some $\psi \in \mathcal{H}$ and $\lambda \in \mathbb{R}$, then

$$F_{\frac{1}{2}}(V\overline{\psi}) = V\overline{(F_{\frac{1}{2}}\psi)} = \lambda(V\overline{\psi}).$$

It remains to show that $V\overline{\psi}$ and ψ are linearly independent. Suppose that there is some $a \in \mathbb{C}$ such that $a\psi = V\overline{\psi}$. Then $\psi = -|a|^2\psi$, which implies $a = 0$. This shows that λ is at least twice degenerate. If the degeneracy is larger, then one chooses an eigenvector in the orthogonal complement of the span of ψ and $V\overline{\psi}$ and runs the argument above. Repeating this procedure proves the lemma. ∎

Proposition 5.2 *Let $F \in \widehat{\mathbb{SF}}_*(\mathcal{H}, I)$ and $U \in \mathbb{US}(\mathcal{H}, I)$. Then*

$$\mathrm{Sf}\big(s \in [0, \tfrac{1}{2}] \mapsto F_s\big) \mod 2$$

is independent of the choice of path $s \in [0, 1] \mapsto F_s$ in $\Theta(F, U, I)$ and thus defines $\mathrm{Sf}_2(F, U) \in \mathbb{Z}_2$, which is called a $\mathbb{Z}_2$ spectral flow. In particular, it can be calculated by a special path

$$\mathrm{Sf}_2(F, U) = \mathrm{Sf}\big(s \in [0, \tfrac{1}{2}] \mapsto F + s\, U^*[F, U]\big) \mod 2.$$

Proof Note that the definition of $\mathrm{Sf}_2(F, U)$ only invokes half of the spectral curves, reflecting that $\sigma(F_s) = \sigma(F_{1-s})$. Let $s \in [0, 1] \mapsto F_s$ and $s \in [0, 1] \mapsto F_s'$ be two

paths in $\Theta(F,U,I)$. Then $r \in [0,1] \mapsto rF_s + (1-r)F_s'$ is a homotopy in $\Theta(F,U,I)$ keeping the initial and final point fixed. It has to be shown that $\mathrm{Sf}(s \in [0,\frac{1}{2}] \mapsto F_s)$ mod 2 is constant during this homotopy. The spectral flow $\mathrm{Sf}(s \in [0,\frac{1}{2}] \mapsto F_s)$ is not necessarily constant because the right end point $F_{\frac{1}{2}}$ is not being kept fixed (even though the left end point F_0 is fixed), but it only changes by multiples of 2 because of the Kramers' degeneracy of $F_{\frac{1}{2}}$ proved in Lemma 5.1.   ■

Remark   If $0 \notin \sigma(F_s)$ for $s = 0, \frac{1}{2}$ and the spectral curves of $s \in [0,\frac{1}{2}] \mapsto F_s$ intersect the segment $[0,1] \times \{0\}$ transversally, then $\mathrm{Sf}_2(F,U)$ can simply be calculated as the number of intersections with the segment modulo 2. Note that $\mathrm{Sf}(F,U) = 0$.

The following proposition states a consequence of non-trivial $\mathbb{Z}_2$-spectral flows. It is readily proved by contraposition.

Proposition 5.3   *Let $F \in \widehat{\mathbb{SF}}_*(\mathcal{H},I)$ and $U \in \mathbb{US}(\mathcal{H},I)$. Suppose that $\mathrm{Sf}_2(F,U) = 1$ and let $s \in [0,1] \mapsto F_s$ be any path satisfying (i)–(iii′). Then there exists at least one evenly degenerate eigenvalue $\lambda \in \sigma(F_{\frac{1}{2}}) \cap (-1,1)$.*

Let us introduce the set of operator pairs for which a $\mathbb{Z}_2$ spectral flow can be defined:

$$\mathbb{P}(\mathcal{H},I) = \left\{ (F,U) \in \widehat{\mathbb{SF}}_*(\mathcal{H},I) \times \mathbb{U}(\mathcal{H},I) \mid [F,U] \text{ compact} \right\}.$$

It carries the subspace topology induced from the norm topology on $\mathbb{B}(\mathcal{H}) \times \mathbb{B}(\mathcal{H})$. As $\mathrm{Sf}(F,U) = 0$ for all $(F,U) \in \mathbb{P}(\mathcal{H},I)$, one has $\mathbb{P}(\mathcal{H},I) \subset \mathbb{P}(\mathcal{H})_0$. Note that indeed each point $(F,U) \in \mathbb{P}(\mathcal{H},I)$ defines a class of paths $\Theta(F,U,I)$ so that it is possible to view the $\mathbb{Z}_2$ spectral flow as a map $\mathrm{Sf}_2 \colon \mathbb{P}(\mathcal{H},I) \to \mathbb{Z}_2$.

