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ANALYSIS OF CONTACT CAUCHY–RIEMANN MAPS II: CANONICAL NEIGHBORHOODS AND EXPONENTIAL CONVERGENCE FOR THE MORSE–BOTT CASE

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Abstract. This is a sequel to the papers Oh and Wang (*Real and Complex Submanifolds*, Springer Proceedings in Mathematics and Statistics **106** (2014), 43–63, eds. by Y.-J. Suh and *et al.* for ICM-2014 satellite conference, Daejeon, Korea, August 2014; arXiv:1212.4817; *Analysis of contact Cauchy–Riemann maps I: a priori C^k estimates and asymptotic convergence*, submitted, preprint, 2012, arXiv:1212.5186v3). In Oh and Wang (*Real and Complex Submanifolds*, Springer Proceedings in Mathematics and Statistics **106** (2014), 43–63, eds. by Y.-J. Suh and *et al.* for ICM-2014 satellite conference, Daejeon, Korea, August 2014; arXiv:1212.4817), the authors introduced a canonical affine connection on M associated to the contact triad (M, λ, J). In Oh and Wang (*Analysis of contact Cauchy–Riemann maps I: a priori C^k estimates and asymptotic convergence*, submitted, preprint, 2012, arXiv:1212.5186v3), they used the connection to establish *a priori* $W^{k,p}$-coercive estimates for maps $w:\dot\Sigma \to M$ satisfying $\overline{\partial}^\pi w = 0$, $d(w^*\lambda \circ j) = 0$ *without involving symplectization*. We call such a pair (w, j) a contact instanton. In this paper, we first prove a canonical neighborhood theorem of the locus Q foliated by closed Reeb orbits of a Morse–Bott contact form. Then using a general framework of the three-interval method, we establish exponential decay estimates for contact instantons (w, j) of the triad (M, λ, J), with λ a Morse–Bott contact form and J a CR-almost complex structure adapted to Q, under the condition that the asymptotic charge of (w, j) at the associated puncture vanishes.

We also apply the three-interval method to the symplectization case and provide an alternative approach via tensorial calculations to exponential decay estimates in the Morse–Bott case for the pseudoholomorphic curves on the symplectization of contact manifolds. This was previously established by Bourgeois (*A Morse–Bott approach to contact homology*, Ph.D. dissertation, Stanford University, 2002) (resp. by Bao (*On J-holomorphic curves in almost complex manifolds with asymptotically cylindrical ends*, Pacific J. Math. **278**(2) (2015), 291–324)), by using special coordinates, for the cylindrical (resp. for the asymptotically cylindrical) ends. The exponential decay result for the Morse–Bott case is an essential ingredient in the setup of the moduli space of pseudoholomorphic curves which plays a central role in contact homology and symplectic field theory (SFT).

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§1. Introduction

Let (M, ξ) be a contact manifold. Each contact form λ of ξ, that is, a one-form with $\ker \lambda = \xi$, canonically induces a splitting

$$TM = \mathbb{R}\{X_\lambda\} \oplus \xi.$$

Here X_λ is the Reeb vector field of λ, which is uniquely determined by the equations

$$X_\lambda \rfloor \lambda \equiv 1, \qquad X_\lambda \rfloor d\lambda \equiv 0.$$

We denote by $\Pi = \Pi_\lambda : TM \to TM$ the idempotent, that is, an endomorphism satisfying $\Pi^2 = \Pi$ such that $\ker \Pi = \mathbb{R}\{X_\lambda\}$ and $\operatorname{Im} \Pi = \xi$. Denote by $\pi = \pi_\lambda : TM \to \xi$ the associated projection.

Definition 1.1. (Contact triad) We call the triple (M, λ, J) a contact triad of (M, ξ) if λ is a contact form of (M, ξ), and J is an endomorphism of TM with $J^2 = -\Pi$ which we call CR-almost complex structure, such that the triple $(\xi, d\lambda|_\xi, J|_\xi)$ defines a Hermitian vector bundle over M.

As long as no confusion arises, we abuse our notation J also for its restriction to ξ.

In [OW2], the authors of the present paper called the pair (w, j) a contact instanton, if (Σ, j) is a (punctured) Riemann surface and $w : \Sigma \to M$ satisfies the following equations

$$\overline{\partial}^\pi w = 0, \qquad d(w^*\lambda \circ j) = 0. \tag{1.1}$$

A priori coercive $W^{k,2}$-estimates for w with $W^{1,2}$-bound was established *without involving symplectization*. Moreover, the study of $W^{1,2}$ (or the derivative) bound and the definition of relevant energy is carried out by the Yong-Geun Oh in [Oh2].

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Furthermore, for the punctured domains $\dot\Sigma$ equipped with cylindrical metric near the puncture, the present authors proved the result of asymptotic subsequence *uniform* convergence to a Reeb orbit (which must be closed when the corresponding charge is vanishing) under the assumption that the π-harmonic energy is finite and the C^0-norm of derivative dw is bounded. (Refer [OW2, Section 6] for precise statement and Section 9 in the current paper for its review.) Based on this subsequence uniform convergence result, the present authors previously proved C^∞ exponential decay in [OW2] when the contact form is nondegenerate. The proof is based on the so-called *three-interval argument* which is essentially different from the proofs for exponential convergence in existing literatures, for example, from those in [HWZ1, HWZ2, HWZ4] which use the method of differential inequality.

The present paper is a sequel to the paper [OW2] and the main purpose thereof is to generalize the exponential convergence result to the Morse–Bott case. In Part 2 of the current paper, we systematically develop the above mentioned three-interval method as a general framework and establish the result for Morse–Bott contact forms. (Corresponding results for pseudoholomorphic curves in symplectizations were provided by various authors including [HWZ3, Bou, Ba] and we suggest readers to compare our method with theirs.)

In general, the exponential convergence result is an important ingredient in the setup of the Fredholm theory and in the relevant gluing construction. In contact geometry, the moduli spaces of pseudoholomorphic curves with noncompact sources are used in defining the contact homology or setting up the framework of the symplectic field theory (SFT) (see e.g., [EGH] for an introduction). In this regard, the Morse–Bott case provides important computable examples in contact geometry and in SFT. (See [Bou] for some examples of such computations based on the Morse–Bott framework of contact homology.) However, there are various subtleties in describing the structure of the Morse–Bott moduli spaces and the corresponding contact homology for the contact forms of Morse–Bott type, which have not been rigorously set up yet. One of the purposes of the current paper is to provide a careful geometric description of the locus of closed Reeb orbits and the corresponding tensorial proof of exponential decay results. Moreover, the abstract framework of the three-interval method we develop in this paper for the exponential decay proof can be easily applied to other evolution type of equations, and provides a general "black box" for the exponential decay.

The proof of the exponential decay result consists of two parts, one geometric and the other analytic. Part 1 is devoted to unveil the geometric structure, the *precontact structure*, carried by the loci Q of the closed Reeb orbits of a Morse–Bott contact form λ (see Section 1.1 for precise definition). We prove a canonical neighborhood theorem of any precontact manifold which is the contact analogue to Gotay's on presymplectic manifolds [G], which we call the *contact thickening* of a precontact manifold. By using this neighborhood theorem, we obtain a canonical splitting of the tangent bundle TM in terms of the precontact structure of Q and its thickening. Then we introduce the class consisting of J's *adapted to* Q (refer Section 8 for definition) besides the standard compatibility requirement to $d\lambda$. At last we split the derivative dw of contact instanton w into various components and study them separately. In this way, we are given the geometric framework which gets us ready to conduct the three-interval method provided in Part 2. Part 2 is then devoted to applying the enhanced version of the three-interval framework in proving the exponential convergence for the Morse–Bott case, which generalizes the one presented for the nondegenerate case in [OW2].

Now we outline the main results in the present paper in more details.

1.1 Structure of the locus of closed Reeb orbits

Assume λ is a fixed contact form of the contact manifold (M, ξ). For a closed Reeb orbit γ of period $T > 0$, one can write $\gamma(t) = \phi^t(\gamma(0))$, where $\phi^t = \phi^t_{X_\lambda}$ is the flow of the Reeb vector field X_λ.

Denote by $\mathrm{Cont}(M, \xi)$ the set of all contact one-forms with respect to the contact structure ξ, and by $\mathcal{L}(M) = C^\infty(S^1, M)$ the space of loops $z : S^1 = \mathbb{R}/\mathbb{Z} \to M$. Consider the bundle $\mathcal{L}$ over the product $(0, \infty) \times \mathcal{L}(M) \times \mathrm{Cont}(M, \xi)$ whose fiber at (T, z, λ) is $C^\infty(z^*TM)$. The assignment

$$\Upsilon : (T, z, \lambda) \mapsto \dot{z} - T\, X_\lambda(z)$$

defines a section of the bundle, where (T, z) is a pair with a loop z parameterized over the unit interval $S^1 = [0, 1]/\sim$ defined by $z(t) = \gamma(Tt)$ for a Reeb orbit γ of period T. Notice that $(T, z, \lambda) \in \Upsilon^{-1}(0) := \mathfrak{Reeb}(M, \xi)$ if and only if there exists some Reeb orbit $\gamma : \mathbb{R} \to M$ with period T, such that $z(\cdot) = \gamma(T\cdot)$. Denote by

$$\mathfrak{Reeb}(M, \lambda) := \{(T, z) \mid (T, z, \lambda) \in \mathfrak{Reeb}(M, \xi)\}$$

for each $\lambda \in \mathrm{Cont}(M, \xi)$. From the formula of a T-periodic orbit (T, γ), $T = \int_\gamma \lambda$. It follows that the period varies smoothly over γ.

The general Morse–Bott condition (Bott's notion [Bot] of clean critical submanifold in general) for λ corresponds to the statement that every connected component of $\mathfrak{Reeb}(M, \lambda)$ is a smooth submanifold of $(0, \infty) \times \mathcal{L}(M)$ and its tangent space at every pair $(T, z) \in \mathfrak{Reeb}(M, \lambda)$ therein coincides with $\ker d_{(T,z)}\Upsilon$. Denote by Q the locus of closed Reeb orbits contained in a fixed connected component of $\mathfrak{Reeb}(M, \lambda)$. Throughout this paper, we also call Q a Morse–Bott submanifold when we want to emphasize its manifold structure.

However, when one tries to set up the moduli space of contact instantons for Morse–Bott contact forms, more requirements are needed and we recall the definition that Bourgeois adopted in [Bou]. (Strictly speaking, we also need to take suitable completions of $\mathcal{L}(M)$ and $\mathrm{Cont}(M, \xi)$ but we ignore this point which does not play any role in our main discussion.)

DEFINITION 1.2. (Equivalent to Definition 1.7 [Bou]) A contact form λ is called be of Morse–Bott type, if it satisfies the following:

(1) every connected component of $\mathfrak{Reeb}(M, \lambda)$ is a smooth submanifold of $(0, \infty) \times \mathcal{L}(M)$ with its tangent space at every pair $(T, z) \in \mathfrak{Reeb}(M, \lambda)$ therein coincides with $\ker d_{(T,z)}\Upsilon$;
(2) the locus Q is embedded;
(3) the 2-form $d\lambda|_Q$ associated to the locus Q of closed Reeb orbits has constant rank.

Here Condition (1) corresponds to Bott's notion of Morse–Bott critical manifolds which we name as *standard Morse–Bott type*. While $\mathfrak{Reeb}(M, \lambda)$ is a smooth submanifold, the orbit locus $Q \subset M$ of $\mathfrak{Reeb}(M, \lambda)$ is in general only an immersed submanifold and could have multiple sheets along the locus of multiple orbits. Therefore, we impose Condition (2). In general, the restriction of the two-form $d\lambda$ to Q has varying rank. It is still not clear whether the exponential estimates we derive in this paper holds in this general context because our proof strongly relies on the existence of canonical model of neighborhoods of Q. For this reason, we also impose Condition (3). We remark that Condition (3) means that the 2-form $d\lambda|_Q$ becomes a presymplectic form.

Depending on the type of the presymplectic form, we say that Q is of prequantization type if the rank of $d\lambda|_Q$ is maximal, and is of isotropic type if the rank of $d\lambda|_Q$ is zero. The general case is a mixture of these two. In particular when $\dim M = 3$, such Q must be either of prequantization type or of isotropic type. This is the case dealt with in [HWZ3]. The general case considered in [Bou], [BEHWZ] includes the mixed type.

DEFINITION 1.3. (Precontact form) We call one-form θ on a manifold Q a *precontact* form if $d\theta$ has constant rank, that is, if $d\theta$ is a presymplectic form.

While the notion of presymplectic manifolds is well established in symplectic geometry, this contact analogue seems to have not been used in literature, at least formally, as far as we know.

With this terminology introduced, we prove the following theorem.

THEOREM 1.4. (Theorem 3.11) *Let λ be a Morse–Bott-type contact form of contact manifold (M, ξ) as defined above. Let Q be an associated Morse–Bott submanifold of closed Reeb orbits. Suppose that Q is embedded and $d\lambda|_Q = i_Q^*(d\lambda)$ has constant rank. Then Q carries:*

(1) *a locally free S^1-action generated by the Reeb vector field $X_\lambda|_Q$;*
(2) *the precontact form θ given by $\theta = i_Q^*\lambda$ and the splitting*

$$\ker d\theta = \mathbb{R}\{X_\theta\} \oplus H, \tag{1.2}$$

such that the distribution $H = \ker d\theta \cap \xi|_Q$ is integrable;

(3) *an S^1-equivariant symplectic vector bundle $(E, \Omega) \to Q$ with*

$$E = (TQ)^{d\lambda}/\ker d\theta, \qquad \Omega = [d\lambda]_E.$$

Here we use the fact that there exists a canonical embedding

$$E = (TQ)^{d\lambda}/\ker d\theta \hookrightarrow T_Q M/TQ = N_Q M, \tag{1.3}$$

and $d\lambda|_{(TQ)^{d\lambda}}$ canonically induces a bilinear form $[d\lambda]_E$ on $E = (TQ)^{d\lambda}/\ker di_Q^*\lambda$ by symplectic reduction.

DEFINITION 1.5. Let (Q, θ) be a precontact manifold equipped with the splitting (1.2). We call such a triple (Q, θ, H) a *Morse–Bott contact setup.*

Denote by $\mathcal{F}$ and $\mathcal{N}$ the foliations associated to the distribution $\ker d\theta$ and H, respectively. We also denote by $T\mathcal{F}$, $T\mathcal{N}$ the associated foliation tangent bundles and $T^*\mathcal{N}$ the foliation cotangent bundle of $\mathcal{N}$.

We prove the following canonical model theorem which describes a natural way of thickening of Morse–Bott contact setup (Q, θ, H) whenever a symplectic vector bundle $E \to Q$ is given.

THEOREM 1.6. *Let (Q,θ,H) be a Morse–Bott contact setup. Let a symplectic vector bundle $(E,\Omega)\to Q$ be given. Then the bundle $F=T^*\mathcal{N}\oplus E$ carries a canonical contact form $\lambda_{F;G}$ defined as in (4.8), for each choice of complement G such that $TQ=T\mathcal{F}\oplus G$. Furthermore, for two such choices of G, G', two induced contact structures are naturally isomorphic.*

Based on this theorem, we denote any such $\lambda_{F;G}$ just by λ_F suppressing G from its notation. This normal form provides a general class of contact manifolds equipped with a contact form of Morse–Bott type.

Finally we prove the following canonical neighborhood theorem for $Q\subset M$ with Q defined above for any Morse–Bott contact form λ of contact manifold (M,ξ).

THEOREM 1.7. (Theorem 5.1) *Let Q be the submanifold foliated by closed Reeb orbits of Morse–Bott-type contact form λ of contact manifold (M,ξ), and (Q,θ) and (E,Ω) be the associated pair defined above. Let (F,λ_F) be the model contact manifold with $F=T^*\mathcal{N}\oplus E$ and λ_F be the contact form on $U_F\subset F$ given in (4.8).*

Then there exist neighborhoods $\mathcal{U}$ of Q and U_F of the zero section o_F and a diffeomorphism $\psi:U_F\to\mathcal{U}$ and a function $f:U_F\to\mathbb{R}$ such that

$$\psi^*\lambda=f\,\lambda_F,\qquad f|_{o_F}\equiv 1,\qquad df|_{o_F}\equiv 0 \tag{1.4}$$

and

$$i_{o_F}^*\psi^*\lambda=\theta,\qquad (\psi^*d\lambda|_{VTF})|_{o_F}=0\oplus\Omega \tag{1.5}$$

where we use the canonical identification of $VTF|_{o_F}\cong T^*\mathcal{N}\oplus E$ on the zero section $o_F\cong Q$.

REMARK 1.8. We would like to remark that while the bundles E and $TQ/T\mathcal{F}$ carry canonical fiberwise symplectic form and so carry canonical orientations induced by $d\lambda$, the bundle $T\mathcal{N}$ may not be orientable in general along a Reeb orbit corresponding to an orbifold point in $P=Q/\sim$.

1.2 The three-interval method of exponential estimates

For the study of the asymptotic behavior of finite π-energy solutions of contact instanton $w:\dot\Sigma\to M$ near a Morse–Bott submanifold Q, we introduce the following class of CR-almost complex structures.

DEFINITION 1.9. (Definition 8.2) Let $Q\subset M$ be a Morse–Bott submanifold foliated by closed Reeb orbits of λ. Suppose J defines a contact triad

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(M, λ, J). We say a CR-almost complex structure J for (M, ξ) is adapted to the submanifold Q or simply is Q-adapted if J satisfies

$$J(TQ) \subset TQ + JT\mathcal{N}. \tag{1.6}$$

Note that this condition is vacuous for the nondegenerate case, but for the general Morse–Bott case, the class of adapted J is strictly smaller than the one of general CR-almost complex structures of the triad. The set of Q-adapted J's is contractible and the proof is given in Appendix A. As far as the applications to contact topology are concerned, requiring this condition is not any restriction but seems to be necessary for the analysis of contact-instanton maps or of the pseudoholomorphic maps in the symplectization (or in the symplectic manifolds with contact-type boundary).

Let $w : \dot{\Sigma} \to M$ be a contact-instanton map, that is, satisfying (1.1) at a cylindrical end $[0, \infty) \times S^1$, which now can be written as

$$\pi \frac{\partial w}{\partial \tau} + J\pi \frac{\partial w}{\partial t} = 0, \qquad d(w^*\lambda \circ j) = 0, \tag{1.7}$$

for $(\tau, t) \in [0, \infty) \times S^1$. We put the following basic hypotheses for the study of exponential convergence.

HYPOTHESIS 1.10. (Hypothesis 9.1)

(1) *Finite π-energy:* $E^\pi(w) := (1/2) \int_{[0,\infty)\times S^1} |d^\pi w|^2 < \infty$;
(2) *finite derivative bound:* $\|dw\|_{C^0([0,\infty)\times S^1)} \leqslant C < \infty$;
(3) *nonvanishing asymptotic action:*

$$\mathcal{T} := \frac{1}{2} \int_{[0,\infty)\times S^1} |d^\pi w|^2 + \int_{\{0\}\times S^1} (w|_{\{0\}\times S^1})^*\lambda \neq 0;$$

(4) *vanishing asymptotic charge:* $\mathcal{Q} := \int_{\{0\}\times S^1} ((w|_{\{0\}\times S^1})^*\lambda \circ j) = 0$.

Under these hypotheses, we establish the following C^∞ uniform exponential convergence of w to a closed Reeb orbit z of period $T = \mathcal{T}$. This result was already known in [HWZ3, Bou, Ba] in the context of pseudoholomorphic curves $u = (w, a)$ in symplectizations. However, we emphasize that our proof presented here, which uses the three-interval framework, is different from the ones [HWZ3, Bou, Ba] even in the symplectization case. Furthermore, when we deal with the case of symplectization, our method completely separates the estimates of w from that of a's. (See Section 13.)

THEOREM 1.11. *Assume (M, λ) is a Morse–Bott contact manifold and w is a contact instanton satisfying the Hypothesis 1.10 at the given end. Then there exists a closed Reeb orbit z with period $T = \mathcal{T}$ and positive constant δ determined by z, such that*

$$\|d(w(\tau, \cdot), z(T\cdot))\|_{C^0(S^1)} < Ce^{-\delta\tau},$$

and

$$\left\|\pi \frac{\partial w}{\partial \tau}(\tau, \cdot)\right\|_{C^0(S^1)} < Ce^{-\delta\tau}, \qquad \left\|\pi \frac{\partial w}{\partial t}(\tau, \cdot)\right\|_{C^0(S^1)} < Ce^{-\delta\tau}$$

$$\left\|\lambda\left(\frac{\partial w}{\partial \tau}\right)(\tau, \cdot)\right\|_{C^0(S^1)} < Ce^{-\delta\tau}, \qquad \left\|\lambda\left(\frac{\partial w}{\partial t}\right)(\tau, \cdot) - T\right\|_{C^0(S^1)} < Ce^{-\delta\tau}$$

$$\|\nabla^l dw(\tau, t)\|_{C^0(S^1)} < C_l e^{-\delta\tau} \quad \text{for any } l \geqslant 1,$$

where d is the distance function induced from the triad metric on M and C, C_l are positive constants which only depend on l.

Now comes the outline of the strategy of our proof of exponential convergence in the present paper. Mundet i Riera and Tian in [MT] elegantly used a discrete method of three-interval arguments in proving exponential decay under the assumption of C^0-convergence already established. However, for most cases of interests, the C^0-convergence is not *a priori* given in the beginning but it is often the case that the C^0-convergence can be obtained only after one proves the exponential convergence of derivatives. (See the proofs of, for example, [HWZ1, HWZ2, HWZ4], [HWZ3], [Bou], [Ba].) To obtain the exponential estimates of derivatives, researchers conduct some brute-force calculations in deriving the needed differential inequality, and then proceed from there toward the final result. Such calculations, especially in coordinates, become quite complicated for the Morse–Bott situation and hide the geometry that explains why such a differential inequality could be expected.

Our proof is divided into two parts by writing $w = (u, s)$ in the normalized contact triad (U_F, λ_F, J_0) (see Definition 8.7) with $U_F \subset F \to Q$ for any given compatible J adapted to Q, where J_0 is canonical normalized CR-almost complex structure associated to J. We also decompose $s = (\mu, e)$ in terms of the splitting $F = T^*\mathcal{N} \oplus E$. In this decomposition, the L^2-exponential estimates for the e-component is an easy consequence of the three-interval method which we formulate above in a general abstract framework (see Theorem 10.11 for the precise statement). This estimate belongs

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to the standard realm of exponential decay proof for the asymptotically cylindrical elliptic equations.

However, the study of L^2-exponential estimates for (u, μ) does not directly belong to this standard realm. Although we still apply similar three-interval method for the study of L^2-exponential convergence, its study is much more subtle than that of the normal component due to the presence of nontrivial kernel of the asymptotic operator B_∞ of the linearization. To handle the (u, μ)-component, we formulate the following general theorem from the abstract framework of the three-interval argument, and refer readers to Sections 10, 11.3 for the precise statement and its proof.

THEOREM 1.12. *Assume $\xi(\tau, t)$ is a section of some vector bundle on $\mathbb{R} \times S^1$ which satisfies the equation*

$$\nabla_\tau^\pi \zeta + J\nabla_t^\pi \zeta + S\zeta = L(\tau, t) \quad \text{with } |L| < Ce^{-\delta_0 \tau}$$

of Cauchy–Riemann type (or more generally any elliptic PDE of evolution type), where S is a bounded symmetric operator.

Suppose that there exists a sequence $\{\bar\zeta_k\}$ (e.g., by performing a suitable rescaling of ζ) such that at least one subsequence converges to a nonzero section $\bar\zeta_\infty$ of a (trivial) Banach bundle over a fixed finite interval, say on $[0, 3]$, that satisfies the ODE

$$\frac{D\bar\zeta_\infty}{d\tau} + B_\infty \bar\zeta_\infty = 0$$

on the associated Banach space.

Then provided $\|\zeta(\tau, \cdot)\|_{L^2(S^1)}$ converges to zero as τ goes to ∞, $\|\zeta(\tau, \cdot)\|_{L^2(S^1)}$ decays exponentially fast with the rate $\delta > 0$ for any constant $\delta < \min\{\lambda_0, \delta_0\}$ where λ_0 is the smallest absolute value of nonzero eigenvalues of B_∞.

REMARK 1.13. For the special case when B_∞ has only trivial kernel, the result can be regarded as the discrete analogue of the differential inequality method used by Robbin and Salamon in [RS].

In this framework, our exponential convergence proof is based on intrinsic geometric tensor calculations which is coordinate-free. As a result, our proof make it manifest that (roughly) the exponential decay occurs whenever the geometric PDE has bounded energy at cylindrical ends and the limiting equation is of linear evolution type $\partial\bar\zeta_\infty/\partial\tau + B_\infty\bar\zeta_\infty = 0$, where B_∞ is an elliptic operator with discrete spectrum. If B_∞ has trivial kernel, the conclusion follows rather immediately from the three-interval argument. Even when B_∞ contains nontrivial kernel, the exponential decay would still follow as long as some geometric condition, like the Morse–Bott assumption in the current case of our interest, enables one to extract some nonvanishing solution of the limit equation $\partial\overline{\zeta}_\infty/\partial\tau + B_\infty\overline{\zeta}_\infty = 0$ that arises in the course of three-interval arguments. Moreover, the decay rate $\delta > 0$ is always provided by the minimal eigenvalue of B_∞.

Now we roughly explain how the nonvanishing limiting solution mentioned above is obtained in the current situation: first, the canonical neighborhood provided in Part 1 is used to split the contact-instanton equations into the vertical and horizontal parts. By this way, only the horizontal equation could be involved with the kernel of B_∞ which by the Morse–Bott condition has nice geometric structure in the sense that the kernel can be excluded by looking at a higher derivative instead of the map itself. Then, to further see the limit of the derivative is indeed nonvanishing, we apply the geometric decomposition to the derivative and study the center of mass on the Morse–Bott submanifold Q. The details are presented in Sections 11.3 and 11.4.

Part 1. Contact Hamiltonian geometry and canonical neighborhoods

The main purpose of this part is to prove a canonical neighborhood theorem for the loci of closed Reeb orbits when the contact form λ of a contact manifold (M, ξ) is of Morse–Bott type. The results of this part provides geometric preparation for the study of asymptotic exponential convergence of contact instanton at a puncture of the domain Riemann surface.

The outline of Part 1 in sectionwise is as follows.