Proposition 5.4   *The spectral flow $\mathrm{Sf}_2 \colon \mathbb{P}(\mathcal{H},I) \to \mathbb{Z}_2$ is locally constant.*

Proof   Let $r \in [0,1] \mapsto (F(r),U(r)) \in \mathbb{P}(\mathcal{H},I)$ be a homotopic change in $\mathbb{P}(\mathcal{H},I)$. Associated with it are the paths $s \in [0,\frac{1}{2}] \mapsto F(r) + s\,U(r)^*[F(r),U(r)]$, for each of which $\mathrm{Sf}_2(F(r),U(r))$ is well defined by Proposition 5.2. Now the homotopic deformations change the spectrum of $F_0 = F$ and $F_{\frac{1}{2}}$, but only by multiples of 2 each by Lemma 5.1. Thus, $\mathrm{Sf}_2(F(r),U(r))$ is constant in r.   ■

6   Odd Symmetric Dilations and their $\mathbb{Z}_2$ Spectral Flow

In agreement with the previously introduced notation, let us denote by $\mathbb{EUS}(\mathcal{H},I)$ the subset of odd symmetric essentially unitary Fredholm operators.

Theorem 6.1   *Let $T \in \mathbb{EUS}(\mathcal{K},I_\mathcal{K})$ be an odd symmetric essentially unitary operator on a separable Hilbert space $\mathcal{K}$ furnished with an orthogonal $I_\mathcal{K}$ satisfying $(I_\mathcal{K})^2 = -\mathbf{1}$. Suppose that $\|T\| \le 1$. Let $U_T \in \mathbb{BS}(\mathcal{H},I_\mathcal{H})$ be an odd symmetric unitary dilation with associated injective partial isometry $\Pi \colon \mathcal{K} \to \mathcal{H}$ satisfying $I_\mathcal{K} = \Pi^* I_\mathcal{H} \Pi$. Here $I_\mathcal{H}$ is an orthogonal on $\mathcal{H}$ with $(I_\mathcal{H})^2 = -\mathbf{1}$. Then, with $F = 2\,\Pi\Pi^* - \mathbf{1}$,*

$$(6.1) \qquad \mathrm{Ind}_2(T) = \mathrm{Sf}_2(F,U_T) = \mathrm{Sf}_2\big(s \in [0,1] \mapsto F + s(U_T)^*[F,U_T]\big).$$

Proof Let $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}'$ be furnished with a real unitary $I_\mathcal{H} = I_\mathcal{K} \oplus I_{\mathcal{K}'}$ satisfying $(I_\mathcal{H})^2 = -\mathbf{1}$. Both $\mathcal{K}$ and $\mathcal{K}'$ are infinite dimensional (and separable), and therefore [SB, Proposition 5] implies that there exists an orthogonal (namely a real unitary) $O: \mathcal{K} \to \mathcal{K}'$ such that $I_{\mathcal{K}'} = O I_\mathcal{K} O^t$. Hence, conjugating U_T with $\mathrm{diag}(\mathbf{1}, O)$ leads to a new odd symmetric dilation still denoted by U_T, which now acts on $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$ furnished with $I_\mathcal{H} = I_\mathcal{K} \oplus I_\mathcal{K}$. From now on the index on I will be suppressed. In the grading of $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$ the dilation must have the form

$$U_T = \begin{pmatrix} T & B \\ I^* B^t I & D \end{pmatrix}.$$

The unitarity of U_T is equivalent to the 4 equations

$$TT^* + BB^* = \mathbf{1}_\mathcal{K}, \quad B^*B + D^*D = \mathbf{1}_\mathcal{K}, \quad T^*B + I^*\overline{B}ID = 0, \quad TI^*\overline{B}I + BD^* = 0.$$

This shows again that B is compact. As $\mathrm{Ind}_2(U_T) = 0$, it thus follows by homotopy that $\mathrm{Ind}_2(D) = \mathrm{Ind}_2(T)$. Let us note that the Halmos dilation U_T^H defined in (2.4) lies in $\mathbb{US}(\mathcal{H}, I)$. As a first step, it will be verified that (6.1) holds for U_T^H. Two cases are considered separately. First, if $\mathrm{Ind}_2(T) = 0$, then by Theorem 4.2(iv) there exists a path in $\mathbb{FS}(\mathcal{K}, I)$ from T to the identity. This leads to a path of Halmos dilations (and thus, in particular, in $\mathbb{US}(\mathcal{H}, I)$) connecting U_T^H to $\mathbf{1}$. Because the $\mathbb{Z}_2$ spectral flow is a homotopy invariant, this implies that

$$\mathrm{Sf}_2\big(s \in [0,1] \mapsto F + s\,(U_T^\mathrm{H})^*[F, U_T^\mathrm{H}] \big) = 0 = \mathrm{Ind}_2(T).$$

Second of all, if $\mathrm{Ind}_2(T) = 1$, again by Theorem 4.2(iv), there exists a homotopy from T to

$$T_0 = \begin{pmatrix} S & 0 \\ 0 & S^* \end{pmatrix}, \qquad I = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix},$$

where S is a unilateral left shift on $\mathcal{K}$ of multiplicity 1 (associated with an orthonormal sequence in $\mathcal{K}$ spanning half of $\mathcal{K}$). It satisfies $S^*S = \mathbf{1} - P$, $SS^* = \mathbf{1}$, and $SP = 0$, where P is a one-dimensional orthogonal projection on $\mathcal{K}$. Thus there exists a path of odd symmetric unitary dilations from U_T^H to

$$U_0 = \begin{pmatrix} S & 0 & 0 & 0 \\ 0 & S^* & 0 & P \\ P & 0 & -S^* & 0 \\ 0 & 0 & 0 & -S \end{pmatrix}.$$

For this dilation, one has

$$F + s\,(U_0)^*[F, U_0] = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & -\mathbf{1} & 0 \\ 0 & 0 & 0 & -\mathbf{1} \end{pmatrix} + 2s \begin{pmatrix} P & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -P \end{pmatrix}.$$

Now one can read off

$$\mathrm{Sf}_2\big(s \in [0,1] \mapsto F + s\,(U_0)^*[F, U_0] \text{ by } 0 \big) = 1 = \mathrm{Ind}_2(T).$$

Invoking again the homotopy invariance of the $\mathbb{Z}_2$ spectral flow and combining with the case $\mathrm{Ind}_2(T) = 0$, one concludes that (6.1) holds for U_T^H.

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As in the first proof of Theorem 2.1, it now remains to construct a homotopy from (F, U_T) to (F, U_T^{H}) in $\mathbb{P}(\mathcal{H}, I)$. Let us begin by factorizing U_T. From $TT^* + BB^* = \mathbf{1}$, it follows that there is a unitary V such that

$$B^* = V(\mathbf{1} - TT^*)^{\frac{1}{2}}.$$

This is not the polar decomposition, in which V would be a partial isometry with $\mathrm{Ker}(V) = \mathrm{Ker}(B^*)$, but rather a modification of it (which is arbitrary). Then

$$(6.2) \quad U_T = \begin{pmatrix} T & (\mathbf{1} - TT^*)^{\frac{1}{2}} V^* \\ I^* \overline{V} I \, I^*((\mathbf{1} - TT^*)^{\frac{1}{2}})^t I & D \end{pmatrix}$$

$$= I^* \begin{pmatrix} I^* & 0 \\ 0 & I^* V^* \end{pmatrix}^t \begin{pmatrix} T & (\mathbf{1} - TT^*)^{\frac{1}{2}} \\ (\mathbf{1} - T^*T)^{\frac{1}{2}} & I^* V^t I D V \end{pmatrix} I \begin{pmatrix} I^* & 0 \\ 0 & I^* V^* \end{pmatrix}.$$