- In Section 2, we review some basic facts related to contact forms of a contact manifold. We first set up a natural isomorphism between TM and T^*M using the contact form λ. This is a contact analogue of the isomorphism between tangent bundle and cotangent bundle for symplectic manifolds. Then we derive explicit formulas of the Reeb vector field $R_{f\lambda}$ and the contact projection $\pi_{f\lambda}$ in terms of X_λ, π_λ and f, respectively.
- In Section 3, we introduce the definition of Morse–Bott contact forms. Then we study the canonical precontact structure associated to the loci of closed Reeb orbits under Morse–Bott assumption.

- In Section 4, we introduce the notion of contact thickening of precontact structure. It is the contact analogue of the symplectic thickening for presymplectic structure constructed in [G], [OP].
- In Section 5, we prove a canonical neighborhood theorem of the loci of closed Reeb orbits Q under the Morse–Bott assumption.
- In Section 6, we derive the linearization formula of a Reeb orbit in the normal form.
- In Section 7, we express the derivative $dw = (du, \nabla_{du} f)$ of any smooth map $w = (u, f)$ from a (punctured) surface into the normal neighborhood F of Q in terms of the splitting

$$TU_F = TQ \oplus F = TQ \oplus (E \oplus JT\mathcal{F}), \quad TQ = T\mathcal{F} \oplus G.$$

- In Sections 8 and 9, we introduce the class of adapted CR-almost complex structure and prove its abundance.

§2. Basics on contact forms

We recall some basic facts on the contact geometry and contact Hamiltonian dynamics especially in relation to the perturbation of contact forms for a given contact manifold (M, ξ).

2.1 λ-dual vector fields and λ-dual one-forms

Let (M, ξ) be a contact manifold and λ be a contact form with $\ker \lambda = \xi$. Consider its associated decomposition

$$TM = \mathbb{R}\{X_\lambda\} \oplus \xi \tag{2.1}$$

and denote by $\pi = \pi_\lambda : TM \to \xi$ the associated projection. This decomposition canonically induces the corresponding dual decomposition

$$T^*M = \xi^\perp \oplus (\mathbb{R}\{X_\lambda\})^\perp \tag{2.2}$$

where $(\cdot)^\perp$ is the annihilator of $(\cdot)$. This gives rise to a decomposition

$$\alpha = \alpha(X_\lambda)\, \lambda + \alpha \circ \pi_\lambda. \tag{2.3}$$

Then we have the following general lemma whose proof immediately follows from (2.2).

LEMMA 2.1. *For any given one-form α, there exists a unique $Y_\alpha \in \xi$ such that*

$$\alpha = Y_\alpha \rfloor d\lambda + \alpha(X_\lambda)\lambda.$$

DEFINITION 2.2. (λ-Dual vector field and one-form) Let λ be a given contact form of (M, ξ). We define the λ-*dual vector field* of a one-form α to be

$$\flat_\lambda(\alpha) := Y_\alpha + \alpha(X_\lambda)\, X_\lambda.$$

Conversely for any given vector field X we define its λ-dual one-form by

$$\sharp_\lambda(X) = X \rfloor d\lambda + \lambda(X)\, \lambda.$$

For the simplicity of notation, we denote $\alpha_X := \sharp_\lambda(X)$. By definition, we have the identity

$$\lambda(\flat_\lambda(\alpha)) = \alpha(X_\lambda). \tag{2.4}$$

The following proposition is immediate from the definitions of the dual vector field and the dual one-forms.

PROPOSITION 2.3. *The map $\flat_\lambda : \Omega^1(M) \to \mathfrak{X}(M)$, $\alpha \mapsto \flat_\lambda(\alpha)$ and the map $\sharp^\lambda : \mathfrak{X}(M) \to \Omega^1(M)$, $X \mapsto \alpha_X$ are inverse to each other. In particular any vector field can be written as $\flat_\lambda(\alpha)$ for a unique one-form α and any one-form can be written as α_X for a unique vector field X.*

By definition, we have $\flat_\lambda(\lambda) = X_\lambda$ which corresponds to the λ-dual to the contact form λ itself for which $Y_\lambda = 0$ by definition. Obviously when an exact one-form α is given, the choice of h with $\alpha = dh$ is unique modulo addition by a constant (on each connected component of M).

To equip readers with some feeling on the above decomposition which is not common in the literature, we now provide the coordinate expression of $\flat_\lambda(\alpha)$ and α_X in the Darboux chart $(q_1, \ldots, q_n, p_1, \ldots, p_n, z)$ with respect to the canonical one-form $\lambda_0 = dz - \sum_{i=1}^n p_i\, dq_i$ on $\mathbb{R}^{2n+1}$ or more generally on the one-jet bundle $J^1(N)$ of a smooth n-manifold N. *However, this coordinate expression will not be used in the rest of the present paper.*

We recall that for this contact form, the associated Reeb vector field is nothing but

$$X_{\lambda_0} = \frac{\partial}{\partial z}.$$

We start with the expression of $\flat_\lambda(\alpha)$ for a given one-form

$$\alpha = \alpha_0\, dz + \sum_{i=1}^n a_i\, dq_i + \sum_{j=1}^n b_j\, dp_j.$$

We denote

$$\flat_\lambda(\alpha) = v_0 \frac{\partial}{\partial z} + \sum_{i=1}^n v_{i;q} \frac{\partial}{\partial q_i} + \sum_{j=1}^n v_{j;p} \frac{\partial}{\partial p_j}.$$

A direct computation using the defining equation of $\flat_\lambda(\alpha)$ leads to

PROPOSITION 2.4. *Consider the standard contact structure*

$$\lambda = dz - \sum_{i=1}^n p_i \, dq_i \quad \textit{on } \mathbb{R}^{2n+1}.$$

Then for the given one-form $\alpha = \alpha_0 \, dz + \sum_{i=1}^n a_i \, dq_i + \sum_{j=1}^n b_j \, dp_j$,

$$\flat_\lambda(\alpha) = \left(\alpha_0 + \sum_{k=1}^n p_k \, b_k\right) \frac{\partial}{\partial z} + \sum_{i=1}^n b_i \frac{\partial}{\partial q_i} + \sum_{j=1}^n (-a_j - p_j \, \alpha_0) \frac{\partial}{\partial p_j}. \tag{2.5}$$

Conversely, for the given

$$X = v_0 \frac{\partial}{\partial z} + \sum_{i=1}^n v_{i;q} \frac{\partial}{\partial q_i} + \sum_{j=1}^n v_{j;p} \frac{\partial}{\partial p_j},$$

we obtain

$$\begin{aligned} \alpha_X = & \left(v_0 - \sum_{j=1}^n p_j \, v_{j;q}\right) dz \\ & - \sum_{i=1}^n \left(v_{i;p} + \left((v_0 - \sum_{j=1}^n p_j \, v_{j;q}) p_i\right)\right) dq_i + \sum_{j=1}^n v_{j;q} \, dp_j. \end{aligned} \tag{2.6}$$

Proof. Here we first recall the basic identity (2.4).

By definition, $\flat_\lambda(\alpha)$ is determined by the equation

$$\alpha = \flat_\lambda(\alpha) \rfloor \sum_{i=1}^n dq_i \wedge dp_i + \lambda(\flat_\lambda(\alpha)) \left(dz - \sum_{i=1}^n p_i \, dq_i\right) \tag{2.7}$$

in the current case. A straightforward computation leads to the formula (2.5). Then (2.6) can be derived either by inverting this formula or by using the defining equation of α_X, which is further reduced to

$$\alpha_X = X \rfloor \, d\lambda + \lambda(X) \, \lambda = X \rfloor \sum_{i=1}^n \, dq_i \wedge dp_i + \left(dz - \sum_{j=1}^n p_j \, dq_j\right)(X) \, \lambda.$$

We omit the details of the computation. ∎

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EXAMPLE 2.5. Again consider the canonical one-form $dz - \sum_{i=1}^n p_i\, dq_i$. Then we obtain the following coordinate expression as a special case of (2.5)

(2.8)
$$\flat_\lambda(dh) = \left(\frac{\partial h}{\partial z} + \sum_{i=1}^n p_i \frac{\partial h}{\partial p_i}\right)\frac{\partial}{\partial z} + \sum_{i=1}^n \frac{\partial h}{\partial p_i}\frac{\partial}{\partial q_i} + \sum_{i=1}^n \left(-\frac{\partial h}{\partial q_i} - p_i \frac{\partial h}{\partial z}\right)\frac{\partial}{\partial p_i}.$$

2.2 Perturbation of contact forms of (M, ξ)

In this section, we exploit the discussion on the λ-dual vector fields and express the Reeb vector fields $X_{f\lambda}$ and the projection $\pi_{f\lambda}$ associated to the contact form $f\lambda$ for a positive function $f > 0$, in terms of those associated to the given contact form λ and the λ-dual vector fields of df.

Recalling Lemma 2.1, we can write

$$dg = Y_{dg} \rfloor\, d\lambda + dg(X_\lambda)\lambda$$

with $Y_{dg} \in \xi$ in a unique way for any smooth function g. Then by definition, we have $Y_{dg} = \pi_\lambda(\flat_\lambda(dg))$.

We first compute the following useful explicit formula for the associated Reeb vector fields $X_{f\lambda}$ in terms of X_λ and Y_{dg}.

PROPOSITION 2.6. (Perturbed Reeb vector field) *Denote*

$$Y_{dg} = \pi_\lambda(\flat_\lambda(dg))$$

as above. Then we have

$$X_{f\lambda} = \frac{1}{f}(X_\lambda + Y_{dg}), \quad g = \log f.$$

Proof. It turns out to be easier to consider $f\, X_{f\lambda}$ which we compute below. First we have

$$f\, X_{f\lambda} = c \cdot X_\lambda + \eta \tag{2.9}$$

with respect to the splitting $TM = \mathbb{R}\{X_\lambda\} \oplus \xi$ for some constant c and $\eta \in \xi$. We evaluate

$$c = \lambda(f X_{f\lambda}) = (f\lambda)(X_{f\lambda}) = 1.$$

It remains to derive the formula for η. Using the formula

$$d(f\lambda) = f\, d\lambda + df \wedge \lambda$$

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and $\lambda(\eta) = 0$, we compute

$$\begin{aligned}
\eta \rfloor\, d\lambda &= (f X_{f\lambda}) \rfloor\, d\lambda \\
&= X_{f\lambda} \rfloor\, d(f\lambda) - X_{f\lambda} \rfloor (df \wedge \lambda) \\
&= -X_{f\lambda} \rfloor (df \wedge \lambda) \\
&= -X_{f\lambda}(f)\lambda + \lambda(X_{f\lambda})\, df \\
&= -\frac{1}{f}(X_\lambda + \eta)(f)\lambda + \frac{1}{f}\lambda(X_\lambda + \eta)\, df \\
&= -\frac{1}{f} X_\lambda(f)\lambda - \frac{1}{f}\eta(f)\lambda + \frac{1}{f}\, df.
\end{aligned}$$

Take value of X_λ for both sides, we get $\eta(f) = 0$, and hence

$$\eta \rfloor\, d\lambda = -\frac{1}{f} X_\lambda(f)\lambda + \frac{1}{f}\, df.$$

Setting $g = \log f$, we can rewrite this into

$$\eta \rfloor\, d\lambda = -dg(X_\lambda)\lambda + dg.$$

In other words, we obtain

$$dg = \eta \rfloor\, d\lambda + dg(X_\lambda)\lambda. \tag{2.10}$$

Therefore, by Lemma 2.1, we have obtained $\eta = Y_{dg}$. Substituting this into (2.9) and dividing it by f, we have finished the proof. ∎

Next we compare the contact projections π_λ with $\pi_{f\lambda}$ associated to λ and $f\lambda$, respectively.

PROPOSITION 2.7. (Perturbed contact projection) *Let (M, ξ) be a contact manifold and let λ be a contact form that is,* $\ker \lambda = \xi$. *Let f be a positive smooth function and $f\,\lambda$ be its associated contact form. Denote by π_λ and $\pi_{f\,\lambda}$ be their associated ξ-projection. Then*

$$\pi_{f\,\lambda}(Z) = \pi_\lambda(Z) - \lambda(Z) Y_{dg} \tag{2.11}$$

for the function $g = \log f$.

Proof. We compute

$$\begin{aligned}\pi_{f\lambda}(Z) &= Z - f\lambda(Z)X_{f\lambda} = Z - \lambda(Z)(fX_{f\lambda})\\ &= Z - \lambda(Z)X_\lambda + (\lambda(Z)X_\lambda - \lambda(Z)(fX_{f\lambda}))\\ &= \pi_\lambda Z + \lambda(Z)(X_\lambda - fX_{f\lambda})\\ &= \pi_\lambda Z - \lambda(Z)Y_{dg}.\end{aligned}$$

This finishes the proof. ∎

2.3 Linearization formula for the perturbed contact form

We next study the relationship between the linearization of $\Upsilon_\lambda(z) = \dot z - TX_{f\lambda}(z)$ which we denote by

$$D^\pi\Upsilon(z)(Z) = \nabla_t^\pi Z - T\nabla_Z X_{f\,\lambda}$$

with respect to the triad connection of (M, λ, J) (see [OW2, Proposition 7.6]) for a given function f. Substituting

$$X_{f\,\lambda} = \frac{1}{f}(X_\lambda + Y_{dg})$$

into this formula, we derive

LEMMA 2.8. (Linearization) *Let ∇ be the triad connection of $(M, f\lambda, J)$. Then for any vector field Z along a Reeb orbit $z = (\gamma(T\cdot), o_{\gamma(T\cdot)})$,*

$$\begin{aligned}D^\pi\Upsilon(z)(Z) = \nabla_t^\pi Z - T\left(\frac{1}{f}\nabla_Z X_\lambda + Z[1/f]X_\lambda\right)\\ - T\left(\frac{1}{f}\nabla_Z Y_{dg} + Z[1/f]Y_{dg}\right)\end{aligned} \tag{2.12}$$

Proof. Let ∇ be the triad connection of $(M, f\lambda, J)$. Then by definition its torsion T satisfies the axiom $T(X_\lambda, Z) = 0$ for any vector field Z on M (see [OW1, Theorem 1]). Using this property, as in [OW2, Section 7], we compute

$$\begin{aligned}D^\pi\Upsilon(z)(Z) &= \nabla_t^\pi Z - T\nabla_Z X_{f\lambda}\\ &= \nabla_t^\pi Z - T\nabla_Z\left(\frac{1}{f}(X_\lambda + Y_{dg})\right)\\ &= \nabla_t^\pi Z - TZ[1/f](X_\lambda + Y_{dg}) - \frac{T}{f}\nabla_Z(X_\lambda + Y_{dg})\end{aligned}$$

$$= \nabla_t^\pi Z - T\left(\frac{1}{f}\nabla_Z X_\lambda + Z[1/f]X_\lambda\right)$$
$$- T\left(\frac{1}{f}\nabla_Z Y_{dg} + Z[1/f]Y_{dg}\right),$$

which finishes the proof. ∎

We note that when $f \equiv 1$, the above formula reduces to the standard formula

$$D^\pi \Upsilon(z)(Z) = \nabla_t^\pi Z - T\nabla_Z X_\lambda$$

which is further reduced to

$$D^\pi \Upsilon(z)(Z) = \nabla_t^\pi Z - \frac{T}{2}(\mathcal{L}_{X_\lambda} J)JZ$$

for any contact triad (M, λ, J). (See [OW2, Section 7] for some discussion on this formula.)

§3. The locus foliated by closed Reeb orbits

3.1 Definition of Morse–Bott contact form

Let (M, ξ) be a contact manifold and λ be a contact form of ξ. We would like to study the linearization of the equation $\dot{x} = X_\lambda(x)$ along a closed Reeb orbit. Let γ be a closed Reeb orbit of period $T > 0$. In other words, $\gamma : \mathbb{R} \to M$ is a solution of $\dot{\gamma} = X_\lambda(\gamma)$ satisfying $\gamma(t + T) = \gamma(t)$.

By definition, we can write $\gamma(t) = \phi^t_{X_\lambda}(\gamma(0))$ for the Reeb flow $\phi^t = \phi^t_{X_\lambda}$ of the Reeb vector field X_λ. In particular $p = \gamma(0)$ is a fixed point of the diffeomorphism ϕ^T when γ is a closed Reeb orbit of period T. Since $\mathcal{L}_{X_\lambda}\lambda = 0$, the contact diffeomorphism ϕ^T canonically induces the isomorphism

$$\Psi_{z;p} := d\phi^T(p)|_{\xi_p} : \xi_p \to \xi_p$$

which is the linearized Poincaré return map ϕ^T restricted to ξ_p via the splitting

$$T_pM = \xi_p \oplus \mathbb{R} \cdot \{X_\lambda(p)\}.$$

Definition 3.1. Let γ be a closed Reeb orbit with period $T > 0$ and denote by $z : S^1 \to M$ the map defined by $z(t) = \gamma(Tt)$. We say a T-closed Reeb orbit (T, z) is *nondegenerate* if the linearized return map $\Psi_{z;p} : \xi_p \to \xi_p$ with $p = \gamma(0)$ has no eigenvalue 1.

Denote $\mathrm{Cont}(M, \xi)$ the set of contact one-forms with respect to the contact structure ξ and $\mathcal{L}(M) = C^\infty(S^1, M)$ the space of loops

$z : S^1 = \mathbb{R}/\mathbb{Z} \to M$. We would like to consider the bundle $\mathcal{L}$ over the product $(0, \infty) \times \mathcal{L}(M) \times \mathrm{Cont}(M, \xi)$ whose fiber at (T, z, λ) is given by the space $C^\infty(z^*TM)$ of sections of the bundle $z^*TM \to S^1$. We consider the assignment

$$\Upsilon : (T, z, \lambda) \mapsto \dot{z} - T\, X_\lambda(z)$$

which is a section. Then $(T, z, \lambda) \in \Upsilon^{-1}(0) := \mathfrak{Reeb}(M, \xi)$ if and only if there exists some closed Reeb orbit $\gamma : \mathbb{R} \to M$ with period T, such that $z(\cdot) = \gamma(T\cdot)$.

We first start with the standard notion, applied to the set of closed Reeb orbits, of Morse–Bott critical manifolds introduced by Bott in [Bot]:

DEFINITION 3.2. We call a contact form λ *standard Morse–Bott type* if every connected component of $\mathfrak{Reeb}(M, \lambda)$ is a smooth submanifold of $(0, \infty) \times \mathcal{L}(M)$ with its tangent space at every pair $(T, z) \in \mathfrak{Reeb}(M, \lambda)$ therein coincides with $\ker d_{(T,z)}\Upsilon$.

The following is an immediate consequence of this definition.

LEMMA 3.3. *Suppose λ is of standard Morse–Bott type, then on each connected component of $\mathfrak{Reeb}(M, \lambda)$, the period remains constant.*

Proof. Let (T_0, z_0) and (T_1, z_1) be two elements in the same connected component of $\mathfrak{Reeb}(M, \lambda)$. We connect them by a smooth one-parameter family (T_s, z_s) for $0 \leqslant s \leqslant 1$. Since $\dot{z}_s = T_s X_\lambda(z_s)$ and then $T_s = \int_{S^1} z_s^*\lambda$, it is enough to prove

$$\frac{d}{ds} T_s = \frac{d}{ds} \int_{S^1} z_s^*\lambda \equiv 0.$$

We compute

$$\begin{aligned}
\frac{d}{ds} z_s^*\lambda &= z_s^*(d(z_s' \rfloor \lambda) + z_s' \rfloor d\lambda) \\
&= d(z_s^*(z_s' \rfloor \lambda)) + z_s^*(z_s' \rfloor d\lambda) \\
&= d(z_s^*(z_s' \rfloor \lambda)).
\end{aligned}$$

Here we use z_s' to denote the derivative with respect to s. The last equality comes from the fact that $\dot{z}_s$ is parallel to X_λ. Therefore, we obtain by Stokes formula that

$$\frac{d}{ds} \int_{S^1} z_s^*\lambda = \int_{S^1} d(z_s^*(z_s' \rfloor \lambda)) = 0$$

and finish the proof. ∎

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Now we prove

LEMMA 3.4. *Let λ be standard Morse–Bott type. Fix a connected component $\mathcal{R} \subset \mathfrak{Reeb}(M, \lambda)$ and denote by $Q \subset M$ the locus of the corresponding closed Reeb orbits. Then Q is a smooth immersed submanifold which carries a natural locally free S^1-action induced by the Reeb flow over one period.*

Proof. Consider the evaluation map $ev_{\mathcal{R}} : \mathfrak{Reeb}(M, \lambda) \to M$ defined by $ev_{\mathcal{R}}(T, z) = z(0)$. It is easy to prove that the map is a local immersion and so Q is an immersed submanifold. Since the closed Reeb orbits have constant period $T > 0$ by Lemma 3.3, the action is obviously locally free and so $ev_{\mathcal{R}}$ is an immersion and so Q is immersed in M. This finishes the proof. □

However, Q may not be embedded in general along the locus of multiple orbits.

Partially following [Bou], from now on in the rest of the paper, *we always assume Q is embedded and compact.* Denote $\omega_Q := i_Q^* d\lambda$ and

$$\ker \omega_Q = \{e \in TQ \mid \omega(e, e') = 0 \text{ for any } e' \in TQ\}.$$

We warn readers that the Morse–Bott condition does not imply that the form ω_Q has constant rank, and hence the dimension of this kernel may vary pointwise on Q. However, if it does, $\ker \omega_Q$ defines an integrable distribution and so defines a foliation, denoted by $\mathcal{F}$, on Q. Since Q is also foliated by closed Reeb orbits and $\mathcal{L}_{X_\lambda} d\lambda = 0$, it follows that $\mathcal{L}_{X_\lambda} \omega_Q = 0$ when we restrict everything on Q. Therefore, each leaf of the foliation consists of closed Reeb orbits. Motivated by this, we also impose the condition that *the two-form ω_Q has constant rank.*

DEFINITION 3.5. (Compare with Definition 1.7 [Bou]) We say that the contact form λ is of Morse–Bott type if it satisfies the following:

(1) every connected component of $\mathfrak{Reeb}(M, \lambda)$ is a smooth submanifold of $(0, \infty) \times \mathcal{L}(M)$ with its tangent space at every pair $(T, z) \in \mathfrak{Reeb}(M, \lambda)$ therein coincides with $\ker d_{(T,z)} \Upsilon$;
(2) Q is embedded;
(3) ω_Q has constant rank on Q.

3.2 Structure of the locus of closed Reeb orbits

Let λ be a Morse–Bott contact form of (M, ξ) and X_λ its Reeb vector field. Let Q be as in Definition 3.5. In general, Q carries a natural locally free S^1-action induced by the Reeb flow $\phi^T_{X_\lambda}$ (see Lemma 3.4). Then by

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the general theory of compact Lie group actions (see [He] for example), the action has a finite number of orbit types which have their minimal periods, T/m for some integer $m \geqslant 1$. The set of orbit spaces Q/S^1 carries natural orbifold structure at each multiple orbit with its isotropy group $\mathbb{Z}/m$ for some m.

REMARK 3.6. Here we would like to mention that the S^1-action induced by $\phi^T_{X_\lambda}$ on Q may not be effective: it is possible that the connected component $\mathfrak{R}$ of $\mathfrak{Reeb}(M, \lambda)$ can consist entirely of multiple orbits.

Now we fix a connected component of Q and just denote it by Q itself. Denote $\theta = i_Q^*\lambda$. We note that the two-form $\omega_Q = d\theta$ is assumed to have constant rank on Q by the definition of Morse–Bott contact form in Definition 3.5.

The following is an immediate consequence of the definition but exhibits a particularity of the null foliation of the presymplectic manifold (Q, ω_Q) arising from the locus of closed Reeb orbits. We note that $\ker \omega_Q$ carries a natural splitting

$$\ker \omega_Q = \mathbb{R}\{X_\lambda\} \oplus (\ker \omega_Q \cap \xi|_Q).$$

LEMMA 3.7. *The distribution* $(\ker \omega_Q) \cap \xi|_Q$ *on* Q *is integrable.*

Proof. Let X, Y be vector fields on Q such that $X, Y \in (\ker \omega_Q) \cap \xi|_Q$. Then $[X, Y] \in \ker \omega_Q$ since ω_Q is a closed two-form whose null distribution is integrable. At the same time, we compute $i_Q^*\lambda([X, Y]) = X[\lambda(Y)] - Y[\lambda(X)] - \omega_Q(X, Y) = 0$ where the first two terms vanish since $X, Y \in \xi$ and the third vanishes because $X \in \ker \omega_Q$. This proves $[X, Y] \in \ker \omega_Q \cap \xi|_Q$, which finishes the proof. ∎

Therefore, $\ker \omega_Q \cap \xi|_Q$ defines another foliation $\mathcal{N}$ on Q, and hence

$$TF = \mathbb{R}\{X_\lambda\} \oplus T\mathcal{N}. \tag{3.1}$$

Note that this splitting is S^1-invariant.

We now recall some basic properties of presymplectic manifold [G] and its canonical neighborhood theorem. Fix an S^1-equivariant splitting of TQ

$$TQ = T\mathcal{F} \oplus G = \mathbb{R}X_\lambda \oplus T\mathcal{N} \oplus G \tag{3.2}$$

by choosing an S^1-invariant complementary subbundle $G \subset TQ$. This splitting is not unique but its choice will not matter for the coming discussions.

The null foliation carries a natural *transverse symplectic form* in general [G]. Since the distribution $T\mathcal{F} \subset TQ$ is preserved by Reeb flow, it generates the S^1-action thereon in the current context. We denote by

$$p_{T\mathcal{N};G} : TQ \to T\mathcal{N}, \qquad p_{G;G} : TQ \to G$$

the projection to $T\mathcal{N}$ and to G, respectively, with respect to the splitting (3.2). We denote by $T^*\mathcal{N} \to Q$ the associated foliation cotangent bundle, that is, the dual bundle of $T\mathcal{N}$.

We now consider the isomorphism

$$\widetilde{d\lambda|_\xi} : \xi \to \xi^* \tag{3.3}$$

and fix a splitting $T_Q M = TQ \oplus N_Q M$ with $T_Q M = TM|_Q$ so that $N_Q M \subset \xi_Q$: this is possible since $\mathbb{R}\{X_\lambda\} \subset TQ$. We can also choose the splitting so that it is S^1-equivariant. (See the proof of Proposition 3.9 below.)