Now recall that if T is odd symmetric, then so is $I^* A^t T I A$ for an arbitrary operator A. This shows, first of all, that $D' = I^* V^t I D V$ is odd symmetric. As all factors in (6.2) are unitary, the matrix in the middle is also an odd symmetric unitary dilation of T. Second, the factorization (6.2) shows explicitly that U_T is odd symmetric (which, of course, was already known). This remains true if V in the outer factors of (6.2) (not in the factor in the middle) is homotopically deformed to $\mathbf{1}$. Hence U_T is homotopic to the odd symmetric unitary dilation

$$U_T' = \begin{pmatrix} T & (\mathbf{1} - TT^*)^{\frac{1}{2}} \\ (\mathbf{1} - T^*T)^{\frac{1}{2}} & D' \end{pmatrix},$$

and, moreover, during the homotopy the off-diagonal operators remain compact. For the sake of simplicity, let us set $D' = D$. Now the unitarity of U_T' implies

$$(6.3) \quad TT^* = D^* D, \quad (T^* + D)(\mathbf{1} - TT^*)^{\frac{1}{2}} = 0.$$

Let us introduce the subspace $\mathcal{V} = \mathrm{Ker}(\mathbf{1} - TT^*)$ and denote by $P_{\mathcal{V}}$ the orthogonal projection on $\mathcal{V}$. Then $I^*(\mathbf{1} - T^*T)^t I = \mathbf{1} - TT^*$ shows $I\overline{\mathcal{V}} = \mathrm{Ker}(\mathbf{1} - T^*T)$. Also note that $\mathcal{V} = \mathrm{Ker}(\mathbf{1} - DD^*)$ and $I\overline{\mathcal{V}} = \mathrm{Ker}(\mathbf{1} - DD^*)$. Furthermore, $T^*: \mathcal{V} \mapsto I\overline{\mathcal{V}}$ because $v = TT^*v$ implies $T^*v = T^*T(T^*v)$. Similarly, $T: I\overline{\mathcal{V}} \mapsto \mathcal{V}$, $D: \mathcal{V} \mapsto I\overline{\mathcal{V}}$ and $D^*: I\overline{\mathcal{V}} \mapsto \mathcal{V}$. Moreover, all four maps are isometric and are therefore also unitary. Finally, $T^*: \mathcal{V}^\perp \to (I\overline{\mathcal{V}})^\perp$ as one checks by writing out the orthogonality relations.

Using the functional calculus of a unitary operator, let us now define unitaries $(T^*)^{\frac{1}{2}}: \mathcal{V} \to I\overline{\mathcal{V}}$ and $D^{-\frac{1}{2}}: I\overline{\mathcal{V}} \to \mathcal{V}$ (here the isomorphism $I\overline{\mathcal{V}} \cong \mathcal{V}$ is suppressed in the notation). Thus $W|_{\mathcal{V}} = \imath D^{-\frac{1}{2}}(T^*)^{\frac{1}{2}}|_{\mathcal{V}}: \mathcal{V} \to \mathcal{V}$ is unitary. It is extended to a unitary on $\mathcal{K}$ by setting $W = W|_{\mathcal{V}} + (\mathbf{1} - P_{\mathcal{V}})$. Let us choose a path $r \in [0, 1] \mapsto W(r)|_{\mathcal{V}}$ of unitaries on $\mathcal{V}$ such that $W(0)|_{\mathcal{V}} = W|_{\mathcal{V}}$ and $W(1)|_{\mathcal{V}} = \mathbf{1}_{\mathcal{V}}$. Set $W(r) = W(r)|_{\mathcal{V}} + (\mathbf{1} - P_{\mathcal{V}})$ and $D(r) = I^* W(r)^t I D W(r)$. First, one readily checks that $D(r)$ is odd symmetric for all r. Obviously, $D(1) = D$. Moreover, with some care one also checks $D(0)|_{\mathcal{V}} = -T^*|_{\mathcal{V}}$ by construction. Finally, $D(r)$ leaves $\mathcal{V}^\perp$ invariant and $D(r)|_{\mathcal{V}^\perp} = -T^*|_{\mathcal{V}^\perp}$ due to the second identity in (6.3). Hence $D(0) = -T^*$ so that U_T' is homotopic to the Halmos projection inside $\mathbb{US}(\mathcal{H}, I)$. ∎