This leads to the further splitting

$$\xi|_Q = T\mathcal{N} \oplus G \oplus N_Q M \tag{3.4}$$

combined with (3.2), which in turn leads to

$$\xi^*|_Q = (G \oplus N_Q M)^\perp \oplus (T\mathcal{N})^\perp \tag{3.5}$$

where $(\cdot)^\perp$ denotes the annihilator of $(\cdot)$. In particular, it induces an isomorphism

$$T^*\mathcal{N} \cong (G \oplus N_Q M)^\perp \subset \xi^*|_Q. \tag{3.6}$$

Now we consider the embedding $T^*\mathcal{N} \to \xi$ defined by the inverse of (3.3), which we denote by

$$(T\mathcal{N})^{\#_{d\lambda}} = (\widetilde{d\lambda})^{-1}(T^*\mathcal{N}). \tag{3.7}$$

Next we consider the symplectic normal bundle $(TQ)^{d\lambda} \subset T_Q M$ defined by

$$(TQ)^{d\lambda} = \{v \in T_q M \mid d\lambda(v, w) = 0, \forall w \in T_q Q\}. \tag{3.8}$$

We define a vector bundle

$$E = (TQ)^{d\lambda}/T\mathcal{F}, \tag{3.9}$$

and then have the natural embedding

$$E = (TQ)^{d\lambda}/T\mathcal{F} \hookrightarrow T_Q M/TQ = N_Q M \tag{3.10}$$

induced by the inclusion map $(TQ)^{d\lambda} \hookrightarrow T_Q M$. The following is straightforward to check.

LEMMA 3.8. *The $d\lambda|_E$ induces a nondegenerate 2-form and so E carries a fiberwise symplectic form, which we denote by Ω.*

We now consider the exact sequence

$$0 \to E \to N_Q M \to N_Q M/E \to 0$$

induced by (3.10). The sequence is S^1-equivariant with respect to the natural S^1-action thereon. We have the canonical isomorphism

$$N_Q M/E \cong \frac{T_Q M}{TQ + (TQ)^{d\lambda}}$$

which is S^1-equivariant. Then we have an S^1-equivariant splitting

$$T_Q M = (TQ + (TQ)^{d\lambda}) \oplus (T\mathcal{N})^{\#_{d\lambda}} \tag{3.11}$$

where $(T\mathcal{N})^{\#_{d\lambda}}$ is the $d\lambda$-dual (3.7). This also induces an embedding

$$T^*\mathcal{N} \to (T\mathcal{N})^{\#_{d\lambda}} \hookrightarrow T_Q M \to N_Q M \tag{3.12}$$

which is also S^1-equivariant.

We now denote $F := T^*\mathcal{N} \oplus E \to Q$. The following proposition provides a local model of the neighborhood of $Q \subset M$.

PROPOSITION 3.9. *We fix the splittings (3.2) and (3.4). Then the sum of (3.12) and (3.10) defines an isomorphism $T^*\mathcal{N} \oplus E \to N_Q M$ depending only on the splittings.*

Proof. A straightforward dimension counting shows that the bundle map indeed is an isomorphism. ∎

Identifying a neighborhood of $Q \subset M$ with a neighborhood of the zero section of F and pulling back the contact form λ to F, we may assume that our contact form λ is defined on a neighborhood of $o_F \subset F$. We also identify with $T^*\mathcal{N}$ and E as their images in $N_Q M$.

PROPOSITION 3.10. *The S^1-action on Q canonically induces the S^1-invariant vector bundle structure on E such that the form Ω is invariant under the S^1-action on E.*

Proof. The action of S^1 on Q by $t \cdot q = \phi^t(q)$ canonically induces a S^1 action on $T_Q M$ by $t \cdot v = (d\phi^t)(v)$, for $v \in T_Q M$. Hence it gives rise to the following identity

$$t^* d\lambda = d\lambda \tag{3.13}$$

since the Reeb flow preserves λ. We first show it is well-defined on $E \to Q$, that is, if $v \in (T_q Q)^{d\lambda}$, then $t \cdot v \in (T_{t\cdot q} Q)^{d\lambda}$. In fact, by using (3.13), for $w \in T_{t\cdot q} Q$,

$$d\lambda(t \cdot v, w) = ((\phi^t)^* d\lambda)(v, (d\phi^t)^{-1}(w)) = d\lambda(v, (d\phi^t)^{-1}(w)).$$

This vanishes, since Q consists of closed Reeb orbits and thus $d\phi^t$ preserves TQ.

Secondly, the same identity (3.13) further indicates that this S^1 action preserves Ω on fibers, that is, $t^*\Omega = \Omega$, and we are done with the proof of this proposition. ∎

Summarizing the above discussion, we have concluded that the base Q is decorated by the one-form $\theta := i_Q^* \lambda$ on the base Q and the bundle E is decorated by the fiberwise symplectic 2-form Ω. They satisfy the following additional properties:

(1) $Q = o_F$ carries an S^1-action which is locally free. In particular Q/S^1 is a smooth orbifold;
(2) the one-form θ is S^1-invariant, and $d\theta$ is a presymplectic form;
(3) the bundle E carries an S^1-action that preserves the fiberwise 2-form Ω and hence induces a S^1-invariant symplectic vector bundle structure on E;
(4) the bundle $F = T^*\mathcal{N} \oplus E \to Q$ carries the direct sum S^1-equivariant vector bundle structure compatible to the S^1-action on Q.

We summarize the above discussions into the following theorem.

THEOREM 3.11. *Consider the locus Q of closed Reeb orbits of a Morse–Bott-type contact form λ. Let $(TQ)^{\omega_Q} \subset TQ$ be the null distribution of $\omega_Q = i_Q^* d\lambda$ and $\mathcal{F}$ be the associated characteristic foliation. Then the restriction* *of λ to Q induces the following geometric structures:*

(1) *$Q = o_F$ carries an S^1-action which is locally free. In particular Q/S^1 is a smooth orbifold. Fix an S^1-invariant splitting (3.2);*

(2) *we have the natural identification*

$$N_Q M \cong T^*\mathcal{N} \oplus E = F, \tag{3.14}$$

as an S^1-equivariant vector bundle, where

$$E := (TQ)^{d\lambda}/T\mathcal{F}$$

is the symplectic normal bundle;

(3) *the two-form $d\lambda$ restricts to a nondegenerate skew-symmetric two-form on G, and induces a fiberwise symplectic form Ω on E defined as above.*

We say that Q is of prequantization type if the rank of $d\lambda|_Q$ is maximal and is of isotropic type if the rank of $d\lambda|_Q$ is zero. The general case will be a mixture of the two.

REMARK 3.12. In particular when dim $M = 3$, such Q must be either of prequantization type or of isotropic type. This is the case that is considered in [HWZ3]. The general case considered in [Bou] and [BEHWZ] includes the mixed type.

§4. Contact thickening of Morse–Bott contact setup

Motivated by the isomorphism in Theorem 3.11, we consider the pair (Q, θ) and the symplectic vector bundle $(E, \Omega) \to Q$ that satisfy the above properties. We assume that Q is compact and connected.

In the next section, we associate the model contact form on the direct sum

$$F = T^*\mathcal{N} \oplus E$$

and prove a canonical neighborhood theorem of the locus of closed Reeb orbits of general contact manifold (M, λ) such that the zero section of F corresponds to Q.

To state our canonical neighborhood theorem, we need to first identify the relevant geometric structure of the canonical neighborhoods. For this purpose, introduction of the following notion is useful.

DEFINITION 4.1. (Precontact form) We call a one-form θ on a manifold Q a *precontact* form if $d\theta$ has constant rank.

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4.1 The S^1-invariant precontact manifold (Q,θ)

First, we consider the pair (Q,θ) such that Q carries a nontrivial S^1-action preserving the one-form θ. After taking the quotient of S^1 by some finite subgroup, we may assume that the action is effective. We also assume that the action is locally free. Then by the general theory of group actions of compact Lie group (see [He] for example), the action is free on a dense open subset and has only a finite number of different types of isotropy groups. In particular the quotient $P := Q/S^1$ becomes a presymplectic orbifold with a finite number of different types of orbifold points. We denote by Y the vector field generating the S^1-action, that is, the S^1-action is generated by its flows.

We require that the circle action preserves θ, that is, $\mathcal{L}_Y\theta = 0$. Since the action is locally free and free on a dense open subset of Q, we can normalize the action so that

$$\theta(Y) \equiv 1. \tag{4.1}$$

We denote this normalized vector field by X_θ. We would like to emphasize that θ may not be a contact form but can be regarded as the connection form of the circle bundle $S^1 \to Q \to P$ over the orbifold P in general. Although P may carry nonempty set of orbifold points, the connection form θ is assumed to be smooth on Q.

Similarly as in Lemma 3.7, we also require the presence of S^1-invariant splitting

$$\ker d\theta = \mathbb{R}\{X_\theta\} \oplus H$$

such that the subbundle H is also integrable.

With these terminologies introduced, we can rephrase Theorem 3.11 as follows.

THEOREM 4.2. *Let Q be the locus foliated by closed Reeb orbits of a contact manifold (M,λ) of Morse–Bott type. Then Q carries a locally free S^1-action and:*

(1) *an S^1-invariant precontact form θ given by $\theta = i_Q^*\lambda$;*
(2) *a splitting*

$$\ker d\theta = \mathbb{R}\{X_\theta\} \oplus H, \tag{4.2}$$

such that the distribution $H = \ker d\theta \cap \xi|_Q$ is integrable;

(3) *an S^1-equivariant symplectic vector bundle $(E, \Omega) \to Q$ with*

$$E = (TQ)^{d\lambda}/\ker d\theta, \qquad \Omega = [d\lambda]_E$$

Here we use the fact that there exists a canonical embedding

$$E = (TQ)^{d\lambda}/\ker d\theta \hookrightarrow T_Q M/TQ = N_Q M$$

and $d\lambda|_{(TQ)^{d\lambda}}$ canonically induces a bilinear form $[d\lambda]_E$ on $E = (TQ)^{d\lambda}/\ker di_Q^*\lambda$ by symplectic reduction.

Definition 4.3. Let (Q, θ) be a precontact manifold equipped with a locally free S^1-action generated by a vector field Y, and with a S^1-invariant one-form θ and the splitting (4.2). Assume $\theta(Y) \equiv 1$. We call such a triple (Q, θ, H) a *Morse–Bott contact setup*.

As before, we denote by $\mathcal{F}$ and $\mathcal{N}$ the associated foliations on Q, and decompose

$$T\mathcal{F} = \mathbb{R}\{X_\theta\} \oplus T\mathcal{N}.$$

We define a one-form Θ_G on $T^*\mathcal{N}$ as follows. For a tangent $\xi \in T_\alpha(T^*\mathcal{N})$, define

$$\Theta_G(\xi) := \alpha(p_{T\mathcal{N};G}\, d\pi_{(T^*\mathcal{N})}(\xi)) \tag{4.3}$$

using the splitting

$$TQ = \mathbb{R}\{X_\theta\} \oplus T\mathcal{N} \oplus G.$$

By definition, it follows $\Theta_G|_{VT(T^*\mathcal{N})} \equiv 0$ and $d\Theta_G(\alpha)$ is nondegenerate on

$$\widetilde{T_q\mathcal{N}} \oplus VT_\alpha T^*\mathcal{N} \cong T_q\mathcal{N} \oplus T_q^*\mathcal{N}$$

which becomes the canonical pairing defined on $T_q\mathcal{N} \oplus T_q^*\mathcal{N}$ under the identification.

4.2 The bundle E

We next examine the structure of the S^1-equivariant symplectic vector bundle (E, Ω).

We denote by $\vec{R}$ the radial vector field which generates the family of radial multiplication

$$(c, e) \mapsto c\, e.$$

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This vector field is invariant under the given S^1-action on E, and vanishes on the zero section. By its definition, $d\pi(\vec{R}) = 0$, that is, $\vec{R}$ is in the vertical distribution, denoted by VTE, of TE.

Denote the canonical isomorphism $V_eTE \cong E_{\pi(e)}$ by $I_{e;\pi(e)}$. It obviously intertwines the scalar multiplication, that is,

$$I_{e;\pi(e)}(\mu\,\xi) = \mu I_{e;\pi(e)}(\xi)$$

for a scalar μ. It also satisfies the following identity (4.4) with respect to the derivative of the fiberwise scalar multiplication map $R_c : E \to E$.

Lemma 4.4. *Let $\xi \in V_eTE$. Then*

$$I_{c\,e;\pi(c\,e)}(dR_c(\xi)) = c\,I_{e;\pi(e)}(\xi) \tag{4.4}$$

on $E_{\pi(c\,e)} = E_{\pi(e)}$ for any constant c.

Proof. We compute

$$\begin{aligned} I_{c\,e;\pi(c\,e)}(dR_c(\xi)) &= I_{c\,e;\pi(c\,e)}\left(\frac{d}{ds}\bigg|_{s=0} c(e+s\xi)\right) \\ &= I_{c\,e;\pi(c\,e)}(R_c(\xi)) = c\,I_{e;\pi(e)}(\xi) \end{aligned}$$

which finishes the proof. □

We then define the fiberwise two-form Ω^v on $VTE \to E$ by

$$\Omega^v_e(\xi_1, \xi_2) = \Omega_{\pi_F(e)}(I_{e;\pi(e)}(\xi_1), I_{e;\pi(e)}(\xi_2))$$

for $\xi_1, \xi_2 \in V_eTE$, and one-form $\vec{R} \rfloor \Omega^v$, respectively.

Now we introduce an S^1-invariant symplectic connection on E and choose the splitting

$$TE = HTE \oplus VTE.$$

Existence of such an invariant connection follows for example, by averaging over the compact group S^1. We denote by $\widetilde{\Omega}$ the resulting two-form on E. We extend the fiberwise form Ω of E into the differential two-form $\widetilde{\Omega}$ on E by setting

$$\widetilde{\Omega}_e(\xi, \zeta) = \Omega^v_e(\xi^v, \zeta^v).$$

Denote by $\vec{R}$ the radial vector field of $E \to Q$ and consider the one-form

$$\vec{R} \rfloor \widetilde{\Omega} \tag{4.5}$$

which is invariant under the action of S^1 on E.

REMARK 4.5. Suppose $d\lambda_E(\cdot, J_E\cdot) =: g_{E;J_E}$ defines a Hermitian vector bundle $(\xi_E, g_{E,J}, J_E)$. Then we can write the radial vector field considered in the previous section as

$$\vec{R}(e) = \sum_{i=1}^{k} r_i \frac{\partial}{\partial r_i}$$

where $(r_1, \ldots, r_k)$ is the coordinates of e for a local frame $\{e_1, \ldots, e_k\}$ of the vector bundle E. By definition, we have

$$I_{e;\pi_E(e)}(\vec{R}(e)) = e. \tag{4.6}$$

Obviously the right-hand side expression does not depend on the choice of local frames. Let (E, Ω, J_E) be a Hermitian vector bundle and define $|e|^2 = g_F(e, e)$. Motivated by the terminology used in [BT], we call the one-form

$$\psi = \psi_\Omega = \frac{1}{r}\frac{\partial}{\partial r}\rfloor \Omega^v \tag{4.7}$$

the *global angular form* for the Hermitian vector bundle (E, Ω, J_E). Note that ψ is defined only on $E \setminus o_E$ although Ω is globally defined.

We state the following lemma.

LEMMA 4.6. *Let Ω be as above. Then:*

(1) $\vec{R}\rfloor\, d\widetilde{\Omega} = 0$;
(2) *for any nonzero constant $c > 0$, we have*

$$R_c^*\widetilde{\Omega} = c^2\, \widetilde{\Omega}.$$

Proof. Notice that $\widetilde{\Omega}$ is compatible with Ω in the sense of symplectic fibration and the symplectic vector bundle connection is nothing but the Ehresmann connection induced by $\widetilde{\Omega}$, which is a symplectic connection now. Since $\vec{R}$ is vertical, the statement (1) immediately follows from the fact that the symplectic connection is vertical closed.

It remains to prove statement (2). Let $e \in E$ and $\xi_1,\ \xi_2 \in T_eE$. By definition, we derive

$$\begin{aligned}(R_c^*\widetilde{\Omega})_e(\xi_1, \xi_2) &= \widetilde{\Omega}_{c\,e}(dR_c(\xi_1^v), dR_c(\xi_2^v)) \\ &= \Omega^v_{c\,e}(dR_c(\xi_1^v), dR_c(\xi_2^v)) = c^2\widetilde{\Omega}_e(\xi_1, \xi_2)\end{aligned}$$

where we use the equality (4.4) and $\pi_F(c\,e) = \pi(e)$ for the fourth equality.

This proves $R_c^*\widetilde{\Omega} = c^2\widetilde{\Omega}$. ∎

It follows from Lemma 4.6(2) that we get $\mathcal{L}_{\vec{R}}\widetilde{\Omega} = 2\widetilde{\Omega}$. By Cartan's formula, we get

$$d(\vec{R}\rfloor\,\widetilde{\Omega}) = 2\widetilde{\Omega}.$$

4.3 Canonical contact form and contact structure on F

Let (Q, θ, H) be a given Morse–Bott contact setup and $(E, \Omega) \to Q$ be any S^1-equivariant symplectic vector bundle equipped with an S^1-invariant symplectic connection on it.

Now we equip the bundle $F = T^*\mathcal{N} \oplus E$ with a canonical S^1-invariant contact form on F. We denote the bundle projections by $\pi_{E;F} : F \to E$ and $\pi_{T^*\mathcal{N};F} : F \to T^*\mathcal{N}$ of the splitting $F = T^*\mathcal{N} \oplus E$, respectively, and provide the direct sum connection on $F = T^*\mathcal{N} \oplus E$.

THEOREM 4.7. *Let (Q, θ, H) be a Morse–Bott contact setup. Denote by $\mathcal{F}$ and $\mathcal{N}$ the foliations associated to the distribution* $\ker d\theta$ *and H, respectively. We also denote by $T\mathcal{F}$, $T\mathcal{N}$ the associated foliation tangent bundles and $T^*\mathcal{N}$ the foliation cotangent bundle of $\mathcal{N}$. Then for any symplectic vector bundle $(E, \Omega) \to Q$ with an S^1-invariant symplectic connection, the following holds:*

(1) *the total space of the bundle $F = T^*\mathcal{N} \oplus E$ carries a canonical contact form $\lambda_{F;G}$ defined as in (4.8), for each choice of complement G such that $TQ = T\mathcal{F} \oplus G$;*
(2) *for any two such choices of G, G', the associated contact forms are canonically gauge-equivalent by a bundle map $\psi_{GG'} : TQ \to TQ$ preserving $T\mathcal{F}$.*

Proof. We define a differential one-form on F explicitly by

$$\lambda_F = \pi_F^*\theta + \pi_{T^*\mathcal{N};F}^*\Theta_G + \tfrac{1}{2}\pi_{E;F}^*(\vec{R}\rfloor\,\widetilde{\Omega}). \tag{4.8}$$

Using Lemma 4.6, we obtain

$$d\lambda_F = \pi_F^*\, d\theta + \pi_{T^*\mathcal{N};F}^*\, d\Theta_G + \pi_{E;F}^*\widetilde{\Omega} \tag{4.9}$$

by taking the differential of (4.8).

A moment of examination of this formula gives rise to the following

PROPOSITION 4.8. *There exists some $\delta > 0$ such that the one-form λ_F is a contact form on the disc bundle $D^\delta(F)$, where*

$$D^\delta(F) = \{(q, v) \in F \mid \|v\| < \delta\}$$

such that $\lambda_F|_{o_F} = \theta$ on $Q \cong o_F \subset F$.

Proof. This immediately follows from the formulas (4.8) and (4.9). ∎

This proves the statement (1).

For the proof of the statement (2), we first note that the bundle E itself does not depend on the choice of G. On the other hand, we put the one-form

$$\lambda_{F;G} = \pi_F^*\theta + \pi_{T^*\mathcal{N};F}^*\Theta_G + \tfrac{1}{2}\pi_{E;F}^*(\vec{R}\rfloor\,\widetilde{\Omega})$$

on E, which depends on G in general because the one-form Θ_G does. Furthermore, we recall that the projection map $\pi_{T^*\mathcal{N};F}$ also depends on the canonical splitting

$$T_QF = HT_QF \oplus VT_QF \cong TQ \oplus F|_Q. \tag{4.10}$$

Now we fix this splitting T_QF and let G, G' be two splittings of TQ

$$TQ = T\mathcal{F} \oplus G = T\mathcal{F} \oplus G'.$$

Since both G, G' are transversal to $T\mathcal{F}$ in TQ, we can represent G' as the graph of the bundle map $A_G : G \to T\mathcal{N}$. Then we consider the bundle isomorphism

$$\psi_{GG'} : TQ/\mathbb{R}\{X_\lambda\} \to TQ/\mathbb{R}\{X_\lambda\}$$

defined by

$$\psi_{GG'} = \begin{pmatrix} \mathrm{Id}_{T\mathcal{N}} & A_G \\ 0 & \mathrm{id}_G \end{pmatrix}$$

under the splitting $TQ = \mathbb{R}\{X_\lambda\} \oplus T\mathcal{N} \oplus G$. Then $\psi_{GG'}(G) = \mathrm{Graph}\, A_G$ and $\psi_{GG'}|_{T\mathcal{N}} = \mathrm{id}_{T\mathcal{N}}$. Therefore, we have $p_{T\mathcal{N};G} = p_{T\mathcal{N};G'} \circ \psi_{GG'}$.

We compute

$$\begin{aligned} \Theta_G(\alpha)(\eta) &= \alpha(p_{T\mathcal{N};G}(d\pi_{T^*\mathcal{N}}(\eta))) \\ &= \alpha(p_{T\mathcal{N};G'} \circ \psi_{GG'}(\eta)) = \Theta_{G'}(\alpha)(\psi_{GG'}(\eta)). \end{aligned}$$

This proves $\Theta_G = \Theta_{G'} \circ \psi_{GG'}$. ∎

Now we study the contact geometry of $(D^\delta(F), \lambda_F)$. We first note that the two-form $d\lambda_F$ is a presymplectic form with one-dimensional kernel such that

$$d\lambda_F|_{VTF} = \widetilde{\Omega}^v|_{VTF}.$$

Denote by $\widetilde{X} := (d\pi_{F;H})^{-1}(X)$ the horizontal lifting of the vector field X on Q, where

$$d\pi_{F;H} := d\pi_F|_H : HTF \to TQ$$

is the bijection of the horizontal distribution and TQ.

LEMMA 4.9. (Reeb vector field) *The Reeb vector field X_F of λ_F is given by*

$$X_F = \widetilde{X}_\theta,$$

where $\widetilde{X}_\theta$ denotes the horizontal lifting of X_θ to F.

Proof. We only have to check the defining property $\widetilde{X}_\theta \rfloor \lambda_F = 1$ and $\widetilde{X}_\theta \rfloor d\lambda_F = 0$. We first look at

$$\begin{aligned}\lambda_F(\widetilde{X}_\theta) &= \pi_F^*\theta(\widetilde{X}_\theta) + \pi_{T^*\mathcal{N};F}^*\Theta_G(\widetilde{X}_\theta) + \tfrac{1}{2}\pi_{E;F}^*(\vec{R} \rfloor \widetilde{\Omega})(\widetilde{X}_\theta)\\ &= \theta(X_\theta) + 0 + 0 = 1.\end{aligned}$$

Here $\widetilde{\Omega}(\vec{R}, \widetilde{X}_\theta) = 0$ by definition of $\widetilde{\Omega}$ since $\widetilde{X}_\theta^v = 0$. Then we calculate

$$\begin{aligned}\widetilde{X}_\theta \rfloor d\lambda_F &= \widetilde{X}_\theta \rfloor (\pi_F^*\, d\theta + \widetilde{\Omega} + \pi_{T^*\mathcal{N};F}^*\, d\Theta_G)\\ &= \widetilde{X}_\theta \rfloor \pi_F^*\, d\theta + \widetilde{X}_\theta \rfloor \widetilde{\Omega} + \widetilde{X}_\theta \rfloor \pi_{T^*\mathcal{N};F}^*\, d\Theta_G = 0.\end{aligned}$$

We only need to explain why the last term $\widetilde{X}_\theta \rfloor \pi_{T^*\mathcal{N};F}^*\, d\Theta_G$ vanishes. In fact $pr_{\mathcal{N};G}\, d\pi_F(\widetilde{X}_\theta) = pr_{\mathcal{N};G}(X_\theta) = 0$. Using this, the definition of Θ_G and the S^1-equivariance of the vector bundle $F \to Q$ and the fact that $\widetilde{X}_\theta$ is the vector field generating the S^1-action, we derive

$$\widetilde{X}_\theta \rfloor \pi_{T^*\mathcal{N};F}^*\, d\Theta_G = 0$$

by a straightforward computation. This finishes the proof. ∎

Now we calculate the contact structure ξ_F.

LEMMA 4.10. (Contact distribution) *At each point $(\alpha, e) \in U_F \subset F$, we define two subspaces of $T_{(\alpha,e)}F$*

$$V := \{\xi_V \in T_{(\alpha,e)}F \mid \xi_V = -\pi_{T^*\mathcal{N};F}^*\Theta_G(\eta)X_F + \widetilde{\eta}, \eta \in \ker\theta\}$$

and

$$W := \{\xi_W \in T_{(\alpha,e)}F \mid \xi_W := -\tfrac{1}{2}\pi_{E;F}^*\widetilde{\Omega}(e, v)X_F + v, v \in VTF\},$$

Then $\xi_F = V \oplus W$.

Proof. By straightforward calculation, both V and W are subspaces of $\xi_F = \ker\lambda_F$.

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For any $\xi \in \xi_F$, we decompose $\xi = \xi^{\mathrm{h}} + \xi^{\mathrm{v}}$ using the decomposition $TF = HTF \oplus VTF$. Since $\pi^*_{T^*\mathcal{N};F}\Theta_G(\widetilde{\eta}) = \Theta_G(\eta)$ and we can write $\xi^{\mathrm{v}} = I_{(e;\pi(e))}(v)$ for a unique $v \in E_{\pi(e)}$. Therefore, we need to find $b \in \mathbb{R}$, $\eta \in \ker\theta$ so that for the horizontal vector $\xi^{\mathrm{h}} = (\widetilde{\eta + bX_\theta})$

$$\lambda_F(\xi) = 0$$
$$-\pi^*_{T^*\mathcal{N};F}\Theta_G(\eta)X_F + \widetilde{\eta} \in V$$
$$-\tfrac{1}{2}\pi^*_{E;F}\widetilde{\Omega}(e,v)X_F + v \in W.$$

Then

$$\begin{aligned}\xi^h &= (\widetilde{\eta + bX_\theta})\\ &= \widetilde{\eta} + bX_F\end{aligned}$$

which determines $\eta \in T_{\pi(e)}\mathcal{N} \oplus G_{\pi(e)}$ uniquely. We need to determine b.

Since

$$\begin{aligned}0 = \lambda_F(\xi) &= \lambda_F(\widetilde{\eta} + bX_F + \xi^{\mathrm{v}})\\ &= b + \lambda_F(\widetilde{\eta}) + \lambda_F(\xi^{\mathrm{v}})\\ &= b + \pi^*_{T^*\mathcal{N};F}\Theta_G(\widetilde{\eta}) + \tfrac{1}{2}\pi^*_{E;F}\widetilde{\Omega}(e,v).\end{aligned}$$

Then we set $\xi_W = -(1/2)\pi^*_{E;F}\widetilde{\Omega}(\vec{R}, v)X_F + v$ for v such that $I_{e;\pi(e)}(v) = \xi^v$ and then finally choose $b = -\pi^*_{T^*\mathcal{N};F}\Theta_G(\widetilde{\eta})$ so that $\xi_V := -\pi^*_{T^*\mathcal{N};F}\Theta_G(\widetilde{\eta})X_F + \widetilde{\eta}$. Therefore, we have proved $\xi_F = V + W$.

To see it is a direct sum, assume

$$-(\pi^*_{T^*\mathcal{N};F}\Theta_G)(\widetilde{\eta})X_F + \widetilde{\eta} - \tfrac{1}{2}\pi^*_{E;F}\widetilde{\Omega}(e,v)X_F + v = 0,$$

for some $\eta \in \xi_\lambda$ and $v \in VTM$. Apply $d\pi$ to both sides, and it follows that

$$-((\pi^*_{T^*|CN;F}\Theta_G)(\widetilde{\eta}) + \tfrac{1}{2}\pi^*_{E;F}\widetilde{\Omega}(e,v)X_\theta) + \eta = 0.$$

Hence $\eta = 0$, and then $v = 0$ follows since X_F is in horizontal part. This finishes the proof. ∎

§5. Canonical neighborhoods of the locus of closed Reeb orbits

Now let Q be the submanifold of (M, λ) that is foliated by the closed Reeb orbits of λ with constant period T. Consider the Morse–Bott contact setup (Q, θ, H) defined as before and the symplectic vector bundle (E, Ω) associated to Q.

Now in this section, we prove the following canonical neighborhood theorem as the converse of Theorem 4.2.

THEOREM 5.1. (Canonical neighborhood theorem) *Let Q be the submanifold of closed Reeb orbits of Morse–Bott-type contact form λ, and (Q,θ) and (E,Ω) be the associated pair and $F=T^*\mathcal{N}\oplus E$ defined above. Then there exist neighborhoods U of Q and U_F of the zero section o_F, and a diffeomorphism $\psi: U_F\to U$ and a function $f: U_F\to\mathbb{R}$ such that*

$$\psi^*\lambda = f\,\lambda_{F;G}\ f|_{o_F}\equiv 1,\ df|_{o_F}\equiv 0$$

and

$$i_{o_F}^*\psi^*\lambda=\theta, \qquad (\psi^*\,d\lambda|_{VTF})|_{o_F}=0\oplus\Omega$$

where we use the canonical identification of $VTF|_{o_F}\cong T^*\mathcal{N}\oplus E$ on the zero section $o_F\cong Q$.

We first identify the local pair $(\mathcal{U},Q)\cong(U_F,Q)$ by a diffeomorphism $\phi:\mathcal{U}\to U_F$ such that

$$\phi|_Q=\mathrm{id}_Q, \qquad d\phi(N_QM)=T^*\mathcal{N}\oplus E.$$

Such a diffeomorphism obviously exists by definition of E and $T^*\mathcal{N}$ via the normal exponential map with respect to any metric g (defined on $\mathcal{U}$) that satisfies the following property: we note that we have the associated short exact sequences

$$0\to T\mathcal{F}\to TQ\to TQ/T\mathcal{F}\to 0 \tag{5.1}$$

$$0\to TQ\to T_QM\to N_QM\to 0 \tag{5.2}$$

$$0\to E\to N_QM\to N_QM/E\to 0 \tag{5.3}$$

which are S^1-equivariant with respect to the above mentioned natural induced S^1-action on Q. We take S^1-equivariant splittings of (5.2), (5.3) in addition to that of (5.1) used in Theorem 1.6. We then choose an S^1-equivariant metric on the vector bundle $N_QM\cong F$ whose associated normal exponential map of $Q\cong o_F$ respects the above chosen splittings.

From now on, we sometimes denote U_F by F in the following context if there is no danger of confusion. Now F carries two contact forms $\psi^*\lambda$, λ_F with $\psi=\phi^{-1}$ and they are the same on the zero section o_F. With this preparation, we derive Theorem 5.1 by the following general submanifold version of Gray's theorem.

THEOREM 5.2. *Let M be an odd-dimensional manifold with two contact forms λ_0 and λ_1 on it. Let Q be a closed manifold of closed Reeb orbits of λ_0 in M and*

$$\lambda_0|_{T_QM} = \lambda_1|_{T_QM}, \qquad d\lambda_0|_{T_QM} = d\lambda_1|_{T_QM} \tag{5.4}$$

where we denote $T_QM = TM|_Q$. Then there exists a diffeomorphism ϕ from a neighborhood $\mathcal{U}$ to $\mathcal{V}$ such that

$$\phi|_Q = \mathrm{id}_Q, \tag{5.5}$$

and a function $f > 0$ such that

$$\phi^*\lambda_1 = f \cdot \lambda_0,$$

and

$$f|_Q \equiv 1, \qquad df|_{T_QM} \equiv 0. \tag{5.6}$$

Proof. By the assumption on λ_0, λ_1, there exists a small tubular neighborhood of Q in M, denote by $\mathcal{U}$, such that the isotopy $\lambda_t = (1-t)\lambda_0 + t\lambda_1$, $t \in [0, 1]$, are contact forms in $\mathcal{U}$: this follows from the requirement (5.4). Moreover, we have

$$\lambda_t|_{T_QM} \equiv \lambda_0|_{T_QM} (=\lambda_1|_{T_QM}) \quad \text{for any } t \in [0, 1].$$

Then the standard Moser's trick will finish up the proof. For reader's convenience, we provide the details here.

We are looking for a family of diffeomorphisms onto its image $\phi_t : \mathcal{U}' \to \mathcal{U}$ for some smaller open subset $\mathcal{U}' \subset \overline{\mathcal{U}'} \subset \mathcal{U}$ such that

$$\phi_t|_Q = \mathrm{id}_Q, \qquad d\phi_t|_{T_QM} = \mathrm{id}_{T_QM}$$

for all $t \in [0, 1]$, together with a family of functions $f_t > 0$ defined on $\overline{\mathcal{U}'}$ such that

$$\phi_t^*\lambda_t = f_t \cdot \lambda_0 \quad \text{on } \mathcal{U}'$$

for $0 \leqslant t \leqslant 1$. We further require $f_t \equiv 1$ on Q and $df_t|_Q \equiv 0$.

Since Q is a closed manifold, it is enough to look for the vector fields Y_t generating ϕ_t via

$$\frac{d}{dt}\phi_t = Y_t \circ \phi_t, \qquad \phi_0 = \mathrm{id}, \tag{5.7}$$

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that satisfies

$$\begin{cases} \phi_t^* \left(\dfrac{d}{dt}\lambda_t + \mathcal{L}_{Y_t}\lambda_t \right) = \dfrac{f_t'}{f_t}\phi_t^*\lambda_t \\ Y_t|_Q \equiv 0, \qquad \nabla Y_t|_{T_Q M} \equiv 0. \end{cases}$$

By Cartan's magic formula, the first equation gives rise to

$$d(Y_t \rfloor \lambda_t) + Y_t \rfloor d\lambda_t = \mathcal{L}_{Y_t}\lambda_t = \left(\frac{f_t'}{f_t} \circ \phi_t^{-1} \right) \lambda_t - \alpha, \tag{5.8}$$

where

$$\alpha = \lambda_1 - \lambda_0 \left(\equiv \frac{d\lambda_t}{dt} \right).$$

Now, we need to show that there exists Y_t such that $(d/dt)\lambda_t + \mathcal{L}_{Y_t}\lambda_t$ is proportional to λ_t. Actually, we can make our choice of Y_t unique if we restrict ourselves to those tangent to $\xi_t = \ker \lambda_t$ by Lemma 2.1.

We require $Y_t \in \xi_t$ and then (5.8) becomes

$$\alpha = -Y_t \rfloor d\lambda_t + \left(\frac{f_t'}{f_t} \circ \phi_t^{-1} \right) \lambda_t. \tag{5.9}$$

This in turn determines ϕ_t by integration. Since $\alpha|_Q = (\lambda_1 - \lambda_0)|_Q = 0$ and $\lambda_t|_{T_Q M} = \lambda_0|_{T_Q M}$, (and hence $f_t \equiv 1$ on Q), it follows that $Y_t = 0$ on Q. Therefore, by compactness of $[0, 1] \times Q$, the domain of existence of the ODE $\dot{x} = Y_t(x)$ includes an open neighborhood of $[0, 1] \times Q \subset \mathbb{R} \times M$ which we may assume is of the form $(-\epsilon, 1 + \epsilon) \times \mathcal{V}$.

Now going back to (5.9), we find that the coefficient $f_t'/f_t \circ \phi_t^{-1}$ is uniquely determined. We evaluate $\alpha = \lambda_1 - \lambda_0$ against the vector fields $X_t := (\phi_t)_* X_{\lambda_0}$, and get

$$\frac{d}{dt} \log f_t = \frac{f_t'}{f_t} = (\lambda_1(X_t) - \lambda_0(X_t)) \circ \phi_t, \tag{5.10}$$

which determines f_t by integration with the initial condition $f_0 \equiv 1$.

It remains to check the additional properties (5.5), (5.6). We set

$$h_t = \frac{f_t'}{f_t} \circ \phi_t^{-1}.$$

LEMMA 5.3.

$$dh_t|_{T_Q M} \equiv 0$$

Proof. By (5.10), we obtain

$$dh_t = d(\lambda_1(X_t)) - d(\lambda_0(X_t)) = \mathcal{L}_{X_t}(\lambda_1 - \lambda_0) - X_t \rfloor\, d(\lambda_1 - \lambda_0).$$

Since $X_t = X_{\lambda_0} = X_{\lambda_1}$ on Q, $X_t \in \xi_1 \cap \xi_0$ and so the second term vanishes.

For the first term, consider $p \in Q$ and $v \in T_pM$. Let Y be a locally defined vector field with $Y(p) = v$. Then we compute

$$\mathcal{L}_{X_t}(\lambda_1 - \lambda_0)(Y)(p) = \mathcal{L}_{X_t}((\lambda_1 - \lambda_0)(Y))(p) - (\lambda_1 - \lambda_0)(\mathcal{L}_{X_t} Y)(p).$$

The second term of the right-hand side vanishes since $\lambda_1 = \lambda_0$ on T_pM for $p \in Q$. For the first one, we note X_t is tangent to Q for all t and $(\lambda_1 - \lambda_0)(Y) \equiv 0$ on Q by the hypothesis $\lambda_0 = \lambda_1$ on T_QM. Therefore, the first term also vanishes. This finishes the proof. □

Now we set $g_t = \log f_t$. Since ϕ_t is a diffeomorphism and $\phi_t(Q) \subset Q$, this implies $dg_t' = 0$ on Q for all t. By integrating $dg_t' = 0$ with $dg_0' = 0$ along Q over time $t = 0$ to $t = 1$, which implies $dg_t = 0$ along Q (meaning $dg_t|_{T_QM} = 0$), that is, $df_t = 0$ on Q. This completes the proof of Theorem 5.2. □

Applying this theorem to λ and λ_F on F with Q as the zero section o_F, we can wrap up the proof of Theorem 5.1

Proof of Theorem 5.1. The requirement (1.4) and the first of (1.5) are immediate translations of Theorem 5.2. For the second requirement in (1.5), we compute

$$\psi^* d\lambda = df \wedge \lambda_F + f\, d\lambda_F.$$

By using $df|_{o_F} = 0$ and $f|_{o_F} = 1$, we derive

$$\psi^*\, d\lambda|_{VTF} = d\lambda_F|_{VTF} = \Omega$$

on o_F. This then finishes the second an hence finishes the proof. □

DEFINITION 5.4. (Normal form of contact form) We call $(U_F, f\,\lambda_F)$ the normal form of the contact form λ associated to the Morse–Bott submanifold Q of closed Reeb orbits.

Note that the contact structures associated to $\psi^*\lambda$ and λ_F are the same which is given by

$$\xi_F = \ker \lambda_F = \ker \psi^*\lambda.$$

This proves the following normal form theorem of the contact structure (M, ξ) in a neighborhood of Q.

PROPOSITION 5.5. *Suppose that $Q \subset M$ be a submanifold of closed Reeb orbits of λ. Then there exists a contactomorphism from a neighborhood $\mathcal{U} \supset Q$ to a neighborhood of the zero section of F equipped with S^1-equivariant contact structure λ_F.*

DEFINITION 5.6. (Normal form of contact structure) We call (F, ξ_F) the normal form of (M, ξ) associated to the Morse–Bott submanifold Q of closed Reeb orbits.

However, the Reeb vector fields of $\psi^*\lambda$ and λ_F coincide only along the zero section in general.

In the rest of the paper, we work with F and for the general contact form λ that satisfies

$$\lambda|_{o_F} \equiv \lambda_F|_{o_F}, \qquad d\lambda|_{VTF}|_{o_F} = \Omega. \tag{5.11}$$

In particular o_F is also the locus of closed Reeb orbits with the same period T of a Morse–Bott contact form λ. We restate the above normal form theorem in this context.

PROPOSITION 5.7. *Let λ be any contact form in a neighborhood of o_F on F satisfying (5.11). Then there exist an open embedding $\psi : \mathcal{U} \to F$ and a function f on $\mathcal{U}$ for open neighborhoods $\mathcal{U}$, F of o_F such that $\psi|_{o_F} = \mathrm{id}_{o_F}$, $\psi^*\lambda = f\,\lambda_F$ with $f|_{o_F} \equiv 1$ and $df|_{o_F} \equiv 0$.*

We denote ξ and X_λ the corresponding contact structure and Reeb vector field of λ, and π_λ, π_{λ_F} the corresponding projection from TF to ξ and ξ_F.

§6. Linearization of closed Reeb orbit on the normal form

In this section, we systematically examine the decomposition of the linearization map of closed Reeb orbits in terms of the coordinate expression of the loops z in F in this normal neighborhood.

For a given map $z : S^1 \to F$, we denote by $x := \pi_F \circ z$. Then we can express

$$z(t) = (x(t), s(t)), \quad t \in S^1$$

where $s(t) \in F_{x(t)}$, that is, s is the section of x^*F.

We regard this decomposition as the map

$$\mathcal{I} : C^\infty(S^1, F) \to \mathcal{H}^F_{S^1}$$

where $\mathcal{H}^F$ is the infinite-dimensional vector bundle

$$\mathcal{H}^F_{S^1} = \bigcup_{x \in C^\infty(S^1,F)} \mathcal{H}^F_{S^1,x}$$

where $\mathcal{H}^F_{S^1,x}$ is the vector space given by

$$\mathcal{H}^F_{S^1,x} = \Omega^0(x^*F)$$

the set of smooth sections of the pullback vector bundle x^*F. This provides a coordinate description of $C^\infty(S^1, F)$ in terms of $\mathcal{H}^F_{S^1}$. We denote the corresponding coordinates $z = (u_z, s_z)$ when we feel necessary to make the dependence of (x, s) on z explicit.

We fix an S^1-invariant connection on F and the associated splitting

$$TF = HTF \oplus VTF \tag{6.1}$$

which is defined to be the direct sum of the connection of $T^*\mathcal{N}$ and the S^1-invariant connection on the symplectic vector bundle (E, Ω). Then we express

$$\dot z = \begin{pmatrix} \widetilde{\dot x} \\ \nabla_t s \end{pmatrix}.$$

Here we regard $\dot x$ as a TQ-valued one-form on S^1 and $\nabla_t s$ is defined to be

$$\nabla_{\dot x} s = (x^*\nabla)_{(\partial/\partial t)} s$$

which we regard as an element of $F_{x(t)}$. Through identification of $H_s TF$ with $T_{\pi_F(s)}Q$ and $V_s TF$ with $F_{\pi(s)}$ or more precisely through the identity

$$\widetilde{\dot x} \circ I_{s;x} = \dot x,$$

we just write

$$\dot z = \begin{pmatrix} \dot x \\ \nabla_t s \end{pmatrix}.$$

Recall that o_F is foliated by the closed Reeb orbits of λ which also form the fibers of the prequantization bundle $Q \to P$.

For a given Reeb orbit $z = (x, s)$, we denote $x(t) = \gamma(T \cdot t)$ where γ is a Reeb orbit of period T of the contact form θ on Q which is nothing but a fiber of the prequantization $Q \to P$. We then decompose

$$D\Upsilon(z)(Z) = (D\Upsilon(z)(Z))^v + (D\Upsilon(z)(Z))^h.$$

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Then the assignment $Z \mapsto (D\Upsilon(z)(Z))^v$ defines an operator from $\Gamma(z^*VTF)$ to $\Gamma(z^*VTF)$. We remark that since $VTF \subset \xi$, we have

$$D\Upsilon(z)(Z) = D^\pi \Upsilon(z)(Z) \tag{6.2}$$

for any vertical vector field Z.

Composing with the map $I_{z;x}$, we have obtained an operator from $\Omega^0(x^*F)$ to $\Omega^0(x^*F)$. We denote this operator by

$$D\upsilon(x) : \Omega^0(x^*F) \to \Omega^0(x^*F). \tag{6.3}$$

Using $X_F = \widetilde{X}_{\lambda,Q}$ and $\nabla_Y X_F = 0$ for any vertical vector field Y, we derive the following proposition from Lemma 2.8. This will be important later for our exponential estimates.

Proposition 6.1. *Let $D\upsilon = D\upsilon(x)$ be the operator defined above. Define the vertical Hamiltonian vector field X_g^Ω by*

$$X_g^\Omega \rfloor \Omega = dg|_{VTF}.$$

Then

$$D\upsilon = \nabla_t^F - T\, D^v X_g^\Omega(z) \tag{6.4}$$

where $z = (x, o_x)$.

Proof. Consider a vertical vector field $Z \in VTF$ along a Reeb orbit z as above and regard it as the section of z^*F defined by

$$s_Z(t) = I_{z;x}(Z(t))$$

where γ is a Reeb orbit with period T of $X_{\lambda,Q}$ on $o_F \cong Q$. Recall the formula

$$\begin{aligned} D\Upsilon(z)(Z) &= D^\pi \Upsilon(z)(Z) \\ &= \nabla_t^\pi Z - T\left(\frac{1}{f}\nabla_Z X_{\lambda_F} + Z[1/f] X_{\lambda_F}\right) \\ &\quad - T\left(\frac{1}{f}\nabla_Z Y_{dg} + Z[1/f] Y_{dg}\right) \end{aligned} \tag{6.5}$$

from (6.2), and Lemma 2.8 which we apply to the vertical vector field Z for the contact manifold (U_F, λ_F).

We recall $f \equiv 1$ on o_F and $df \equiv 0$ on $TF|_{o_F}$. Therefore, we have $Z[1/f] = 0$. Furthermore, recall $X_F = \widetilde{X}_{\lambda,Q}$ and

$$\nabla_Z X_F = D^v X_F(s_Z) = D^v \widetilde{X}_\theta(s_Z) = 0$$

on o_F. On the other hand, by definition, we derive

$$I_{z;x}\left(\frac{1}{f}\nabla_Z Y_{dg}\right) = D^v X_g^\Omega(s_Z).$$

By substituting this into (6.5) and composing with $I_{z;x}$, we have finished the proof. ∎

By construction, it follows that the vector field along z defined by

$$t \mapsto \phi^t_{X_\theta}(v), \quad t \in [0, 1]$$

for any $v \in T_{z(0)}Q$ lie in $\ker D\Upsilon(z)$. By the Morse–Bott hypothesis, this set of vector fields exhausts $\ker D\Upsilon(z)$. We denote by $\delta > 0$ the gap between 0 and the first nonzero eigenvalue of $D\Upsilon(z)$. Then we obtain the following

COROLLARY 6.2. *Let $z = (x_z, o_{x_z})$ be a Reeb orbit. Then for any section $s \in \Omega^0(x^*F)$, we have*

$$\|\nabla_t^F s - TD^v X_g^\Omega(z)(s)\|^2 \geqslant \delta^2 \|s\|_2^2. \tag{6.6}$$

This inequality plays a crucial role in the study of exponential convergence of contact instantons in the Morse–Bott context studied later in the present paper.

§7. Normal coordinates of dw in $(U_F, f\lambda_F)$

We fix the splitting $TF = HTF \oplus VTF$ given in (6.1) and consider the decomposition of $w = (u, s)$ according to the splitting. For a given map $w : \dot\Sigma \to F$, we denote by $u := \pi_F \circ w$. Then we can express

$$w(z) = (u(z), s(z)), \quad z \in \Sigma$$

where $s(z) \in F_{u(z)}$, that is, s is the section of u^*F.

We regard this decomposition as the map

$$\mathcal{I} : \mathcal{F}(\Sigma, F) \to \mathcal{H}^F_\Sigma$$

where $\mathcal{H}^F$ is the infinite-dimensional vector bundle

$$\mathcal{H}^F_\Sigma = \bigcup_{u \in \mathcal{F}(\Sigma, F)} \mathcal{H}^F_{\Sigma, u}$$

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where $\mathcal{H}^F_{\Sigma,u}$ is the vector space given by

$$\mathcal{H}^F_{\Sigma,u} = \Omega^0(u^*F)$$

the set of smooth sections of the pullback vector bundle u^*F. This provides a coordinate description of $\mathcal{F}(\Sigma, F)$ in terms of $\mathcal{H}^F_{\Sigma}$. We denote the corresponding coordinates $w = (u_w, s_w)$ when we feel necessary to make the dependence of (u, s) on w explicit.

In terms of the splitting (6.1), we express

$$dw = \begin{pmatrix} \widetilde{du} \\ \nabla_{du} s \end{pmatrix}.$$

Here we regard du as a TQ-valued one-form on $\dot\Sigma$ and $\nabla_{du} s$ is defined to be

$$\nabla_{du(\eta)} s(z) = (u^*\nabla)_\eta s$$

for a tangent vector $\eta \in T_z\Sigma$, which we regard as an element of $F_{u(z)}$. Through identification of HTF_s with $T_{\pi_F(s)}Q$ and VTF_s with $F_{\pi_F(s)}$ or more precisely through the identity

$$I_{w;u}(\widetilde{du}) = du,$$

we just write

$$dw = \begin{pmatrix} du \\ \nabla_{du} s \end{pmatrix}$$

from now on, unless it is necessary to emphasize the fact that dw *a priori* has values in $TF = HTF \oplus VTF$, not $TQ \oplus F$.

To write them in terms of the coordinates $w = (u, s)$, we first derive the formula for the projection $d^\pi w = d^{\pi_\lambda} w$ with $\lambda = f\,\lambda_F$. For this purpose, we recall the formula for $X_{f\lambda_F}$ from Proposition 2.6 in Section 2.2

$$X_{f\lambda_F} = \frac{1}{f}(X_\lambda + Y_{dg}), \qquad Y_{dg} := \pi_{\lambda_F}(\flat_{\lambda_F}(dg))$$

for $g = \log f$. We decompose

$$Y_{dg} = (Y_{dg})^v + (Y_{dg})^h$$

into the vertical and the horizontal components. This leads us to the decomposition

$$f\,X_{f\lambda_F} = (Y_{dg})^h + X_{\lambda_F} + (Y_{dg})^v \tag{7.1}$$

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in terms of the splitting

$$TF = (\widetilde{\xi_\lambda \cap TQ}) \oplus \mathbb{R}\{X_F\} \oplus VTF, \qquad HTF = (\widetilde{\xi_\lambda \cap TQ}) \oplus \mathbb{R}\{X_F\}.$$

Recalling $d\lambda_F = \pi_F^* d\theta + \pi_{T^*\mathcal{N};F}^* d\Theta_G + \pi_{E;F}^* \widetilde{\Omega}$, and since $d\Theta_G$ vanishes on VTF, we have derived

LEMMA 7.1. *At each $s \in F$,*

$$(Y_{dg})^v(s) = X_{g|_{F_{\pi(s)}}}^{\Omega^v(s)}. \tag{7.2}$$

Now we are ready to derive an important formula that will play a crucial role in our exponential estimates in later sections. Recalling the canonical isomorphism

$$I_{s;\pi_F(s)}; VTF_s \to F_{\pi(s)}$$

we introduced in Section 2, we define the following *vertical derivative*.

DEFINITION 7.2. Let X be a vector field on $F \to Q$. The *vertical derivative*, denoted by $D^v X : F \to F$ is the map defined by

$$D^v X(q)(f) := \left.\frac{d}{dr}\right|_{r=0} I_{rf;\pi(rf)}(X^v(rf)). \tag{7.3}$$

PROPOSITION 7.3. *Let (E, Ω, J_E) be the Hermitian vector bundle for Ω defined as before. Let $g = \log f$ and $X_g^{d\lambda_E}$ be the contact Hamiltonian vector field as above. Then we have*

$$J_E D^v Y_{dg} = \mathrm{Hess}^v g(q, o_q).$$

In particular, $J_E D^v X_g^{d\lambda_E} : E \to E$ is a symmetric endomorphism with respect to the metric $g_E = \Omega(\cdot, J_E \cdot)$.