7 The $\mathbb{Z}_2$ Mapping Cone of a Fredholm Module

Let $\mathcal{A}$ be a C*-algebra of bounded operators on a complex Hilbert space $\mathcal{H}$ and suppose that, given an involutive *-antiautomorphism, $\tau\colon \mathcal{A} \to \mathcal{A}$. Then $\mathcal{A}_\tau = \{A \in \mathcal{A} \mid \tau(A) = A\}$ is a real C*-algebra that has 8 Real K-groups [Sch]. Here the focus will be on τ of the form $\tau(A) = I^*\overline{A}I$ with I and complex conjugation as in Section 4, and the real C*-algebra will then be denoted by $\mathcal{A}_I$. Furthermore, only the K-group $K_0(\mathcal{A}_I)$ will be considered. It is by definition the set of homotopy classes of orthogonal projections (or invertible self-adjoints) in matrix algebras over $\mathcal{A}_I$, to which I is extended by $I \otimes \mathbf{1}$. Otherwise stated, $K_0(\mathcal{A}_I)$ is the set of homotopy classes of odd symmetric projections in $\mathcal{A}$. The reason for picking out this example is that there may be $\mathbb{Z}_2$ invariants contained in $K_0(\mathcal{A}_I)$. It is also possible to study classes of even, symmetric, invertible self-adjoints (giving classes in $K_0(\mathcal{A}_J)$ with $J^2 = \mathbf{1}$), or even and odd symmetric skew-adjoints, and furthermore classes of unitaries (or even invertibles) that are even or odd real or symmetric, but this full-fledged theory requires various elements of Real K-theory and will not be developed here.

Topological content will be extracted from a class $[P]_0 \in K_0(\mathcal{A}_I)$ again by use of Fredholm modules that have a supplementary symmetry property. An (ungraded version of an) even $\mathbb{Z}_2$-Fredholm module $(\mathcal{H}, F)$ is defined to be an odd symmetric unitary operator $F = I^*F^tI$ such that $[A, F] \in \mathbb{K}(\mathcal{H})$ for all $A \in \mathcal{A}_I$. A pairing will be defined by

$$\langle\!\langle (\mathcal{H}, F), [P]_0 \rangle\!\rangle_0 = \mathrm{Ind}_2(PFP),$$

where $P \in \mathcal{A}_I$, or matrix algebras thereof. Note that indeed PFP is odd symmetric so that the $\mathbb{Z}_2$ index is well defined, and due to its homotopy, invariance the pairing is independent of the choice of representative. As shown in Theorem 6.1, $\mathrm{Ind}_2(PFP)$ can be calculated as the $\mathbb{Z}_2$-valued spectral flow of any odd symmetric dilation of PFP:

$$\langle\!\langle (\mathcal{H}, F), [P]_0 \rangle\!\rangle_0 = \mathrm{Sf}_2(2P - \mathbf{1}, F).$$

Let us now sketch a K-theoretic interpretation of this $\mathbb{Z}_2$ spectral flow by using an adequate $\mathbb{Z}_2$-mapping cone. It is defined by

$$\mathcal{M}_I = \big\{ (A_s)_{s\in[0,1]} \in C([0,1], \mathcal{A} + \mathbb{K}(\mathcal{H})) \mid$$
$$I^*\overline{A_s}I = F^*A_{1-s}F, A_s - A_0 \in \mathbb{K}(\mathcal{H}), A_0 \in \mathcal{A}_I \big\}.$$

This is the fixed point set in $\mathcal{M}$ with respect to the anti-linear involution (also denoted by τ)

$$\tau((A_s)_{s\in[0,1]}) = (I^*F^t\overline{A_{1-s}F}I)_{s\in[0,1]}.$$

Hence $\mathcal{M}_I$ is indeed a real C*-algebra. Furthermore, one has an exact sequence of real C*-algebras

$$0 \longrightarrow S_I\mathbb{K}(\mathcal{H}) \hookrightarrow \mathcal{M}_I \xrightarrow{\mathrm{ev}} \mathcal{A}_I \longrightarrow 0,$$

where

$$S_I\mathbb{K}(\mathcal{H}) = \big\{ (K_s)_{s\in[0,1]} \in C_0((0,1), \mathbb{K}(\mathcal{H})) \mid I^*\overline{K_s}I = F^*K_{1-s}F \big\}.$$

Indeed, for any $A \in \mathcal{A}_I$ a lift into $\mathcal{M}_I$ is given as before by $\mathrm{Lift}(A)_s = (1-s)A + sF^*AF$, and this lift can again be chosen self-adjoint for self-adjoint A. Based on this construction, one can now transpose the ideas of Section 3.