Proof. Let $q \in Q$ and $e_1, e_2 \in E_q$. We compute

$$\begin{aligned}
\langle D^v Y_{dg}(q)e_1, e_2\rangle &= \Omega(D^v Y_{dg}(q)e_1, J_E e_2) \\
d\lambda_E(D^v Y_{dg}(q)e_1, J_E e_2) &= \Omega\left(\left.\frac{d}{dr}\right|_{r=0} I_{re_1;q}((Y_{dg})^v(re_1)), J_E e_2\right) \\
&= \Omega\left(\left.\frac{d}{dr}\right|_{r=0} I_{re_1;q}(X_g^\Omega(re_1)), J_E e_2\right).
\end{aligned}$$

Here $d/dr|_{r=0} X_g^\Omega(re_1)$ is nothing but

$$DX_g^\Omega(q)(e_1)$$

where $DX_g^{\Omega}(q)$ is the linearization of the Hamiltonian vector field of $g|_{E_q}$ of the symplectic inner product $\Omega(q)$ on E_q. Therefore, it lies at the symplectic Lie algebra $sp(\Omega)$ and so satisfies

$$\Omega(DX_g^{\Omega}(q)(e_1), e_2) + \Omega(e_1, DX_g^{\Omega}(q)(e_2)) = 0 \tag{7.4}$$

which is equivalent to saying that $J_E DX_g^{\Omega}(q)$ is symmetric with respect to the inner product $g_E = \Omega(\cdot, J_E \cdot)$. But we also have

$$J_E DX_g^{\Omega}(q) = D\,\mathrm{grad}_{g_E(q)}\, g|_{E_q} = \mathrm{Hess}^v g(q).$$

On the other hand, (7.4) also implies

$$\Omega(DX_g^{\Omega}(q)(J_E e_1), e_2) - \Omega(DX_g^{\Omega}(q)(e_2), J_E e_1) = 0$$

with e_1 replaced by Je_1 therein. The first term becomes

$$\langle DX_g^{\Omega}(q)(e_1), e_2\rangle$$

and the second term can be written as

$$\begin{aligned}\Omega(DX_g^{\Omega}(q)(J_E e_2), e_1) &= -\Omega(J_E e_2, DX_g^{\Omega}(q)(e_1))\\ &= \Omega(e_2, J_E DX_g^{\Omega}(q)(e_1))\\ &= \langle e_2, DX_g^{\Omega}(q)(e_1)\rangle.\end{aligned}$$

Combining the two, we have finished the proof. □

§8. CR-almost complex structures adapted to Q

We would like to emphasize that we have not involved any almost complex structure yet. Now we involve J in our discussion.

Let J be any CR-almost complex structure compatible to λ in that (M, λ, J) defines a contact triad and denote by g the triad metric. Then we can realize the normal bundle $N_Q M = T_Q M/TQ$ as the metric normal bundle

$$N_Q^g M = \{v \in T_Q M \mid d\lambda(v, Jw) = 0, \forall w \in TQ\}.$$

We start with the following obvious lemma

LEMMA 8.1. *Consider the foliation $\mathcal{N}$ of (Q, ω_Q), where $\omega_Q = i_Q^* d\lambda$. Then $JT\mathcal{N}$ is perpendicular to TQ with respect to the triad metric of (M, λ, J). In particular $JT\mathcal{N} \subset N_Q^g M$.*

Proof. The first statement follows from the property that $T\mathcal{N}$ is isotropic with respect to ω_Q. ∎

Now, we introduce the concept of almost complex structures adapted to the locus Q of closed Reeb orbits of M.

DEFINITION 8.2. Let $Q \subset M$ be the locus of closed Reeb orbits of Morse–Bott contact form λ. Suppose J defines a contact triad (M, λ, J). We say a CR-almost complex structure J for (M, ξ) is adapted to the submanifold Q if J satisfies

$$J(TQ) \subset TQ + JT\mathcal{N}. \tag{8.1}$$

PROPOSITION 8.3. *The set of adapted J relative to Q is nonempty and is a contractible infinite-dimensional manifold.*

Proof. For the existence of a J adapted to Q, we recall the splitting

$$\begin{aligned} T_qQ &= \mathbb{R}\{X_\lambda(q)\} \oplus T_q\mathcal{N} \oplus G_q, \\ T_qM &\cong (\mathbb{R}\{X_\lambda(q)\} \oplus T_q\mathcal{N} \oplus G_q) \oplus (T_q^*\mathcal{N} \oplus E_q) \\ &= \mathbb{R}\{X_\lambda(q)\} \oplus (T_q\mathcal{N} \oplus T_q^*\mathcal{N}) \oplus G_q \oplus E_q \end{aligned}$$

on each connected component of Q. Therefore, we can find J so that it is compatible on $T\mathcal{N} \oplus T^*\mathcal{N}$ with respect to $-d\Theta_G|_{T\mathcal{N}\oplus T^*\mathcal{N}}$, and compatible on G with respect to ω_Q and on E with respect to Ω. It follows that any such J is adapted to Q. This proves the first statement.

The proof of the second statement will be postponed until appendix. ∎

We note that each summand $T_q\mathcal{N} \oplus T_q^*\mathcal{N}$, G_q and E_q in the above splitting of T_QM is symplectic with respect to $d\lambda$.

We recall the embeddings $T^*\mathcal{N}$ and E into N_QM and the identification $N_QM \cong T^*\mathcal{N} \oplus E$ discussed in Section 3.2.

LEMMA 8.4. *For any adapted J, the identification of the normal bundle*

$$N_QM \to N_Q^gM; \qquad [v] \mapsto \widetilde{d\lambda}(-Jv) \tag{8.2}$$

naturally induces the following identifications:

(1) $T^*\mathcal{N} \cong JT\mathcal{N}$;
(2) $\mathrm{Image}(E \hookrightarrow N_QM) = (TQ)^{d\lambda} \cap (JT\mathcal{N})^{d\lambda}$.

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Proof. (1) follows by looking at the metric $\langle\cdot,\cdot\rangle = d\lambda(\cdot, J\cdot)$. Now restrict it to $(TQ)^{d\lambda}$, we can identify E with the complement of $T\mathcal{F}$ with respect to this metric, which is just $(TQ)^{d\lambda} \cap (JT\mathcal{N})^{d\lambda}$. ∎

LEMMA 8.5. *For any adapted J, $JE \subset E$ in the sense of the identification of E with the subbundle of $T_Q M$ given in the above lemma.*

Proof. Take $v \in (TQ)^{d\lambda} \cap (JT\mathcal{N})^{d\lambda}$, then for any $w \in TQ$,

$$d\lambda(Jv, w) = -d\lambda(v, Jw) = 0$$

since $JTQ \subset TQ + JT\mathcal{N}$. Hence $Jv \in (TQ)^{d\lambda}$.

For any $w \in T\mathcal{N}$,

$$d\lambda(Jv, Jw) = d\lambda(v, w) = 0$$

since $v \in (TQ)^{d\lambda}$ and $w \in TQ$. Hence $Jv \in (JT\mathcal{N})^{d\lambda}$, and we are done. ∎

REMARK 8.6.

(1) We would like to mention that in the nondegenerate case the adaptedness is automatically satisfied by any compatible CR-almost complex structure $J \in \mathcal{J}(M, \lambda)$, because in that case P is a point and $HTF = \mathbb{R} \cdot \{X_F\}$ and $VTF = TF = \xi_F$.

(2) However, for the general Morse–Bott case, the set of adapted CR-almost complex structure is strictly smaller than $\mathcal{J}(M, \lambda)$. It appears that for the proof of exponential convergence result of closed Reeb orbits in the Morse–Bott case, this additional restriction of J to those adapted to the Morse–Bott submanifold of closed Reeb orbits in the above sense facilitates geometric computation considerably. (Compare our computations with those given in [Bou], [BEHWZ].)

(3) When $T\mathcal{N} = \{0\}$, (Q, λ_Q) carries the structure of prequantization $Q \to P = Q/S^1$.

We specialize to the normal form $(U_F, f\lambda_F)$. We note that the complex structure $J_F : \xi_F \to \xi_F$ canonically induces one on the vector bundle $VTF \to F$

$$J_F^v; VTF \to VTF$$

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satisfying $(J_F^v)^2 = -\operatorname{id}_{VTF}$. For any given J adapted to $o_F \subset F$, it has the decomposition

$$J_{U_F}|_{o_F} = \begin{pmatrix} \widetilde{J}_G & 0 & 0 & 0 \\ 0 & 0 & I & D \\ C & -I & 0 & 0 \\ 0 & 0 & 0 & J_E \end{pmatrix}$$

on the zero section with respect to the splitting

$$TF|_{o_F} \cong \mathbb{R}\{X_F\} \oplus G \oplus (T\mathcal{N} \oplus T^*\mathcal{N}) \oplus E.$$

Here we note that $C \in \operatorname{Hom}(G, T^*\mathcal{N})$ and $D \in \operatorname{Hom}(E, T\mathcal{N})$, which depend on J. Indeed it is easy to see $C = 0 = D$ from a consideration of the equation $J_U^2 = -\operatorname{Id}$.

Using the splitting

$$\begin{aligned} TF &\cong \mathbb{R}\{X_{\lambda_F}\} \oplus \widetilde{G} \oplus (\widetilde{T\mathcal{N}} \oplus VT\mathcal{N}^*) \oplus VTF \\ &\cong \mathbb{R}\{X_{\lambda_F}\} \oplus \widetilde{G} \oplus (T\mathcal{N} \oplus T^*\mathcal{N}) \oplus E \end{aligned}$$

on $o_F \cong Q$, we lift J_F to a λ_F-compatible almost complex structure on the total space F, which we denote by J_0. We note that the triad (F, λ_F, J_0) is naturally S^1-equivariant by the S^1-action induced by the Reeb flow on Q.

DEFINITION 8.7. (Normalized contact triad (F, λ_F, J_0)) We call the S^1-invariant contact triad (F, λ_F, J_0) the normalized contact triad adapted to Q.

Now we are ready to give the proof of the following.

PROPOSITION 8.8. *Consider the contact triad (U_F, λ_F, J_0) for an adapted J and its associated triad connection. Then the zero section $o_F \cong Q$ is totally geodesic and so naturally induces an affine connection on Q. Furthermore, the induced connection on Q preserves $T\mathcal{F}$ and the splitting*

$$T\mathcal{F} = \mathbb{R}\{X_{\lambda_F}\} \oplus T\mathcal{N}.$$

Proof. We note that λ_F is invariant under the reflection of the vector bundle $F \to Q$ by definition (4.8) of λ_F and so is J_0 by the construction given above. Therefore, the triad metric of (U_F, λ_F, J_0) is invariant under the reflection. This implies that the associated triad connection, which preserves the triad metric by one of the definition properties [OW1], makes the zero section totally geodesic since it is the fixed point set of the reflection which is an isometry with respect to the triad metric. Therefore, it canonically restricts to an affine connection on $o_F \cong Q$.

It remains to show that this connection preserves the splitting $T\mathcal{F} = \mathbb{R}\{X_{\lambda,Q}\} \oplus T\mathcal{N}$. For the simplicity of notation, we denote $\lambda_F = \lambda$ in the rest of this proof.

Let $q \in Q$ and $v \in T_qQ$. We pick a vector field Z that is tangent to Q and S^1-invariant and satisfies $Z(q) = v$. Such a vector field exists because $\mathcal{F}$ is the null foliation of $\omega_Q = i_Q^* d\lambda$, and Q carries the S^1-action induced by the Reeb flow of λ. If Z is a multiple of X_λ, then we can choose $Z = c\,X_\lambda$ for some constant and so $\nabla_Z X_\lambda = 0$ by the axiom $\nabla_{X_\lambda} X_\lambda = 0$ of contact triad connection. Then for $Y \in \xi \cap T\mathcal{F}$, we compute

$$\nabla_{X_\lambda} Y = \nabla_Y X_\lambda + [X_\lambda, Y] \in \xi$$

by an axiom of the triad connection. On the other hand, for any Z tangent to Q, we derive

$$d\lambda(\nabla_{X_\lambda} Y, Z) = -d\lambda(Y, \nabla_{X_\lambda} Z) = 0$$

since $Q = o_F$ is totally geodesic and so $\nabla_{X_\lambda} Z \in TQ$. This proves that $\nabla_{X_\lambda} T\mathcal{N} \subset T\mathcal{N}$.

For $v \in T_qQ \cap \xi_q$, we have $\nabla_Z X_\lambda \in \xi \cap TQ$. On the other hand,

$$\nabla_Z X_\lambda = \nabla_{X_\lambda} Z + [Z, X_\lambda] = \nabla_{X_\lambda} Z$$

since $[Z, X_\lambda] = 0$ by the S^1-invariance of Z. Now let $W \in T\mathcal{N}$ and compute

$$\langle \nabla_Z X_\lambda, W\rangle = d\lambda(\nabla_Z X_\lambda, J_0 W).$$

On the other hand $J_0 W \in T^*\mathcal{N} \subset T_{o_F} F$ since $\mathcal{F}$ is (maximally) isotropic with respect to $d\lambda$. Therefore, we obtain

$$d\lambda(\nabla_Z X_\lambda, J_0 W) = -\pi^*_{T^*\mathcal{N};F} \Theta_G(q, 0, 0)(\nabla_Z X_\lambda, J_0 W) = 0.$$

This proves $\nabla_Z X_\lambda$ is perpendicular to $T\mathcal{N}$ with respect to the triad metric and so must be parallel to X_λ. On the other hand,

$$\langle \nabla_Z W, X_\lambda\rangle = -\langle \nabla_Z X_\lambda, W\rangle = 0$$

and hence if $W \in T\mathcal{N}$, it must be perpendicular to X_λ. Furthermore, we have

$$d\lambda(\nabla_Z W, V) = -d\lambda(W, \nabla_Z V) = 0$$

for any V tangent to Q since $\nabla_Z V \in TQ$ as Q is totally geodesic. This proves $\nabla_Z W$ indeed lies in $\xi \cap T\mathcal{F} = T\mathcal{N}$, which finishes the proof. $\square$

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Part 2. Exponential estimates for contact instantons: Morse–Bott case

In this part, we develop the three-interval method of proving C^∞ exponential convergence to closed Reeb orbits of any (charge vanishing) contact instanton with finite π-energy and bounded gradient of any Morse–Bott contact form, and use it at each puncture of domain Riemann surface.

The contents of this part are as follows:

- in Section 9, we briefly review the subsequence convergence result for contact instantons with finite π-energy and bounded gradient. This is the starting point for applying the three-interval method introduced in Section 10 and afterward;
- in Section 10, an abstract three-interval method framework is presented;
- in Section 11, we focus on the prequantization case and use the three-interval machinery introduced in Section 10 to prove exponential convergence. The proof is divided into several steps which are organized into different subsections;
- in Section 12, we prove exponential decay for general cases;
- in Section 13, we explain how to apply this method to symplectic manifolds with asymptotically cylindrical ends.

§9. Subsequence convergence on the adapted contact triad (U_F, λ, J)

We first introduce the subsequence convergence result for Morse–Bott contact instantons of finite π-energy and finite gradient bound. The proofs are almost word by word the same as the nondegenerate case considered in [OW2]. For readers' convenience, we include details here.

We fix a punctured Riemann surface $(\dot\Sigma, j)$ with l-punctures and associate it with a metric h which is cylindrical at each end. To be precise, it means that there exists a compact set $K_\Sigma \subset \dot\Sigma$, such that $\dot\Sigma - \operatorname{Int}(K_\Sigma)$ is the disjoint union of l^+-positive half cylinders and l^--negative half cylinders with $l = l^+ + l^-$, that is

$$\dot\Sigma - \operatorname{Int}(K_\Sigma) = \left(\bigsqcup_{i=1,\ldots,l^+} C_i^+\right) \sqcup \left(\bigsqcup_{i=1,\ldots,l^-} C_i^-\right), \quad l = l^+ + l^-$$

where

$$C_i^+ = [0, \infty) \times S^1, \qquad C_i^- = (-\infty, 0] \times S^1$$

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equipped with the cylindrical metric $h|_{C_i^\pm} = d\tau^2 + dt^2$ thereon. For any smooth map $w : \dot\Sigma \to M$, the π-harmonic energy $E^\pi(w)$ is defined as

$$E^\pi(w) := E^\pi_{(\lambda,J;\dot\Sigma,h)}(w) = \frac{1}{2}\int_{\dot\Sigma} |d^\pi w|^2 \tag{9.1}$$

where the norm is taken by h and the triad metric on M.

We put the following hypotheses on the asymptotic study of Morse–Bott contact instantons:

HYPOTHESIS 9.1. *Consider the contact triad (M, λ, J) with Morse–Bott contact form λ. Let $w : \dot\Sigma \to M$ be a contact instanton, that is, satisfy the contact-instanton equations (1.1) defined on a punctured Riemann surface with cylindrical ends $(\dot\Sigma, j, h)$. We assume that w satisfies*

(1) $E^\pi(w) := E^\pi_{(\lambda,J;\dot\Sigma,h)}(w) < \infty$, *that is, finite π-energy;*
(2) $\|dw\|_{L^\infty(\dot\Sigma)} < \infty$, *that is, finite gradient bound.*

The following two asymptotic invariants associated to each puncture play essential roles in the study of asymptotic behavior of a Morse–Bott contact instanton satisfying Hypothesis 9.1.

DEFINITION 9.2. Let $w : \dot\Sigma \to M$ be as in Hypothesis 9.1. At each puncture, we define the asymptotic contact action $\mathcal{T}_{C_i^\pm}(w)$ and the asymptotic contact charge $\mathcal{Q}_{C_i^\pm}(w)$ for a contact instanton w satisfying Hypothesis 9.1 as

$$\mathcal{T}_{C_i^\pm}(w) := \frac{1}{2}\int_{C_i^\pm} |d^\pi w|^2 + \int_{\{0\}\times S^1} (w|_{\{0\}\times S^1})^*\lambda \tag{9.2}$$

$$\mathcal{Q}_{C_i^\pm}(w) := \int_{\{0\}\times S^1} ((w|_{\{0\}\times S^1})^*\lambda \circ j) \tag{9.3}$$

where $C^\pm$ is the cylindrical end associated to the given puncture.

The following remark shows that both $\mathcal{T}$ and $\mathcal{Q}$ are translation invariant.

REMARK 9.3. For any contact instanton w satisfying Hypothesis 9.1 at a puncture $[0,\infty) \times S^1$, we have

$$\mathcal{T}(w) = \frac{1}{2}\int_{[s,\infty)\times S^1} |d^\pi w|^2 + \int_{\{s\}\times S^1} (w|_{\{s\}\times S^1})^*\lambda, \quad \text{for any } s \geqslant 0,$$

which is due to $(1/2)|d^\pi w|^2\, dA = d(w^*\lambda)$ and Stokes' formula; and

$$\mathcal{Q}(w) = \int_{\{s\}\times S^1} (w|_{\{s\}\times S^1})^*\lambda \circ j, \quad \text{for any } s \geqslant 0,$$

which is due to $d(w^*\lambda \circ j) = 0$.

Since our main interest lies on the asymptotic behavior of a fixed contact instanton w at a given puncture, we assume the domain of w is a positive half cylinder $[0, \infty) \times S^1$ without loss of generality. (The case of negative half cylinder can be treated in the same way.) We simply denote the asymptotic contact action and charge at this puncture by $\mathcal{T}$ and by $\mathcal{Q}$, respectively.

THEOREM 9.4. (Subsequence convergence [OW2]) *Let (M, λ, J) be any, not necessarily Morse–Bott, contact triad. Assume $w : [0, \infty) \times S^1 \to M$ is a contact instanton, that is it satisfies the contact-instanton equations (1.1), and satisfies Hypothesis 9.1. Then for any sequence $s_k \to \infty$, there exists a subsequence, still denoted by s_k, and a Reeb trajectory γ, not necessarily closed, such that*

$$\lim_{k\to\infty} w(s_k + \tau, t) = \gamma(-\mathcal{Q}\tau + \mathcal{T}t)$$

in the $C^l(K \times S^1, M)$ sense for any $l \geqslant 0$, where $K \subset \mathbb{R}$ is an arbitrary compact set.

Furthermore, when (M, λ, J) is of Morse–Bott type and w has nonvanishing period $\mathcal{T} \neq 0$, then there exists a connected submanifold Q foliated by closed Reeb orbits of period $\mathcal{T}$, so that the limit becomes $\gamma(-\mathcal{Q}\tau + \mathcal{T}\, t)$, where z is a closed Reeb orbit over Q.

The first part of the theorem was proved in [OW2, Section 6]. For reader's convenience, we include its complete proof in Appendix B. Similar statement for Morse–Bott case in the context of symplectization was proved in [HWZ3, Proposition 2.1] (see also [HWZ1, HWZ2, Proposition 2.1]).

COROLLARY 9.5. *Assume $w : [0, \infty) \times S^1 \to M$ is a contact instanton, that is it satisfies the contact-instanton equations (1.1) in a Morse–Bott contact triad (M, λ, J), and satisfies Hypothesis 9.1. Then*

$$\lim_{s\to\infty} \left|\pi \frac{\partial w}{\partial \tau}(s + \tau, t)\right| = 0, \qquad \lim_{s\to\infty} \left|\pi \frac{\partial w}{\partial t}(s + \tau, t)\right| = 0$$

$$\lim_{s\to\infty} \lambda \left(\frac{\partial w}{\partial \tau}\right)(s + \tau, t) = -\mathcal{Q}, \qquad \lim_{s\to\infty} \lambda \left(\frac{\partial w}{\partial t}\right)(s + \tau, t) = \mathcal{T}$$

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and

$$\lim_{s\to\infty} |\nabla^l\, dw(s+\tau, t)| = 0 \quad \textit{for any } l \geqslant 1.$$

All the limits are uniform for (τ, t) *in* $K \times S^1$ *with compact* $K \subset \mathbb{R}$.

From now on, we consider J as a CR-almost complex structure adapted to Q, which in turn induces a CR-almost complex structure on a neighborhood U_F of the zero section of F. Denote by (U_F, λ, J) the corresponding adapted contact triad.

When restricted to each connected component of the loci of closed Reeb orbits, there exists a uniform constant $\tau_0 > 0$ such that the image of w lies in a tubular neighborhood of Q whenever $\tau > \tau_0$. In other words, it is enough to restrict ourselves to study contact-instanton maps from half cylinder $[0, \infty) \times S^1$ to the canonical neighborhood (U_F, λ, J) defined in Definition 5.4 for the purpose of the study of asymptotic behavior at the end.

With the normal form we developed in Part 1, we express w as $w = (u, s)$ where $u := \pi \circ w : [0, \infty) \times S^1 \to Q$ and $s = (\mu, e)$ is a section of the pullback bundle $u^*(JT\mathcal{N}) \oplus u^*E \to [0, \infty) \times S^1$. Recall from Section 7 and express

$$dw = \begin{pmatrix} du \\ \nabla_{du} s \end{pmatrix} = \begin{pmatrix} du \\ \nabla_{du}\mu \\ \nabla_{du} e \end{pmatrix},$$

we reinterpret the convergence of w stated in Theorem 9.4 in terms of the coordinate $w = (u, s) = (u, (\mu, e))$.

COROLLARY 9.6. *Let* $w = (u, s) = (u, (\mu, e))$ *satisfy the same assumption as in Theorem 9.4. Then for any sequence* $s_k \to \infty$, *there exists a subsequence, still denoted by* s_k, *and a Reeb orbit* γ *on* Q *(may depend on the choice of subsequences) with action* $\mathcal{T}$ *and charge* $\mathcal{Q}$, *such that*

$$\lim_{k\to\infty} u(\tau + s_k, t) = \gamma(-\mathcal{Q}\,\tau + \mathcal{T}\, t)$$

in $C^l(K \times S^1, M)$ *sense for any* l, *where* $K \subset [0, \infty)$ *is an arbitrary compact set. Furthermore, we have*

$$\begin{aligned} &\lim_{s\to\infty} |\mu(s+\tau, t)| = 0, && \lim_{s\to\infty} |e(s+\tau, t)| = 0 \\ &\lim_{s\to\infty} |d^{\pi_\lambda} u(s+\tau, t)| = 0, && \lim_{s\to\infty} u^*\theta(s+\tau, t) = -\mathcal{Q}\, d\tau + \mathcal{T}\, dt \\ &\lim_{s\to\infty} |\nabla_{du} e(s+\tau, t)| = 0, && \end{aligned}$$

and

$$\lim_{s\to\infty} |\nabla^k d^{\pi_\lambda} u(s+\tau,t)| = 0, \qquad \lim_{s\to\infty} |\nabla^k u^*\theta(s+\tau,t)| = 0$$

$$\lim_{s\to\infty} |\nabla^k_{du} e(s+\tau,t)| = 0 \tag{9.4}$$

for all $k \geqslant 1$, and all the limits are uniform for (τ,t) on $K \times S^1$ with compact $K \subset [0,\infty)$.

In particular,

$$\lim_{s\to\infty} du(s+\tau,t) = (-\mathcal{Q}\, d\tau + \mathcal{T}\, dt) \otimes X_\theta$$

uniformly for (τ,t) in C^∞ topology on $K \times S^1$ for any given compact $K \subset [0,\infty)$.

In the rest of the present paper, we add the following technical assumption of vanishing charge.

Hypothesis 9.7. (Charge vanishing)

$$\mathcal{Q} := \int_{\{0\}\times S^1} ((w|_{\{0\}\times S^1})^* \lambda \circ j) = 0. \tag{9.5}$$

Then the uniform convergence proved in this section ensures all the basic requirements (including the uniformly local tameness, precompactness, uniformly local coercive property and the locally asymptotically cylindrical property) of applying the three-interval method to prove exponential decay of w at the end which we introduce in details in following sections.

§10. Abstract framework of the three-interval method

In this section, we introduce a new method in proving exponential decay using the abstract framework of the three-interval method. In later Section 11.2, we apply the scheme to the normal bundle part. We remark that the method can deal with the case with an exponentially decaying perturbation too (see Theorem 10.11).

The three-interval method is based on the following analytic lemma.

Lemma 10.1. [MT, Lemma 9.4] *For a sequence of nonnegative numbers $\{x_k\}_{k=0,1,\ldots,N}$, if there exists some constant $0 < \gamma < 1/2$ such that*

$$x_k \leqslant \gamma(x_{k-1} + x_{k+1})$$

for every $1 \leqslant k \leqslant N-1$, *then it follows*

$$x_k \leqslant x_0 \xi^{-k} + x_N \xi^{-(N-k)}, \quad k = 0, 1, \ldots, N,$$

where $\xi := (1 + \sqrt{1-4\gamma^2})/2\gamma$.

REMARK 10.2.

(1) If we write $\gamma = \gamma(c) := 1/(e^c + e^{-c})$ where $c > 0$ is uniquely determined by γ, then the conclusion can be written into the exponential form

$$x_k \leqslant x_0 e^{-ck} + x_N e^{-c(N-k)}.$$

(2) For an infinite nonnegative sequence $\{x_k\}_{k=0,1,\ldots}$, if we have a uniform bound of in addition, then the exponential decay follows as

$$x_k \leqslant x_0 e^{-ck}.$$

The analysis of proving the exponential decay will be carried on a Banach bundle $\mathcal{E} \to [0, \infty)$ modeled by the Banach space $\mathbb{E}$, for which we mean every fiber $\mathbb{E}_\tau$ is identified with the Banach space $\mathbb{E}$ smoothly depending on τ. We omit this identification if there is no way of confusion.

First we emphasize the base $[0, \infty)$ is noncompact and carries a natural translation map for any positive number r, which is $\sigma_r : \tau \mapsto \tau + r$. We introduce the following definition which ensures us to study the sections in local trivialization after taking a subsequence.

DEFINITION 10.3. Let $\mathcal{E}$ be a Banach bundle modeled with a Banach space $\mathbb{E}$ over $[0, \infty)$. Let $[a, b] \subset [0, \infty)$ be any given bounded interval and let $s_k \to \infty$ be any given sequence. A *tame family of trivialization* over $[a, b]$ relative to the sequence s_k is defined to be a sequence of trivializations $\{\Phi_k\} : \mathcal{E}|_{[a,b]} \to [a, b] \times \mathbb{E}$

$$\Phi_k : \sigma_{s.}^* \mathcal{E}|_{[a+s_k, b+s_k]} \to [a, b] \times \mathbb{E}$$

for $k \geqslant 0$ satisfying the following: there exists a sufficiently large $k_0 > 0$ such that for any $k \geqslant k_0$ the bundle map

$$\Phi_{k_0+k} \circ \Phi_{k_0}^{-1} : [a, b] \times \mathbb{E} \to [a, b] \times \mathbb{E}$$

satisfies

$$\|\nabla_\tau^l (\Phi_{k_0+k} \circ \Phi_{k_0}^{-1})\|_{\mathcal{L}(\mathbb{E},\mathbb{E})} \leqslant C_l < \infty \tag{10.1}$$

for constants $C_l = C_l(|b - a|)$ depending only on $|b - a|$, $l = 0, 1, \ldots$.

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We call $\mathcal{E}$ *uniformly locally tame*, if it carries a tame family of trivializations over $[a, b]$ relative to the sequence s_k for any given bounded interval $[a, b] \subset [0, \infty)$ and a sequence $s_k \to \infty$.

DEFINITION 10.4. Suppose $\mathcal{E}$ is uniformly locally tame. We say a connection ∇ on $\mathcal{E}$ is *uniformly locally tame* if the push-forward $(\Phi_k)_* \nabla_\tau$ can be written as

$$(\Phi_k)_* \nabla_\tau = \frac{d}{d\tau} + \Gamma_k(\tau)$$

for any tame family $\{\Phi_k\}$ so that $\sup_{\tau \in [a,b]} \|\Gamma_k(\tau)\|_{\mathcal{L}(\mathbb{E},\mathbb{E})} < C$ for some $C > 0$ independent of k's.

DEFINITION 10.5. Consider a pair $\mathcal{E}_2 \subset \mathcal{E}_1$ of uniformly locally tame bundles, and a bundle map $B : \mathcal{E}_2 \to \mathcal{E}_1$. We say B is *uniformly locally bounded*, if for any compact set $[a, b] \subset [0, \infty)$ and any sequence $s_k \to \infty$, there exists a subsequence, still denoted by s_k, a sufficiently large $k_0 > 0$ and tame families $\Phi_{1,k}$, $\Phi_{2,k}$ such that for any $k \geqslant 0$

$$\sup_{\tau \in [a,b]} \|\Phi_{i,k_0+k} \circ B \circ \Phi_{i,k_0}^{-1}\|_{\mathcal{L}(\mathbb{E}_2,\mathbb{E}_1)} \leqslant C \tag{10.2}$$

where C is independent of k.

For a given locally tame pair $\mathcal{E}_2 \subset \mathcal{E}_1$, we denote by $\mathcal{L}(\mathcal{E}_2, \mathcal{E}_1)$ the set of bundle homomorphisms which are uniformly locally bounded.

LEMMA 10.6. *If $\mathcal{E}_1$, $\mathcal{E}_2$ are uniformly locally tame, then so is $\mathcal{L}(\mathcal{E}_2, \mathcal{E}_1)$.*

DEFINITION 10.7. Let $\mathcal{E}_2 \subset \mathcal{E}_1$ be as above and let $B \in \mathcal{L}(\mathcal{E}_2, \mathcal{E}_1)$. We say B is *precompact* on $[0, \infty)$ if for any locally tame families Φ_1, Φ_2, there exists a further subsequence such that $\Phi_{1,k_0+k} \circ B \circ \Phi_{1,k_0}^{-1}$ converges to some $B_{\Phi_1\Phi_2;\infty} \in \mathcal{L}(\Gamma([a, b] \times \mathbb{E}_2), \Gamma([a, b] \times \mathbb{E}_1))$.

Assume B is a bundle map from $\mathcal{E}_2$ to $\mathcal{E}_1$ which is uniformly locally bounded, where $\mathcal{E}_1 \supset \mathcal{E}_2$ are uniformly locally tame with tame families $\Phi_{1,k}$, $\Phi_{2,k}$. We can write

$$\Phi_{2,k_0+k} \circ (\nabla_\tau + B) \circ \Phi_{1,k_0}^{-1} = \frac{\partial}{\partial\tau} + B_{\Phi_1\Phi_2,k}$$

as a linear map from $\Gamma([a, b] \times \mathbb{E}_2)$ to $\Gamma([a, b] \times \mathbb{E}_1)$, since ∇ is uniformly locally tame.

Next we introduce the following notion of coerciveness.

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DEFINITION 10.8. Let $\mathcal{E}_1$, $\mathcal{E}_2$ be as above and $B : \mathcal{E}_2 \to \mathcal{E}_1$ be a uniformly locally bounded bundle map. We say the operator

$$\nabla_\tau + B : \Gamma(\mathcal{E}_2) \to \Gamma(\mathcal{E}_1)$$

is *uniformly locally coercive*, if the following holds:

(1) for any pair of bounded closed intervals I, I' with $I \subset \operatorname{Int} I'$,

$$\|\zeta\|_{L^2(I,\mathcal{E}_2)} \leqslant C(I, I')(\|\nabla_\tau \zeta + B\zeta\|_{L^2(I',\mathcal{E}_1)} + \|\zeta\|_{L^2(I',\mathcal{E}_1)}) \tag{10.3}$$

for a constant $C(I, I')$ depending only on I, I' but independent of ζ;

(2) if for given bounded sequence $\zeta_k \in \Gamma(\mathcal{E}_2)$ satisfying

$$\nabla_\tau \zeta_k + B\zeta_k = L_k$$

with $|L_k(\tau)|_{\mathcal{E}_1}$ bounded on a given compact subset $K \subset [0, \infty)$, there exists a subsequence, still denoted by ζ_k, that uniformly converges in $\mathcal{E}_2$.

REMARK 10.9. Let $E \to [0, \infty) \times S$ be a (finite-dimensional) vector bundle and denote by $W^{k,2}(E)$ the set of $W^{k,2}$-section of E and $L^2(E)$ the set of L^2-sections. Let $D : L^2(E) \to L^2(E)$ be a first-order elliptic operator with cylindrical end. Denote by $i_\tau : S \to [0, \infty) \times S$ the natural inclusion map. Then there is a natural pair of Banach bundles $\mathcal{E}_2 \subset \mathcal{E}_1$ over $[0, \infty)$ associated to E, whose fiber is given by $\mathcal{E}_{1,\tau} = L^2(i_\tau^* E)$, $\mathcal{E}_{2,\tau} = W^{1,2}(i_\tau^* E)$. Furthermore, assume $\mathcal{E}_i$ for $i = 1, 2$ is uniformly local tame if S is a compact manifold (without boundary). Then D is uniformly locally coercive, which follows from the elliptic bootstrapping and the Sobolev's embedding.

Finally we introduce the notion of asymptotically cylindrical operator B.

DEFINITION 10.10. We call B *locally asymptotically cylindrical* if the following holds: any subsequence limit $B_{\Phi_1\Phi_2;\infty}$ appearing in Definition 10.7 is a *constant* section, and $\|B_{\Phi_1\Phi_2,k} - \Phi_{2,k_0+k} \circ B \circ \Phi_{1,k_0}^{-1}\|_{\mathcal{L}(\mathbb{E}_i,\mathbb{E}_i)}$ converges to zero as $k \to \infty$ for both $i = 1, 2$.

Now we specialize to the case of Hilbert bundles $\mathcal{E}_2 \subset \mathcal{E}_1$ over $[0, \infty)$ and assume that $\mathcal{E}_1$ carries a connection which is compatible with the Hilbert inner product of $\mathcal{E}_1$. We denote by ∇_τ the associated covariant derivative. We assume that ∇_τ is uniformly locally tame.

Denote by $L^2([a, b]; \mathcal{E}_i)$ the space of L^2-sections ζ of $\mathcal{E}_i$ over $[a, b]$, that is, those satisfying

$$\int_a^b |\zeta(\tau)|^2_{\mathcal{E}_i}\, dt < \infty,$$

where $|\zeta(\tau)|_{\mathcal{E}_i}$ is the norm with respect to the given Hilbert bundle structure of $\mathcal{E}_i$.

THEOREM 10.11. (Three-interval method) *Assume $\mathcal{E}_2 \subset \mathcal{E}_1$ is a pair of Hilbert bundles over $[0, \infty)$ with fibers $\mathbb{E}_2$ and $\mathbb{E}_1$, and $\mathbb{E}_2 \subset \mathbb{E}_1$ is dense. Let B be a section of the associated bundle $\mathcal{L}(\mathcal{E}_2, \mathcal{E}_1)$ and $L \in \Gamma(\mathcal{E}_1)$. We assume the following:*

(1) *there exists a covariant derivative ∇_τ that preserves the Hilbert structure;*
(2) *$\mathcal{E}_i$ for $i = 1, 2$ are uniformly locally tame;*
(3) *B is precompact, uniformly locally coercive and asymptotically cylindrical;*
(4) *every subsequence limit B_∞ is a self-adjoint unbounded operator on $\mathbb{E}_1$ with its domain $\mathbb{E}_2$, and satisfies* $\ker B_\infty = \{0\}$*;*
(5) *there exists some positive number δ such that any subsequence limiting operator B_∞ of the above mentioned precompact family has all their eigenvalues λ satisfying $|\lambda| > \delta$;*
(6) *there exists some $R_0 > 0$, $C_0 > 0$ and $\delta_0 > \delta$ such that*

$$|L(\tau)|_{\mathcal{E}_{1,\tau}} \leqslant C_0 e^{-\delta_0 \tau}$$

for all $\tau \geqslant R_0$.

Then for any (smooth) section $\zeta \in \Gamma(\mathcal{E}_2)$ with

$$\sup_{\tau \in [R_0, \infty)} |\zeta(\tau, \cdot)|_{\mathcal{E}_{2,\tau}} < \infty \tag{10.4}$$

and satisfying the equation

$$\nabla_\tau \zeta + B(\tau)\zeta(\tau) = L(\tau), \tag{10.5}$$

there exist some constants R, $C > 0$ such that for any $\tau > R$,

$$|\zeta(\tau)|_{\mathcal{E}_{1,\tau}} \leqslant C e^{-\delta \tau}.$$

Proof. We divide $[0, \infty)$ into the union of unit intervals $I_k := [k, k+1]$ for $k = 0, 1, \ldots$. We first prove the exponential decay of $\|\zeta\|^2_{L^2(I_k;\mathcal{E}_1)}$ to zero as $k \to \infty$. By Lemma 10.1 and Remark 10.2, it is enough to prove that for the function $\gamma(c) = 1/(e^c + e^{-c})$ as in Remark 10.2 we have

$$\|\zeta\|^2_{L^2(I_k;\mathcal{E}_1)} \leqslant \gamma(2\delta)(\|\zeta\|^2_{L^2(I_{k-1};\mathcal{E}_1)} + \|\zeta\|^2_{L^2(I_{k+1};\mathcal{E}_1)}), \tag{10.6}$$

for every $k = 1, 2, \ldots$ for some choice of $0 < \delta < 1$. For the simplicity of notation and also because we use only the norms on $\mathcal{E}_1$ (but applied to $\zeta \in \mathcal{E}_2$) in the discussion below, we just denote

$$L^2([a, b]) := L^2([a, b]; \mathcal{E}_1), \qquad L^\infty([a, b]) := L^\infty([a, b]; \mathcal{E}_1)$$

for any given interval $[a, b]$.

If inequality (10.6) does not hold for every k, we collect all the k's that reverse the direction of the inequality. If such k's are finitely many, that is, (10.6) holds after some large k_0, then we still get the exponential estimate as the theorem claims.

Otherwise, there are infinitely many such three-intervals, which we enumerate by $I_I^{l_k} := [l_k, l_k + 1]$, $I_{II}^{l_k} := [l_k + 1, l_k + 2]$, $I_{III}^{l_k} := [l_k + 2, l_k + 3]$, $k = 1, 2, \ldots$, such that

$$\|\zeta\|^2_{L^2(I^k_{II})} > \gamma(2\delta)(\|\zeta\|^2_{L^2(I^k_I)} + \|\zeta\|^2_{L^2(I^k_{III})}). \tag{10.7}$$

Before we deal with this case, we first remark that this hypothesis in particular implies $\zeta \not\equiv 0$ on $I^{l_k} := I_I^{l_k} \cup I_{II}^{l_k} \cup I_{III}^{l_k}$, that is, $\|\zeta\|_{L^\infty(I^{l_k})} \neq 0$.

If there exists some uniform constant $C_1 > 0$ such that on each such three-intervals

$$\|\zeta\|_{L^\infty(I^{l_k})} < C_1 e^{-\delta l_k}, \tag{10.8}$$

it follows that

$$\|\zeta\|_{L^2([l_k+1, l_k+2])} \leqslant C_1 e^{-\delta l_k} = C e^{-\delta(l_k+1)}. \tag{10.9}$$

Here $C = C_1 e^{-\delta}$ is purely a constant depending only on δ which will be determined at the end. From now on, various constants C appearing below may vary but be independent of ζ.

Recall that under our assumption we have infinitely many intervals that satisfy (10.6), and the exponential inequality (10.8). If the union of such intervals $[l_k + 1, l_k + 2]$ is connected after some point, then we already get our conclusion from (10.9) since every interval becomes a middle interval (of the form $[l_k + 1, l_k + 2]$) in such three-intervals.

Otherwise the set of k's that satisfy (10.6) form a sequence of clusters,

$$I^{l_k+1}, I^{l_k+2}, \ldots, I^{l_k+N_k}$$

for the sequence $l_1, l_2, \ldots, l_k, \ldots$ such that $l_{k+1} > l_k + N_k$ and (10.6) holds on each element contained in each cluster.

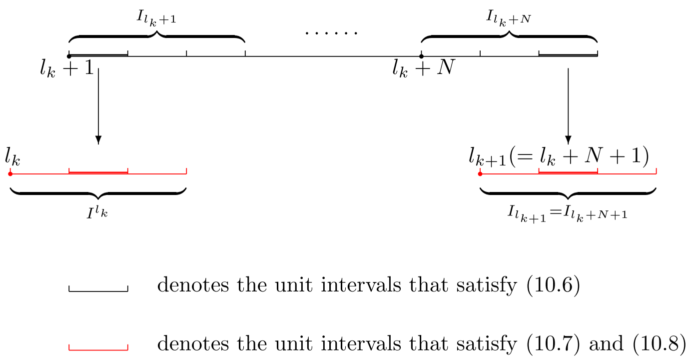

Figure 1.
The three-interval method.

We remark that each cluster has the farthest left interval $[l_k+1, l_k+2]$ as the middle interval in I^{l_k}, and the farthest right interval $[l_k+N+2, l_k+N+3]$ as the middle interval in $I^{l_{k+1}}$. (See Figure 1.)

Then from (10.9), we derive

$$\|\zeta\|_{L^2([l_k+1,l_k+2])} \leqslant Ce^{-\delta l_k},$$
$$\|\zeta\|_{L^2([l_k+N+2,l_k+N+3])} \leqslant Ce^{-\delta l_{k+1}} = Ce^{-\delta(l_k+N+1)}.$$

Combining them and Lemma 10.1, we get the following estimate for $l_k+1 \leqslant l \leqslant l_k+N+2$,

$$\begin{aligned}\|\zeta\|_{L^2([l,l+1])} &\leqslant \|\zeta\|_{L^2([l_k+1,l_k+2])}e^{-\delta(l-(l_k+1))}\\ &\quad + \|\zeta\|_{L^2([l_k+N+2,l_k+N+3])}e^{-\delta(l_k+N+2-l)}\\ &\leqslant Ce^{-\delta l_k}e^{-\delta(l-(l_k+1))} + Ce^{-\delta(l_k+N+1)}e^{-\delta(l_k+N+2-l)}\\ &= Ce^{\delta}(e^{-\delta l} + e^{-\delta(2l_k+2N+4-l)}) \leqslant (2Ce^{\delta})e^{-\delta l}.\end{aligned}$$

Thus on each such cluster, we have exponential decay with the presumed rate δ as claimed in the theorem.

Now if there is no such uniform $C = C_1$ for which (10.8) *holds*, then we can find a sequence of constants $C_k \to \infty$ and a subsequence of such three-intervals $\{I^{l_k}\}$, still denoted by l_k, such that

$$\|\zeta\|_{L^\infty(I^{l_k})} \geqslant C_k e^{-\delta l_k}. \tag{10.10}$$

We can further choose a subsequence, but still denoted by l_k, so that $l_k+3 < l_{k+1}$, that is, the intervals do not intersect one another.

We translate the sections $\zeta_k := \zeta|_{[l_k, l_k+3]}$ and consider the sections $\widetilde{\zeta}_k$ defined on $[0,3]$ given by

$$\widetilde{\zeta}_k(\tau, \cdot) := \zeta(\tau + l_k, \cdot).$$

Then (10.10) becomes

$$\|\widetilde{\zeta}_k\|_{L^\infty([0,3])} \geqslant C_k e^{-\delta l_k}. \tag{10.11}$$

If we consider the translations of L given by $\widetilde{L}_k(\tau, t) = L(\tau + l_k, t)$, then

$$|\widetilde{L}_k(\tau, t)| < C e^{-\delta l_k} e^{-\delta\tau} \leqslant C e^{-\delta l_k} \tag{10.12}$$

for $\tau \geqslant 0$. It follows that $\widetilde{\zeta}_k$ satisfies the equation

$$\nabla_\tau \widetilde{\zeta}_k + B(\tau + l_k, \cdot)\widetilde{\zeta}_k = \widetilde{L}_k(\tau, t). \tag{10.13}$$

We now rescale (10.13) by dividing it by $\|\widetilde{\zeta}_k\|_{L^\infty([0,3])}$, which cannot vanish by the standing hypothesis as we remarked right below (10.7), and consider the rescaled sequence

$$\overline{\zeta}_k := \widetilde{\zeta}_k / \|\widetilde{\zeta}_k\|_{L^\infty([0,3])}.$$

We have now

$$\begin{gathered} \|\overline{\zeta}_k\|_{L^\infty([0,3])} = 1 \\ \nabla_\tau \overline{\zeta}_k + B(\tau + l_k, t)\overline{\zeta}_k = \frac{\widetilde{L}_k}{\|\widetilde{\zeta}_k\|_{L^\infty([0,3])}} \\ \|\overline{\zeta}_k\|^2_{L^2([1,2])} \geqslant \gamma(2\delta)(\|\overline{\zeta}_k\|^2_{L^2([0,1])} + \|\overline{\zeta}_k\|^2_{L^2([2,3])}). \end{gathered} \tag{10.14}$$

From (10.11) and (10.12), we get

$$\frac{\|\widetilde{L}_k\|_{L^\infty([0,3])}}{\|\zeta_k\|_{L^\infty([0,3])}} \leqslant \frac{C}{C_k},$$

and then by our assumption that $C_k \to \infty$, we prove that the right-hand side of (10.14) converges to zero as $k \to \infty$.

Since B is assumed to be precompact, we get a limiting operator B_∞ after taking a subsequence (in a trivialization).

On the other hand, since B is locally coercive, there exists $\overline{\zeta}_\infty$ such that $\overline{\zeta}_k \to \overline{\zeta}_\infty$ uniformly converges in $\mathcal{E}_2$ and $\overline{\zeta}_\infty$ satisfies

$$\nabla_\tau \overline{\zeta}_\infty + B_\infty \overline{\zeta}_\infty = 0 \quad \text{on } [0,3], \tag{10.15}$$

and

$$\|\overline{\zeta}_\infty\|^2_{L^2([1,2])} \geqslant \gamma(2\delta)(\|\overline{\zeta}_\infty\|^2_{L^2([0,1])} + \|\overline{\zeta}_\infty\|^2_{L^2([2,3])}). \tag{10.16}$$

Since $\|\overline{\zeta}_\infty\|_{L^\infty([0,3]\times S^1)} = 1$, $\overline{\zeta}_\infty \not\equiv 0$. Recall that B_∞ is assumed to be a (unbounded) self-adjoint operator on $\mathbb{E}_1$ with its domain $\mathbb{E}_2$. Let $\{e_i\}$ be its orthonormal eigenbasis of $\mathbb{E}_1$ with respect to B_∞. We consider the eigenfunction expansion of $\overline{\zeta}_\infty(\tau, \cdot)$ and write

$$\overline{\zeta}_\infty(\tau) = \sum_{i=1}^{\infty} a_i(\tau)\, e_i$$

for each $\tau \in [0, 3]$, where e_i are the eigenfunctions of B associated to the eigenvalue λ_i with

$$-\infty < \cdots \leqslant \lambda_{-k} \leqslant \lambda_{-k+1} \leqslant \cdots < 0 < \lambda_1 \leqslant \cdots \leqslant \lambda_i \leqslant \cdots < \infty.$$

By plugging $\overline{\zeta}_\infty$ into (10.15), we derive

$$a_i'(\tau) + \lambda_i a_i(\tau) = 0, \quad i \in \mathbb{Z} \setminus \{0\}.$$

It follows that

$$a_i(\tau) = c_i e^{-\lambda_i \tau}, \quad i \in \mathbb{Z} \setminus \{0\}$$

for some constants c_i and hence

$$\|a_i\|^2_{L^2([1,2])} = \gamma(2\lambda_i)(\|a_i\|^2_{L^2([0,1])} + \|a_i\|^2_{L^2([2,3])}) \tag{10.17}$$

with the function determined by the function

$$\gamma(2c) := \frac{\int_1^2 e^{-2c\tau}\, d\tau}{\int_0^1 e^{-2c\tau}\, d\tau + \int_2^3 e^{-2c\tau}\, d\tau}.$$

Equivalently, we obtain

$$\gamma(c) = \frac{e^{-c} - e^{-2c}}{1 - e^{-c} + e^{-2c} - e^{-3\lambda_i}} = \frac{1}{e^c + e^{-c}}.$$

(This is how the function γ becomes relevant to this three-interval argument. We note that γ is an even function.) We compute

$$\begin{aligned}
\|\overline{\zeta}_\infty\|^2_{L^2([k,k+1])} &= \int_{[k,k+1]} \|\overline{\zeta}_\infty\|^2_{L^2(S^1)}\, d\tau \\
&= \int_{[k,k+1]} \sum_i |a_i(\tau)|^2\, d\tau = \sum_i \|a_i\|^2_{L^2([k,k+1])}.
\end{aligned}$$

By the monotonically decreasing property of γ for $c>0$, this and (10.17) give rise to

$$\|\overline{\zeta}_\infty\|^2_{L^2([1,2])} < \gamma(2\delta)(\|\overline{\zeta}_\infty\|^2_{L^2([0,1])} + \|\overline{\zeta}_\infty\|^2_{L^2([2,3])})$$

for any δ satisfying $0<\delta<\min\{|\gamma_{-1}|,\gamma_1\}$. Since $\overline{\zeta}_\infty \not\equiv 0$, this contradicts to (10.16), if we choose $0<\delta<\min\{|\gamma_{-1}|,\gamma_1\}$ at the beginning. This finishes the proof of the exponential decay

$$\|\zeta\|_{L^2(I_k;\mathcal{E}_1)} \leqslant Ce^{-\delta\tau} \tag{10.18}$$

as $k\to\infty$.

Now we show this indicates the exponential decay of $\|\zeta\|_{\mathcal{E}_{1,\tau}}$. Using (10.5) and (10.3), we also derive

$$\begin{aligned}
&\|\nabla_\tau\zeta(\tau)\|_{L^2(I_k,\mathcal{E}_1)}\\
&\quad\leqslant \|B(\tau)\zeta(\tau)\|_{L^2(I_k,\mathcal{E}_1)} + \|L(\tau)\|_{L^2(I_k,\mathcal{E}_1)}\\
&\quad\leqslant \sup_{\tau\in[R_0,\infty)} \|B(\tau)\|_{\mathbb{L}(\mathcal{E}_2,\mathcal{E}_1)}\|\zeta(\tau)\|_{L^2(I_k,\mathcal{E}_2)} + \|L(\tau)\|_{L^2(I_k,\mathcal{E}_1)}\\
&\quad\leqslant C_2'C(I_k,I_k')(\|(\nabla+B)\zeta\|_{L^2(I_k',\mathcal{E}_1)} + \|\zeta\|_{L^2(I_k',\mathcal{E}_1)})\\
&\qquad + \|L(\tau)\|_{L^2(I_k,\mathcal{E}_1)}\\
&\quad\leqslant (C_2'C(I_k,I_k')+1)\|L(\tau)\|_{L^2(I_k',\mathcal{E}_1)} + C_2'C(I_k,I_k')\|\zeta\|_{L^2(I_k,\mathcal{E}_1)}\\
&\quad\leqslant C_2''e^{-\delta k}.
\end{aligned}\tag{10.19}$$

Here we have chosen $I_k' = [k-1/3, k+4/3]$,

$$C_2' = \sup_{\tau\in[R_0,\infty)} \|B(\tau)\|_{\mathbb{L}(\mathcal{E}_2,\mathcal{E}_1)} \qquad \text{and} \qquad C_2'' = (C_2'C(I_k,I_k')e^{\delta/3}+1).$$

Combining (10.18) and (10.19), we have derived $\|\zeta\|_{W^{1,2}(I_k,\mathcal{E}_1)} \leqslant C_2e^{-\delta k}$ for all k with $C_2 = \max\{C, C_2''\}$.