References

[APS]   M. F. Atiyah, V. K. Patodi, and I. M. Singer, *Spectral asymmetry and Riemannian geometry. III*. Math. Proc. Cambridge Phil. Soc. **79**(1976), no. 1, 71–99.
http://dx.doi.org/10.1017/S0305004100052105

[AS]   M. F. Atiyah and I. M. Singer, *Index theory for skew-adjoint Fredholm operators*. Inst. Haute Études Sci. Publ. Math. **37**(1969), 5–26.

[ASV]   J. C. Avila, H. Schulz-Baldes, and C. Villegas-Blas, *Topological invariants of edge states for periodic two-dimensional models*. Math. Phys. Anal. Geom. **16**(2013), no. 2, 137–170.
http://dx.doi.org/10.1007/s11040-012-9123-9

[ASS]   J. Avron, R. Seiler, and B. Simon, *The index of a pair of projections*. J. Funct. Anal. **120**(1994), no. 1, 220–237.   http://dx.doi.org/10.1006/jfan.1994.1031

[BCP]   M.-T. Benameur, A. L. Carey, J. Phillips, A. Rennie, F. A. Sukochev, and K. P. Wojciechowski, *An analytic approach to spectral flow in von Neumann algebras*. In: Analysis, geometry and topology of elliptic operators, World Sci. Publ., Hackensack, NJ, 2006, pp. 297–352.

[CP]   A. Carey and J. Phillips, *Unbounded Fredholm modules and spectral flow*. Canad. J. Math. **50**(1998), no. 4, 673–718.   http://dx.doi.org/10.4153/CJM-1998-038-x

[CPR]   A. Carey, J. Phillips, and A. Rennie, *Noncommutative Atiyah-Patodi-Singer boundary conditions and index pairings in KK-theory*. J. Reine Angew. Math. **643**(2010), 59–109.

[Con]   A. Connes, *Noncommutative geometry*. Academic Press Inc., San Diego, CA, 1994.

[DS]   G. De Nittis and H. Schulz-Baldes, *Spectral flows associated to flux tubes*. arxiv:1405.2054

[Hal]   P. R. Halmos, *Normal dilations and extension of operators*. Summa Brasil Math. **2**(1950), 125–134.

[HR]   N. Higson and J. Roe, *Analytic K-homology*. Oxford Mathematical Monographs, Oxford Science Publications, Oxford University Press, Oxford, 2000.

[Kat]   T. Kato, *Perturbation theory for linear operators*. Die Grundlehren der mathematischen Wissenschaften, 132, Springer-Verlag, New York, 1966.

[Phi1]   J. Phillips, *Self-adjoint Fredholm operators and spectral flow*. Canad. Math. Bull. **39**(1996), no. 4, 460–467.   http://dx.doi.org/10.4153/CMB-1996-054-4

[Phi2]   ———, *Spectral flow in type I and II factors—a new approach*. In: Cyclic cohomology and noncommutative geometry (Waterloo, ON, 1995), Fields Inst. Commun., 17, American Mathematical Society, Providence, RI, 1997, pp. 137–153.

[Put]   I. F. Putnam, *An excision theorem for the K-theory of C^*-algebras*. J. Operator Theory **38**(1997), no. 1, 151–171.

[Sch]   H. Schröder, *K-theory for real C^*-algebras and applications*. Pitman Research Notes in Mathematics Series, 290, Longman Scientific & Technical, Harlow; John Wiley & Sons, New York, 1993.

[SB]   H. Schulz-Baldes, $\mathbb{Z}_2$ *indices of odd symmetric Fredholm operators*. arxiv:1311.0379v2

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