By applying Sobolev's inequality for the section $I_k\to\mathcal{E}_1$

$$\max_{\tau\in I_k} |\zeta(\tau)|_{\mathcal{E}_{1,\tau}} \leqslant C_3\|\zeta\|_{W^{1,2}(I_k,\mathcal{E}_1)} \tag{10.20}$$

with C_3 the Sobolev constant on I_k. This now finishes the proof. □

REMARK 10.12. Since $\mathcal{E}_1$ may not be finitely dimensional, application of the Sobolev inequality (10.20) may not be standard to some readers. For readers' convenience, we give a direct proof of this inequality (10.20) in Appendix C.

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§11. Exponential convergence: the prequantization case

To make the main arguments transparent in the scheme of our exponential estimates, we start with the case of prequantization, that is, the case without $\mathcal{N}$ and the normal form contains E only. The general case will be dealt with in the next section.

We put the basic hypothesis that

$$|e(\tau, t)| < \delta \tag{11.1}$$

for all $\tau \geqslant \tau_0$ in our further study, where δ is given as in Proposition 4.8. From Corollary 9.6 and the remark after it, we can locally work with everything in a neighborhood of zero section in the normal form $(U_E, f\lambda_E, J)$.

11.1 Computational preparation

For a smooth function h, we can express its gradient vector field $\operatorname{grad} h$ with respect to the metric $g_{(\lambda_E,J_0)} = d\lambda_E(\cdot, J_0\cdot) + \lambda_E \otimes \lambda_E$ in terms of the λ_E-contact Hamiltonian vector field $X_h^{d\lambda_E}$ and the Reeb vector field X_E as

$$\operatorname{grad} h = -J_0 X_h^{d\lambda_E} + X_E[h]\, X_E. \tag{11.2}$$

Note the first term $-J_0 X_h^{d\lambda_E} =: \operatorname{grad} h^\pi$ is the π_{λ_E}-component of $\operatorname{grad} h$.

Consider the vector field Y along u given by $Y(\tau, t) := \nabla_\tau^\pi e$ where $w = (u, e)$ in the coordinates defined in Section 4. The vector field $e_\tau = e(\tau, t)$ as a vector field along $u(\tau, t)$ is nothing but the map $(\tau, t) \mapsto I_{w(\tau,t);u(\tau,t)}(\vec{R}(w(\tau, t)))$ as a section of u^*E. In particular

$$e(\infty, t) = I_{w(\infty,t);u(\infty,t)}(\vec{R}(w(\infty, t))) = I_{z(t);x(Tt)}(\vec{R}(o_{x(Tt)})) = o_{x(Tt)}.$$

Obviously, $I_{w(\tau,t);u(\tau,t)}(\vec{R}(w(\tau, t)))$ is pointwise perpendicular to $o_E \cong Q$. In particular,

$$(\Pi_{x(T\cdot)}^{u_\tau})^{-1} e_\tau \in (\ker D\Upsilon(z))^\perp \tag{11.3}$$

where e_τ is the vector field along the loop $u_\tau \subset o_E$ and we regard $(\Pi_{x(T\cdot)}^{u_\tau})^{-1} e_\tau$ as a vector field along $z = (x(T\cdot), o_{x(T\cdot)})$.

For further detailed computations, one needs to decompose the contact-instanton map equation

$$\bar\partial_J^{f\lambda_E} w = 0, \qquad d(w^*(f\, \lambda_E)) \circ j = 0. \tag{11.4}$$

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The second equation does not depend on the choice of endomorphisms J and becomes

$$d(w^*\lambda_E \circ j) = -dg \wedge (\lambda_E \circ j), \qquad g = \log f \tag{11.5}$$

which is equivalent to

$$d(u^*\theta \circ j + \Omega(e, \nabla^E_{du\circ j} e)) = -dg \wedge (u^*\theta \circ j + \Omega(e, \nabla^E_{du\circ j} e)). \tag{11.6}$$

On the other hand, by the formula (2.11), the first equation $\overline{\partial}^{f\lambda_E}_J w = 0$ becomes

$$\overline{\partial}^{\pi\lambda_E}_{J_0} w = (w^*\lambda_E\, Y_{dg})^{(0,1)} + (J - J_0) d^{\pi\lambda_E} w \tag{11.7}$$

where $(w^*\lambda_E\, Y_{dg})^{(0,1)}$ is the $(0,1)$-part of the one-form $w^*\lambda_E\, Y_{dg}$ with respect to J_0.

In terms of the coordinates, the equation can be rewritten as

$$\begin{aligned}
&\begin{pmatrix} \overline{\partial}^{\pi_\theta} u - (\Omega(\vec{R}(u,e), \nabla^E_{du} e)\, X_E(u,e))^{(0,1)} \\ (\nabla^E_{du} e)^{(0,1)} \end{pmatrix} \\
&\quad = \begin{pmatrix} ((u^*\theta + \Omega(\vec{R}(u,e), \nabla^E_{du} e))\, d\pi_E((Y_{dg})^h))^{(0,1)} \\ ((u^*\theta + \Omega(\vec{R}(u,e), \nabla^E_{du} e)) X^\Omega_g(u,e))^{(0,1)} \end{pmatrix} + (J - J_0) d^{\pi\lambda_E} w.
\end{aligned}$$

Here $(\Omega(\vec{R}(u,e), \nabla^E_{du} e)\, X_E(u,e))^{(0,1)}$ is the $(0,1)$-part with respect to J_Q and $(\nabla^E_{du} e)^{(0,1)}$ is the $(0,1)$-part with respect to J_E. From this, we have derived

LEMMA 11.1. *In coordinates $w = (u,e)$, (11.4) is equivalent to*

$$\nabla''_{du} e = (w^*\lambda_E\, X^\Omega_g(u,e))^{(0,1)} + I_{w;u}((J - J_0) d^{\pi\lambda_E} w)^v \tag{11.8}$$

$$\begin{aligned}
\overline{\partial}^\pi u &= (w^*\lambda_E\, (d\pi_E(Y_{dg})^h))^{(0,1)} + (\Omega(e, \nabla^E_{du} e)\, X_\theta(u(\tau,e))^h)^{(0,1)} \\
&\quad + d\pi_E((J - J_0) d^{\pi\lambda_E} w)^h
\end{aligned} \tag{11.9}$$

and

$$d(w^*\lambda_E \circ j) = -dg \wedge w^*\lambda_E \circ j \tag{11.10}$$

with the insertions of

$$w^*\lambda_E = u^*\theta + \Omega(e, \nabla^E_{du} e).$$

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Note that with insertion of (11.10), we obtain

$$\begin{aligned}\overline{\partial}^\pi u &= (u^*\theta\, d\pi_E((Y_{dg})^h))^{(0,1)}\\ &\quad + (\Omega(e, \nabla^E_{du} e)\, (X_\theta(u(\tau, e)) + d\pi_E(Y_{dg}))^h)^{(0,1)}\\ &\quad + d\pi_E((J - J_0) d^{\pi_{\lambda_E}} w)^h. \end{aligned} \tag{11.11}$$

Now let $w = (u, e)$ be a contact instanton in terms of the decomposition as above.

LEMMA 11.2. *Let e be an arbitrary section over a smooth map $u : \Sigma \to Q$. Then*

$$I_{w;u}(((J - J_0) d^{\pi_{\lambda_E}} w)^v) = L_1(u, e)(e, (d^\pi u, \nabla_{du} e)) \tag{11.12}$$

$$d\pi_E(((J - J_0) d^{\pi_{\lambda_E}} w)^h) = L_2(u, e)(e, (d^\pi u, \nabla_{du} e)) \tag{11.13}$$

where $L_1(u, e)$ is a (u, e)-dependent bilinear map with values in $\Omega^0(u^*E)$ and $L_2(u, e)$ is a bilinear map with values in $\Omega^0(u^*TQ)$. They also satisfy

$$|L_i(u, e)| = O(1). \tag{11.14}$$

An immediate corollary of this lemma is

COROLLARY 11.3.

$$\begin{aligned} |I_{w;u}(((J - J_0) d^{\pi_{\lambda_E}} w)^v)| &\leqslant O(1)|e|(|d^\pi u| + |\nabla e|)\\ |\nabla (I_{w;u}(((J - J_0) d^{\pi_{\lambda_E}} w)^v))| &\leqslant O(1)(|du| + |\nabla e|)^2 |e|\\ &\quad + |\nabla e|(|du| + |\nabla e|) + |e||\nabla^2 e|) \,. \end{aligned}$$

Next, we give the following lemmas whose proofs are straightforward from the definition of X_g^Ω.

LEMMA 11.4. *Suppose $d_{C^0}(w(\tau, \cdot), z(\cdot)) \leqslant \iota_g$. Then*

$$\begin{aligned} X_g^\Omega(u, e) &= D^v X_g^\Omega(u, 0) e + M_1(u, e)(e, e)\\ d\pi_E(Y_{dg}) &= M_2(u, e)(e)\\ \Omega(e, \nabla^E_{du} e)\, X_g^\Omega(u, e) &= N(u, e)(e, \nabla_{du} e, e) \end{aligned}$$

where $M_1(u, e)$ is a smoothly (u, e)-dependent bilinear map on $\Omega^0(u^*E)$ and $M_2 : \Omega^0(u^*E) \to \Omega^0(u^*TQ)$ is a linear map, $N(u, e)$ is a (u, e)-dependent trilinear map on $\Omega^0(u^*E)$. They also satisfy

$$|M_i(u, e)| = O(1), \qquad |N(u, e)| = O(1).$$

LEMMA 11.5.

$$(X_g^{d\lambda_E})^h(u,e) = K(u,e)\,e$$

where $K(u,e)$ is a (u,e)-dependent linear map from $\Omega^0(u^*E)$ to $\Omega^0(u^*TQ)$ satisfying

$$|M_1(u,e)| = O(1), \qquad |N(u,e)| = O(1), \qquad |K(u,e)| = o(1).$$

11.2 L^2-exponential decay of the normal bundle component e

Combining Lemmas 11.2, 11.4 and 11.5, we can write (11.8) as

$$\nabla''_{du} e - (u^*\theta\, DX_g^{\Omega}(u)(e))^{(0,1)} = K(e, \nabla_{du} e, d^\pi u). \tag{11.15}$$

By evaluating (11.15) against $\partial/\partial\tau$, we derive

$$\begin{aligned}&\nabla_\tau e + J_E(u)\nabla_t e - \theta\left(\frac{\partial u}{\partial \tau}\right) DX_g^{\Omega}(u)(e)\\ &- J_E\theta\left(\frac{\partial u}{\partial t}\right) DX_g^{\Omega}(u)(e) = K\left(e, \nabla_\tau e, \pi_\theta \frac{\partial u}{\partial \tau}\right).\end{aligned} \tag{11.16}$$

First notice that

LEMMA 11.6.

$$\left|K\left(e, \nabla_\tau e, \frac{\partial u}{\partial \tau}\right)\right|_{L^\infty} = o(|e|).$$

Proof. We consider (11.16) as an equation for e. Clearly this is a quasi-linear elliptic equation of e when u is fixed. We also recall $K(e, \nabla_{du} e, du)$ has the form

$$L_1(u,e)\left(e, \left(\nabla_\tau e, \frac{\partial u}{\partial \tau}\right)\right)$$

where $L_1(u,e)$ is a bilinear map with $|L_1(u,e)| = O(1)$ by (11.12) which satisfies the inequality

$$\left|L_1(u,e)\left(e, \left(\nabla_\tau e, \frac{\partial u}{\partial \tau}\right)\right)\right| \leqslant O(1)|e|(|d^\pi u| + |\nabla_\tau e|).$$

(See Corollary 11.3.) Now the lemma immediately follows from the convergence $|\pi(\partial u/\partial\tau)|$, $|\nabla_\tau e| \to 0$ established in Corollary 9.6. ∎

Denote by $B(\tau)$ a τ-family of operators

$$B(\tau) : W^{1,2}(u(\tau,\cdot)^*E) \to L^2(u(\tau,\cdot)^*E)$$

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defined by

$$B(\tau)e := J_E(u)\nabla_t e - \theta\left(\frac{\partial u}{\partial \tau}\right) DX_g^\Omega(u)(e) - J_E\theta\left(\frac{\partial u}{\partial t}\right) DX_g^\Omega(u)(e)$$
$$- K\left(e, \nabla_\tau e, \frac{\partial u}{\partial \tau}\right).$$

Then (11.16) for e with u fixed can be rewritten as

$$\nabla_\tau e(\tau) + B(\tau)e(\tau) = 0.$$

Once we know $u(\tau, \cdot) \to z_\infty$ as $\tau \to \infty$ for some Reeb orbit z_∞, we can use the exponential map from z_∞ to $u(\tau, \cdot)$ for any sufficiently large τ and its associated parallel transport to regard $B(\tau)$ as a τ-family of linear operators

$$W^{1,2}(z_\infty^* E) \to L^2(z_\infty^* E)$$

along the limiting closed Reeb orbit z_∞. (See [OW2, Section 8] for a detailed discussion on this process.)

Lemma 11.7. *Let τ_k be a sequence with $\tau_k \to \infty$, and also denote by τ_k a subsequence thereof appearing in Theorem 9.4. Under the abovementioned identification, the operator $B(\tau_k)$ converges to the linearized operator*

$$B_\infty = J_E(z_\infty(t))(\nabla_t - \mathcal{T} DX_g^\Omega(u_\infty))$$

as $k \to \infty$.

Proof. Reorganize (11.16) into

$$\nabla_{\partial u/\partial\tau} e + J_E\left(\nabla_{\partial u/\partial t} e - \lambda_E\left(\frac{\partial u}{\partial t}\right) X_g^\Omega(u, e)\right)$$
$$- \lambda_E\left(\frac{\partial u}{\partial \tau}\right) X_g^\Omega(u, e) - K\left(e, \nabla_\tau e, \frac{\partial u}{\partial \tau}\right) = 0.$$

We first note that $\nabla_{(\partial u/\partial t)(\tau_k,)} \to \mathcal{T}\nabla_{\dot z_\infty}$ in the operator norm under the above mentioned identification. (See [OW1, Proposition 8.2] and its proof for the precise explanation of this statement.) We now estimate the two terms in the second line. For the first term, we have

$$\left|\lambda_E\left(\frac{\partial u}{\partial \tau}\right) X_g^\Omega(u, e)\right| \leqslant \left|\lambda_E\left(\frac{\partial u}{\partial \tau}\right)\right| |X_g^\Omega(u, e)| = o(1)|e|,$$

where the last estimate follows from Corollary 9.6 and Lemma 11.3. For the second term, Lemma 11.6 implies that is of order $o(|e|)$. This now completes the proof. ∎

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Note that so far this convergence can only be expected in the subsequence sense. Fortunately this weak convergence is already enough to conduct our scheme of three-interval argument, when combined with the uniform local *a priori* estimate from [OW2].

Next we briefly explain how the current situation fits into the general framework set up in Section 10. We refer readers to [OW2, Section 8] for further details of this verification.

We first consider two Banach spaces $\mathcal{E}_{1,\tau} \supset \mathcal{E}_{2,\tau}$ defined by

$$\mathcal{E}_{1,\tau} := L^2(\iota_\tau^* u^* E), \qquad \mathcal{E}_{2,\tau} := W^{1,2}(\iota_\tau^* u^* E),$$

where the maps

$$\iota_\tau^* : S^1 \to [0,\infty) \times S^1, \qquad t \mapsto (\tau, t)$$

are embeddings at $\tau \in [0,\infty)$. This family defines the bundle $\mathcal{E}_i$ over $[0,\infty)$ whose fiber at τ is given by $\mathcal{E}_{1,\tau}$: its local triviality can be again proved by the parallel transport over a sufficient small interval $(-\varepsilon + \tau, \tau + \varepsilon)$ at each given $\tau \in [0,\infty)$.

We denote the translation map $\sigma_s(\tau) = \tau + s$ by $\sigma_s : [a,b] \to [0,\infty)$ for each $s \in [a,b]$ for any given bounded interval $[a,b] \subset [0,\infty)$. Using the exponential map over the limiting Reeb orbit z_∞ and the associated parallel transport for all sufficiently large k's the sequence of Banach bundles

$$\mathcal{E}_{i;k} := \sigma_{\tau_k}^* \mathcal{E}_i \to [a,b]$$

have global trivializations

$$\Phi_{i;k} : \mathcal{E}_{i;k} \to L^2(z_\infty^* E) \times [a,b].$$

The uniform convergence proved in Theorem 9.4 and Corollary 9.6 ensures the uniformly local tameness of $\mathcal{E}_i$ and also the precompactness, uniformly local coerciveness of B. In particular, since

$$\theta\left(\frac{\partial u}{\partial \tau}\right)(\tau,\cdot) \to \mathcal{Q} = 0, \qquad \theta\left(\frac{\partial u}{\partial t}\right)(\tau,\cdot) \to \mathcal{T} \quad \tau \to \infty$$

uniformly over $[a,b]$, we also conclude that the operator B_∞ defined as

$$B_\infty = J_E(z_\infty(t))(\nabla_t - \mathcal{T} D X_g^\Omega(u_\infty)),$$

is the limit of $B(\tau)$ for $\tau \in [a,b]$ with respect to the subsequence $\{\tau_k\}$, as shown in Lemma 11.7. Moreover, we also notice that B_∞ is invariant under

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the τ-translations, and it shows that B is asymptotically cylindrical. Also, it follows from the Morse–Bott condition and Corollary 6.2 that the operator B_∞ is an unbounded self-adjoint operator with trivial kernel. The C^1-bound of e from (11.1) and (9.4) guarantees that the uniform bound of $W^{1,2}(S^1)$-norm of $e(\tau)$.

The above discussion verifies that (11.16) can be fit into the general abstract framework of Theorem 10.11 applied to $\zeta = e$. Therefore, we immediately obtain the following L^2-exponential estimate.

PROPOSITION 11.8. *There exists a sufficiently large $\tau_0 > 0$ and constants C_0, δ_0 such that*

$$\|e(\tau)\|_{L^2(S^1)} < C_0 e^{-\delta_0 \tau}$$

for all $\tau \geqslant \tau_0$.

11.3 L^2-exponential decay of the tangential component du I

We summarize previous geometric calculations, especially the equation (11.11), into the following basic equation which we study using the three-interval argument in this section.

LEMMA 11.9. *We can write the equation (11.11) into the form*

$$\pi_\theta \frac{\partial u}{\partial \tau} + J(u)\pi_\theta \frac{\partial u}{\partial t} = L(\tau, t), \tag{11.17}$$

so that $\|L(\tau, \cdot)\|_{L^2(S^1)} \leqslant C e^{-\delta\tau}$.

Proof. We recall $|du| \leqslant C$ which follows from Corollary 9.6. Furthermore, since $X_{dg}^{d\lambda_E}|_Q \equiv 0$, it follows

$$|(u^*\theta\, d\pi_E((X_{dg}^{d\lambda_E})^h))^{(0,1)}| \leqslant C|e|.$$

Furthermore, by the adaptedness of J and by the definition of the associated J_0, we also have $(J - J_0)|_Q \equiv 0$ and so

$$|d\pi_E((J - J_0)d^{\pi_{\lambda_E}} w)^h| \leqslant C|e|.$$

It is manifest that the second term above also carries similar estimate. Combining them, we have established that the right-hand side is bounded by $C|e|$ from above. Then the required exponential inequality follows from that of e established in Proposition 11.8. ∎

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In the rest of this section and Section 11.4, we give the proof of the following

PROPOSITION 11.10. *There exists some constant $C_0 > 0$ and $\delta_0 > 0$ such that*

$$\left\|\pi_\theta \frac{\partial u}{\partial \tau}\right\|_{L^2} < C_0\, e^{-\delta_0 \tau}.$$

The proof basically follows the same three-interval argument as in the proof of Theorem 10.11. However, since the current case is much more subtle, we would like to highlight the following points before we start:

(1) unlike the normal component e whose governing equation (11.16) is a (inhomogeneous) quasilinear elliptic equation, (11.17) is only (inhomogeneous) quasilinear *degenerate elliptic*: the limiting operator B of its linearization contains nontrivial kernel;
(2) nonlinearity of the equation makes somewhat cumbersome to formulate the abstract framework of three-interval argument as in Theorem 10.11 although we believe it is doable. Since this is not the main interest of ours, we directly deal with (11.17) in the present paper postponing such an abstract framework elsewhere in the future;
(3) for the normal component, we directly establish the exponential estimates of the map e itself. On the other hand, for the tangential component, partly due to the absence of direct linear structure of u and also due to the presence of nontrivial kernel of the asymptotic operator, we prove the exponential decay of the *derivative* $\pi_\theta(\partial u/\partial \tau)$ first and then prove the exponential convergence to some Reeb orbit afterward;
(4) to obtain the exponential decay of the *derivative* term, we need to exclude the possibility of a kernel element for the limit obtained in the three-interval argument. In Section 11.4 we use the techniques of the center of mass as an intrinsic geometric coordinates system to exclude the possibility of the vanishing of the limit. This idea appears in [MT] and [OZ] too;
(5) unlike [HWZ3] and [Bou], our proof directly obtains L^2-exponential decay instead of showing C^0 convergence first and getting exponential decay afterward.

Starting from now until the end of Section 11.4, we give the proof of Proposition 11.10.

Divide $[0,\infty)$ into the union of unit intervals $I_k = [k, k+1]$ for $k = 0, 1, \ldots,$ and denote by $Z_k := [k, k+1] \times S^1$. In the context below, we also denote by Z^l the union of three-intervals $Z^l_I := [k, k+1] \times S^1$, $Z^l_{II} := [k+1, k+2] \times S^1$ and $Z^l_{III} := [k+2, k+3] \times S^1$.

Consider $x_k := \|\pi_\theta(\partial u/\partial\tau)\|^2_{L^2(Z_k)}$ as symbols in Lemma 10.1. As in the proof of Theorem 10.11, we still use the three-interval inequality as the criterion and consider two situations:

(1) if there exists some constant $\delta > 0$ such that

$$\left\|\pi_\theta \frac{\partial u}{\partial \tau}\right\|^2_{L^2(Z_k)} \leqslant \gamma(2\delta)\left(\left\|\pi_\theta \frac{\partial u}{\partial \tau}\right\|^2_{L^2(Z_{k-1})} + \left\|\pi_\theta \frac{\partial u}{\partial \tau}\right\|^2_{L^2(Z_{k+1})}\right) \tag{11.18}$$

holds for every k, then from Lemma 10.1, we are done with the proof;

(2) otherwise, we collect all the three-intervals Z^{l_k} against (11.18), that is,

$$\left\|\pi_\theta \frac{\partial u}{\partial \tau}\right\|^2_{L^2(Z^{l_k}_{II})} > \gamma(2\delta)\left(\left\|\pi_\theta \frac{\partial u}{\partial \tau}\right\|^2_{L^2(Z^{l_k}_{I})} + \left\|\pi_\theta \frac{\partial u}{\partial \tau}\right\|^2_{L^2(Z^{l_k}_{III})}\right). \tag{11.19}$$

In the rest of the proof, we deal with this case.

First, if there exists some uniform constant $C_1 > 0$ such that on each such three-interval

$$\left\|\pi_\theta \frac{\partial u}{\partial \tau}\right\|_{L^\infty([l_k+0.5, l_k+2.5]\times S^1)} < C_1 e^{-\delta l_k}, \tag{11.20}$$

then through the same estimates and analysis as for Theorem 10.11, we obtain the exponential decay of $\|\pi_\theta(\partial u/\partial\tau)\|$ with the presumed rate δ as claimed.

REMARK 11.11. Here we look at the L^∞-norm on smaller intervals $[l_k + 0.5, l_k + 2.5] \times S^1$ instead of the whole $Z^{l_k} = [l_k, l_k + 3] \times S^1$ is out of consideration for the elliptic bootstrapping argument in Lemma 11.14. However, the change here does not change any argument, since smaller ones are already enough to cover the middle intervals (see Figure 1).

Following the same scheme as for Theorem 10.11, we are going to deal with the case when there is no uniform bound C_1. Then there exists a sequence of constants $C_k \to \infty$ and a subsequence of such three-intervals

$\{Z^{l_k}\}$ (still use l_k to denote them) such that

$$\left\|\pi_\theta \frac{\partial u}{\partial \tau}\right\|_{L^\infty([l_k+0.5,l_k+2.5]\times S^1)} \geqslant C_k e^{-\delta l_k}. \tag{11.21}$$

By Theorem 9.4 and the local uniform C^1-estimate, we can take a subsequence, still denoted by ζ_k, such that $u(Z^{l_k})$ lives in a neighborhood of some closed Reeb orbit z_∞. Next, we translate the sequence $u_k := u|_{Z^{l_k}}$ to $\widetilde{u}_k : [0,3]\times S^1 \to Q$ by defining $\widetilde{u}_k(\tau,t) = u_k(\tau + l_k, t)$. As before, we also define $\widetilde{L}_k(\tau,t) = L(\tau + l_k, t)$. From (11.22), we now have

$$\pi_\theta \frac{\partial \widetilde{u}_k}{\partial \tau} + J(\widetilde{u}_k)\pi_\theta \frac{\partial \widetilde{u}_k}{\partial t} = \widetilde{L}_k(\tau,t). \tag{11.22}$$

Recalling that Q carries a natural S^1-action induced from the Reeb flow, we equip Q with a S^1-invariant metric and its associated Levi–Civita connection. In particular, the vector field X_λ restricted to Q is a Killing vector field of the metric and satisfies $\nabla_{X_\lambda} X_\lambda = 0$.

Now since the image of $\widetilde{u}_k$ live in neighborhood of a fixed Reeb orbit z in Q, we can express

$$\widetilde{u}_k(\tau,t) = \exp^Z_{z_k(\tau,t)} \zeta_k(\tau,t) \tag{11.23}$$

for the normal exponential map $\exp^Z : NZ \to Q$ of the locus Z of z, where $z_k(\tau,t) = \pi_N(\widetilde{u}_k(\tau,t))$ is the normal projection of $\widetilde{u}_k(\tau,t)$ to Z and $\zeta_k(\tau,t) \in N_{z_k(\tau,t)}Z = \zeta_{z_k(\tau,t)} \cap T_{z_k(\tau,t)}Q$. Then

LEMMA 11.12.

$$\begin{aligned} \pi_\theta \frac{\partial \widetilde{u}_k}{\partial \tau} &= \pi_\theta (d_2 \exp^Z)(\nabla^{\pi_\theta}_\tau \zeta_k) \\ \pi_\theta \frac{\partial \widetilde{u}_k}{\partial t} &= \pi_\theta (d_2 \exp^Z)(\nabla^{\pi_\theta}_t \zeta_k). \end{aligned} \tag{11.24}$$

Proof. To simplify notation, we omit k here. For each fixed (τ,t), we compute

$$D_1 \exp^Z(z(\tau,t))\,(X_\lambda(z(\tau,t))) = \left.\frac{d}{ds}\right|_{s=0} \exp^Z_{\alpha(s)} \Pi^{\alpha(s)}_{z(\tau,t)}(X_\lambda(z(\tau,t)))$$

for a curve $\alpha : (-\varepsilon,\varepsilon) \to Q$ satisfying $\alpha(0) = z(\tau,t)$, $\alpha'(0) = X_\lambda(z(\tau,t))$. For example, we can take $\alpha(s) = \phi^s_{X_\lambda}(z(\tau,t))$.

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On the other hand, we compare the initial conditions of the two geodesics $a \mapsto \exp^Z_{\alpha(s)} a\Pi^{\alpha(s)}_x(X_\lambda(x))$ and $a \mapsto \phi^s_{X_\lambda}(\exp^Z_x a(X_\lambda(x)))$ with $x = z(\tau, t)$. Since $\phi^s_{X_\lambda}$ is an isometry, we derive

$$\phi^s_{X_\lambda}(\exp^Z_x a(X_\lambda(x)) = \exp^Z_x a(d\phi^s_{X_\lambda}(X_\lambda(x))).$$

Furthermore, we note that $d\phi^s_{X_\lambda}(X_\lambda(x)) = X_\lambda(x)$ at $s = 0$ and the field $s \mapsto d\phi^s_{X_\lambda}(X_\lambda(x))$ is parallel along the curve $s \mapsto \phi^s_{X_\lambda}(x)$. Therefore by definition of $\Pi^{\alpha(s)}_x(X_\lambda(z(\tau, t)))$, we derive

$$\Pi^{\alpha(s)}_x(X_\lambda(x)) = d\phi^s_{X_\lambda}(X_\lambda(x)).$$

Combining this discussion, we obtain

$$\exp^Z_{\alpha(s)} \Pi^{\alpha(s)}_{z(\tau,t)}(X_\lambda(z(\tau, t))) = \phi^s_{X_\lambda}(\exp^Z_{z(\tau,t)}(X_\lambda(z(\tau, t))))$$

for all $s \in (-\varepsilon, \varepsilon)$. Therefore, we obtain

$$\left.\frac{d}{ds}\right|_{s=0} \exp^Z_{\alpha(s)} \Pi^{\alpha(s)}_{z(\tau,t)}(X_\lambda(z(\tau, t))) = X_\lambda(\exp^Z_{z(\tau,t)}(X_\lambda(z(\tau, t)))).$$

This shows $(D_1 \exp^Z)(X_\lambda) = X_\lambda(\exp^Z_{z(\tau,t)}(X_\lambda(z(\tau, t))))$.

To see $\pi_\theta(D_1 \exp^Z)(\partial z/\partial\tau) = 0$, just note that $\partial z/\partial\tau = k(\tau, t)X_\lambda(z(\tau, t))$ for some function k, which is parallel to X_λ and $z(\tau, t) \in Z$. Using the definition of $D_1 \exp^Z(x)(v)$ for $v \in T_xQ$ at $x \in Q$, we compute

$$\begin{aligned}(D_1 \exp^Z)\left(\frac{\partial z}{\partial\tau}\right)(\tau, t) &= D_1 \exp^Z(z(\tau, t))(k(\tau, t)X_\lambda(z(\tau, t)))\\ &= k(\tau, t)D_1 \exp^Z(z(\tau, t))(X_\lambda(z(\tau, t))),\end{aligned}$$

and hence the π_θ projection vanishes.

At last write

$$\pi_\theta \frac{\partial\widetilde{u}}{\partial\tau} = \pi_\theta(d_2 \exp^Z)(\nabla^{\pi_\theta}_\tau \zeta) + \pi_\theta(D_1 \exp^Z)\left(\frac{\partial z}{\partial\tau}\right)$$

and we are done with the first identity claimed.

The second one is proved exactly the same way. ∎

Further noting that $\pi_\theta(d_2 \exp^Z_{z_k(\tau,t)}) : \zeta_{z_k(\tau,t)} \to \zeta_{z_k(\tau,t)}$ is invertible, using this lemma and (11.22), we now have the equation of ζ_k

$$\nabla^{\pi_\theta}_\tau \zeta_k + \overline{J}(\tau,t)\nabla^{\pi_\theta}_t \zeta_k = [\pi_\theta(d_2 \exp^Z)]^{-1}\widetilde{L}_k, \tag{11.25}$$

where we set $[\pi_\theta(d_2 \exp)]^{-1} J(\widetilde{u}_k)[\pi_\theta(d_2 \exp^Z)] =: \overline{J}(\tau,t)$.

Next, we rescale this equation by the norm $\|\zeta_k\|_{L^\infty([0,3]\times S^1)}$:

Lemma 11.13. *The norm $\|\zeta_k\|_{L^\infty([0,3]\times S^1)}$ is not zero.*

Proof. Suppose to the contrary that $\zeta_k \equiv 0$. This then implies $\widetilde{u}_k(\tau,t) \equiv z_k(\tau,t)$ for all $(\tau,t) \in [0,3]\times S^1$. Therefore, $\partial\widetilde{u}_k/\partial\tau$ is parallel to X_θ on $[0,3]\times S^1$. In particular $\pi_\theta(\partial\widetilde{u}_k/\partial\tau) \equiv 0$. This violates the inequality

$$\left\|\pi_\theta \frac{\partial\widetilde{u}_k}{\partial\tau}\right\|_{L^2([1,2]\times S^1)} > \gamma\left(\left\|\pi_\theta \frac{\partial\widetilde{u}_k}{\partial\tau}\right\|_{L^2([0,1]\times S^1)} + \left\|\pi_\theta \frac{\partial\widetilde{u}_k}{\partial\tau}\right\|_{L^2([2,3]\times S^1)}\right).$$

Therefore, the lemma holds. ∎

Now the rescaled sequence $\overline{\zeta}_k := \zeta_k/\|\zeta_k\|_{L^\infty([0,3]\times S^1)}$ satisfies

$$\|\overline{\zeta}_k\|_{L^\infty([0,3]\times S^1)} = 1,$$

and

$$\nabla^{\pi_\theta}_\tau \bar{\zeta}_k + \overline{J}(\tau,t)\nabla^{\pi_\theta}_t \bar{\zeta}_k = \frac{[\pi_\theta(d_2 \exp^Z)]^{-1}\widetilde{L}_k}{\|\zeta_k\|_{L^\infty([0,3]\times S^1)}} \tag{11.26}$$

$$\|\nabla^{\pi_\theta}_\tau \bar{\zeta}_k\|^2_{L^2([1,2]\times S^1)} \geqslant \gamma(2\delta)(\|\nabla^{\pi_\theta}_\tau \bar{\zeta}_k\|^2_{L^2([0,1]\times S^1)} + \|\nabla^{\pi_\theta}_\tau \bar{\zeta}_k\|^2_{L^2([2,3]\times S^1)}).$$

The next step is to focus on the right-hand side of (11.26).

Lemma 11.14. *The right-hand side of (11.26) converges to zero as $k \to \infty$.*

Proof. Since the left-hand side of (11.25) is an elliptic (Cauchy–Riemann type) operator, we have the elliptic estimates

$$\begin{aligned}\|\nabla^{\pi_\theta}_\tau \zeta_k\|_{W^{l,2}([0.5,2.5]\times S^1)} &\leqslant C_1(\|\zeta_k\|_{L^2([0,3]\times S^1)} + \|\widetilde{L}_k\|_{L^2([0,3]\times S^1)})\\ &\leqslant C_2(\|\zeta_k\|_{L^\infty([0,3]\times S^1)} + \|\widetilde{L}_k\|_{L^\infty([0,3]\times S^1)}).\end{aligned}$$

The Sobolev's embedding theorem further gives

$$\|\nabla^{\pi_\theta}_\tau \zeta_k\|_{L^\infty([0.5,2.5]\times S^1)} \leqslant C(\|\zeta_k\|_{L^\infty([0,3]\times S^1)} + \|\widetilde{L}_k\|_{L^\infty([0,3]\times S^1)}).$$

Hence we have

$$\frac{\|\widetilde{L}_k\|_{L^\infty([0,3]\times S^1)}}{\|\nabla_\tau^{\pi_\theta}\zeta_k\|_{L^\infty([0.5,2.5]\times S^1)}} \geqslant \frac{\|\widetilde{L}_k\|_{L^\infty([0,3]\times S^1)}}{C(\|\zeta_k\|_{L^\infty([0,3]\times S^1)} + \|\widetilde{L}_k\|_{L^\infty([0,3]\times S^1)})}$$
$$= \frac{1}{C\left(\frac{\|\zeta_k\|_{L^\infty([0,3]\times S^1)}}{\|\widetilde{L}_k\|_{L^\infty([0,3]\times S^1)}} + 1\right)}.$$

We use our standing assumption (11.21) with $C_k \to \infty$ as $k \to \infty$ and Lemma 11.12, the left- hand side converges to zero as $k \to \infty$, so we get

$$\frac{\|\widetilde{L}_k\|_{L^\infty([0,3]\times S^1)}}{\|\zeta_k\|_{L^\infty([0,3]\times S^1)}} \to 0, \quad \text{as } k \to \infty.$$

Thus the right-hand side of (11.26) converges to zero. ∎

Then with the same argument as in the proof of Theorem 10.11, after taking a subsequence, we obtain a limiting section $\overline{\zeta}_\infty$ of $z_\infty^*\zeta_\theta$ satisfying

$$\nabla_\tau\overline{\zeta}_\infty + B_\infty\overline{\zeta}_\infty = 0. \tag{11.27}$$

$$\|\nabla_\tau\overline{\zeta}_\infty\|^2_{L^2([1,2]\times S^1)} \geqslant \gamma(2\delta)(\|\nabla_\tau\overline{\zeta}_\infty\|^2_{L^2([0,1]\times S^1)} + \|\nabla_\tau\overline{\zeta}_\infty\|^2_{L^2([2,3]\times S^1)}). \tag{11.28}$$

Here to make it compatible with the notation used in Theorem 10.11, we denote by B_∞ the limit operator of $B := \overline{J}(\tau, t)\nabla_t^{\pi_\theta}$. When applied to horizontal part as in the current case of study, the operator is nothing but the linearization of Reeb orbit z_∞ followed by action of J.

Write

$$\overline{\zeta}_\infty = \sum_{j=0,\ldots,k} a_j(\tau)e_j + \sum_{i\geqslant k+1} a_i(\tau)e_i,$$

where $\{e_i\}$ is the basis consisting of the eigenfunctions associated to the eigenvalue λ_i for $j \geqslant k+1$ with

$$0 < \lambda_{k+1} \leqslant \lambda_{k+2} \leqslant \cdots \leqslant \lambda_i \leqslant \cdots \to \infty$$

and e_j for $j = 1, \ldots, k$ are eigenfunctions of eigenvalue zero. By plugging $\overline{\zeta}_\infty$ into (10.15), we derive

$$a_j'(\tau) = 0, \quad j = 1, \ldots, k$$
$$a_i'(\tau) + \lambda_i a_i(\tau) = 0, \quad i = k+1, \ldots$$

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and it follows that

$$a_j = c_j, \quad j = 1, \ldots, k$$
$$a_i(\tau) = c_i e^{-\lambda_i \tau}, \quad i = k+1, \ldots.$$

By the same calculation in the proof of Theorem 10.11, it follows

$$\|\nabla_\tau^{\pi_\theta} \overline{\zeta}_\infty\|^2_{L^2([1,2]\times S^1)} < \gamma(2\delta)(\|\nabla_\tau^{\pi_\theta} \overline{\zeta}_\infty\|^2_{L^2([0,1]\times S^1)} + \|\nabla_\tau^{\pi_\theta} \overline{\zeta}_\infty\|^2_{L^2([2,3]\times S^1)}).$$

As a conclusion of this section, it remains to show

LEMMA 11.15.

$$\nabla_\tau^{\pi_\theta} \overline{\zeta}_\infty \neq 0.$$

This lemma will then lead to contradiction and hence finish the proof of Proposition 11.10. The proof of this nonvanishing is given in the next section via the study of the center of mass.

11.4 L^2-exponential decay of the tangential component du II: study of center of mass

We equip the submanifold with an S^1-invariant metric. Then its associated Levi–Civita connection ∇ is also S^1-invariant. Denote by $\exp : U_{o_Q} \subset TQ \to Q \times Q$ the exponential map, and this defines a diffeomorphism between the open neighborhood U_{o_Q} of the zero section o_Q of TQ and some open neighborhood U_Δ of the diagonal $\Delta \subset Q \times Q$. Denote its inverse by

$$E : U_\Delta \to U_{o_Q}; \qquad E(x, y) = \exp_x^{-1}(y).$$

We refer readers to [K] for the detailed study of the various basic derivative estimates of this map.

The following lemma is a variation of the well-known center of mass techniques from Riemannian geometry with the contact structure being taken into consideration (by introducing the reparameterization function h in the following statement).

LEMMA 11.16. *Let $(Q, \theta = \lambda|_Q)$ be the submanifold foliated by closed Reeb orbits of period T. Then there exists some $\delta > 0$ depending only on (Q, θ) such that for any C^{k+1} loop $\gamma : S^1 \to M$ with $d_{C^{k+1}}(\gamma, \mathfrak{Reeb}(Q, \theta))$ $< \delta$, there exists a unique point $m(\gamma) \in Q$, and a reparameterization map*

$h: S^1 \to S^1$ which is C^k close to id_{S^1}, such that

$$\int_{S^1} E(m, (\phi_{X_\theta}^{Th(t)})^{-1}(\gamma(t)))\, dt = 0 \tag{11.29}$$

$$E(m, (\phi_{X_\theta}^{Th(t)})^{-1}(\gamma(t))) \in \xi_\theta(m) \quad \textit{for all } t \in S^1. \tag{11.30}$$

Proof. Consider the functional

$$\Upsilon : C^\infty(S^1, S^1) \times Q \times C^\infty(S^1, Q) \to TQ \times \mathcal{R}$$

defined as

$$\begin{aligned}&\Upsilon(h, m, \gamma)\\ &:= \left(\left(m, \int_{S^1} E(m, (\phi_{X_\theta}^{Th(t)})^{-1}(\gamma(t))\, dt\right), \theta(E(m, (\phi_{X_\theta}^{Th(t)})^{-1}(\gamma(t))))\right),\end{aligned}$$

where $\mathcal{R}$ denotes the trivial bundle over $\mathbb{R} \times Q$ over Q.

If γ is a Reeb orbit with period T, then $h = \mathrm{id}_{S^1}$ and $m(\gamma) = \gamma(0)$ will solve the equation

$$\Upsilon(h, m, \gamma) = (o_Q, o_{\mathcal{R}}).$$

From straightforward calculations

$$\begin{aligned}&D_h \Upsilon|_{(\mathrm{id}_{S^1}, \gamma(0), \gamma)}(\eta)\\ &= \left(\left(m, \int_{S^1} d_2 E|_{(\gamma(0), \gamma(0))}(\eta(t) T X_\theta)\, dt\right), \theta(d_2 E|_{(\gamma(0), \gamma(0))}(\eta(t) T X_\theta))\right)\\ &= \left(\left(m, \left(T \int_{S^1} \eta(t)\, dt\right) \cdot X_\theta(\gamma(0))\right), T\eta(t)\right)\end{aligned}$$

and

$$\begin{aligned}&D_m \Upsilon|_{(\mathrm{id}_{S^1}, \gamma(0), \gamma)}(v)\\ &= \left(\left(v, \int_{S^1} D_1 E|_{(\gamma(0), \gamma(0))}(v)\, dt\right), \theta(D_1 E|_{(\gamma(0), \gamma(0))}(v))\right)\\ &= \left(\left(v, \int_{S^1} D_1 E|_{(\gamma(0), \gamma(0))}(v)\, dt\right), \theta(D_1 E|_{(\gamma(0), \gamma(0))}(v))\right)\\ &= ((v, v), \theta(v)),\end{aligned}$$

we claim that $D_{(h,m)}\Upsilon$ is transversal to $o_{TM} \times o_{\mathcal{R}}$ at the point $(\mathrm{id}_{S^1}, \gamma(0), \gamma(t))$, where γ is a Reeb orbit of period T. To see this, notice

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that for any point in the set

$$\Delta := \left\{ (aX_\theta + \mu, f) \in TQ \times C^\infty(S^1, \mathbb{R}) | a = \int_{S^1} f(t)\, dt \right\},$$

one can always find its preimage as follows: for any given $(aX_\theta + \mu, f) \in TQ \times C^\infty(S^1, \mathbb{R})$ with $a = \int_{S^1} f(t)\, dt$, the pair

$$v = a \cdot X_\theta + \mu$$
$$\eta(t) = \frac{1}{T}(f(t) - a)$$

lives in the preimage. This proves surjectivity of the partial derivative $D_{(h,m)}\Upsilon|_{(\mathrm{id}_{S^1}, \gamma(0), \gamma)}$. Then applying the implicit function theorem, we have finished the proof. ∎

Using the center of mass, we can derive the following proposition which will be used to exclude the possibility of the vanishing of $\nabla_\tau^{\pi_\theta} \overline{\zeta}_\infty$.

PROPOSITION 11.17. *Recall the rescaling sequence $\widetilde{\zeta}_k / L_k$ we take in the proof Proposition 11.10 above, and assume for a subsequence,*

$$\frac{\widetilde{\zeta}_k}{L_k} \to \overline{\zeta}_\infty$$

in L^2. Then $\int_{S^1} (d\phi_{X_\theta}^{Tt})^{-1}(\overline{\zeta}_\infty(\tau, t))\, dt = 0$ where $(d\phi_{X_\theta}^t)^{-1}(\overline{\zeta}_\infty(\tau, t)) \in T_{z(0)}Q$.

Proof. By the construction of the center of mass applying to maps $u_k(\tau, \cdot) : S^1 \to Q$ for $\tau \in [0, 3]$, we have obtained

$$\int_{S^1} E(m_k(\tau), (\phi_{X_\theta}^{Th_k(\tau,t)})^{-1} u_k(\tau, t))\, dt = 0.$$

If we write $u_k(\tau, t) = \exp_{z_\infty(\tau,t)} \zeta_k(\tau, t)$ where z_∞ is the limit of z_k defined in (11.23), it follows that

$$\begin{aligned} &\int_{S^1} E(m_k(\tau), \exp_{(\phi_{X_\theta}^{Th_k(\tau,t)})^{-1} z_\infty(\tau,t)} d(\phi_{X_\theta}^{Th_k(\tau,t)})^{-1} \zeta_k(\tau, t))\, dt \\ &= \int_{S^1} E(m_k(\tau), (\phi_{X_\theta}^{Th_k(\tau,t)})^{-1} \exp_{z_\infty(\tau,t)} \zeta_k(\tau, t))\, dt = 0. \end{aligned} \tag{11.31}$$

Recall the following lemma whose proof is direct and we skip.

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LEMMA 11.18. *Let Π_y^x is the parallel transport along the short geodesic from y to x. Then there exists some sufficiently small $\delta > 0$ depending only on the given metric on Q and a constant $C = C(\delta) > 0$ such that $C(\delta) \to 1$ as $\delta \to 0$ and*

$$|E(x, \exp_y^Z(\cdot)) - \Pi_y^x| \leqslant C\, d(x, y).$$

In particular $|E(x, \exp_y^Z(\cdot))| \leqslant |\Pi_y^x| + C\, d(x, y)$.

Applying this lemma to (11.31), we obtain

$$\left| \int_{S^1} \Pi_{m_k(\tau)}^{(\phi_{X_\theta}^{Th_k(\tau,t)})^{-1} z_\infty(\tau,t)} (d\phi_{X_\theta}^{Th(t)})^{-1} \zeta_k(\tau, t)\, dt \right|$$
$$\leqslant \int_{S^1} C\, d(m_k(\tau), (\phi_{X_\theta}^{Th_k(\tau,t)})^{-1} z_\infty(\tau, t))((d\phi_{X_\theta}^{Th(t)})^{-1} \zeta_k(\tau, t))\, dt.$$

We rescale ζ_k by using L_k and derive that

$$\left| \int_{S^1} \Pi_{m_k(\tau)}^{(\phi_{X_\theta}^{Th_k(\tau,t)})^{-1} z_\infty(\tau,t)} (d\phi_{X_\theta}^{Th_k(\tau,t)})^{-1} \frac{\zeta_k(\tau, t)}{L_k}\, dt \right|$$
$$\leqslant \int_{S^1} C\, d(m_k(\tau), (\phi_{X_\theta}^{Th_k(\tau,t)})^{-1} z_\infty(\tau, t)) \left| (d\phi_{X_\theta}^{Th_k(\tau,t)})^{-1} \frac{\zeta_k(\tau, t)}{L_k} \right| dt.$$

Take $k \to \infty$, and since that $m_k(\tau) \to z_\infty(0)$ and $h_k \to \mathrm{id}_{S^1}$ uniformly, we get $d(m_k(\tau), (\phi_{X_\theta}^{Th_k(\tau,t)})^{-1} z_\infty(\tau, t)) \to 0$ uniformly over $(\tau, t) \in [0, 3] \times S^1$ as $k \to \infty$. Therefore, the right-hand side of this inequality goes to 0. On the other hand by the same reason, we obtain

$$\Pi_{m_k(\tau)}^{(\phi_{X_\theta}^{Th_k(\tau,t)})^{-1} z_\infty(\tau,t)} (d\phi_{X_\theta}^{Th_k(\tau,t)})^{-1} \frac{\zeta_k(\tau, t)}{L_k} \to (d\phi_{X_\theta}^{Tt})^{-1} \overline{\zeta}_\infty$$

uniformly and hence we obtain $\int_{S^1} (d\phi_{X_\theta}^{Tt})^{-1} \overline{\zeta}_\infty\, dt = 0$. ∎

Using this proposition, we now prove $\nabla_\tau^{\pi\theta} \overline{\zeta}_\infty \neq 0$, which is the last piece of finishing the proof of Proposition 11.10.

Proof of Lemma 11.15. Suppose to the contrary, that is, $\nabla_\tau^{\pi\theta} \overline{\zeta}_\infty = 0$, then we would have $J(z_\infty(t)) \nabla_t \overline{\zeta}_\infty = 0$ from (11.27) and the remark right after it.

Fix a basis $\{v_1, \ldots, v_{2k}\}$ of $\xi_{z(0)} \subset T_{z(0)} Q$ and define $e_i(t) = d\phi_{X_\theta}^t(v_i)$ for $i = 1, \ldots, 2k$. Since $\nabla_t^{\pi\theta}$ is nothing but the linearization of the Reeb orbit z_∞, the Morse–Bott condition implies $\ker B_\infty = \mathrm{span}\{e_i(t)\}_{i=1}^{2k}$. Then

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one can express $\overline{\zeta}_\infty(t) = \sum_{i=1}^{2k} a_i(t) e_i(t)$. Moreover, since e_i are parallel (with respect to the S^1-invariant connection) by construction, it follows a_i are constants, $i = 1, \ldots, 2k$. Then we can write $\overline{\zeta}_\infty(t) = \phi^t_{X_\theta}(v)$, where $v = \sum_{i=1}^{2k} a_i v_i$, and further it follows that $\int_{S^1} (d\phi^t_{X_\theta})^{-1}(\overline{\zeta}_\infty(t))\, dt = v$. On the other hand, from Proposition 11.17, $v = 0$ and further $\overline{\zeta}_\infty \equiv 0$, which contradicts with $\|\overline{\zeta}_\infty\|_{L^\infty([0,3]\times S^1)} = 1$.

Thus finally we conclude that $\nabla^{\pi_\theta}_\tau \overline{\zeta}_\infty$ cannot be zero. ∎

This now concludes the proof of Proposition 11.10.

11.5 L^2-exponential decay of the Reeb component of dw

We again consider the equation

$$\nabla''_{du} e - (u^* DX^\Omega_g(u)(e))^{(0,1)} = K(e, \nabla_{du} e, du) \tag{11.32}$$

as an equation for e. Clearly this is a quasilinear elliptic equation of e when u is fixed. Applying the uniform (local) elliptic estimates to (11.32), the L^2-exponential decays of e and convergence of $\partial u/\partial\tau$ to 0 then lead to the L^2-exponential decay of $\nabla_{du} e$. Combining these, we have obtained L^2-exponential estimates of the $\pi\, dw$.

Now we consider the original map $w = (u, e)$ which satisfies (11.4)

$$\overline{\partial}^\lambda_J w = 0, \qquad d(w^*\lambda \circ j) = 0$$

where $\lambda = f\lambda_E$. We recall that this system is an elliptic system and the corresponding uniform local *a priori* estimates was established in [OW2]. Then by the elliptic bootstrapping argument using the local uniform *a priori* estimates on the cylindrical region, we obtain higher-order $W^{k,2}$-exponential decay of $\pi\, dw$ for all $k \geqslant 0$ under Hypothesis 9.7.

Next, in the rest of this subsection, we prove the exponential decay of the Reeb component $w^*\lambda$. For this purpose, we define a complex-valued function

$$\alpha(\tau, t) = \left(w^*\lambda\left(\frac{\partial}{\partial t}\right) - \mathcal{T}\right) + \sqrt{-1}\left(w^*\lambda\left(\frac{\partial}{\partial\tau}\right)\right).$$

The following lemma is easy to prove.

Lemma 11.19. *Let $\zeta = \pi(\partial w/\partial\tau)$ on the cylindrical ends. Then*

$$* \, d(w^*\lambda) = |\zeta|^2.$$

Combining this lemma together with the equation $d(w^*\lambda \circ j) = 0$, we notice that α satisfies the equations

$$\overline{\partial}\alpha = \nu, \quad \nu = \tfrac{1}{2}|\zeta|^2 + \sqrt{-1} \cdot 0, \tag{11.33}$$

where $\overline{\partial} = (1/2)((\partial/\partial\tau) + \sqrt{-1}(\partial/\partial t))$ the standard Cauchy–Riemann operator for the standard complex structure $J_0 = \sqrt{-1}$.

Notice that from previous section we have already established the $W^{1,2}$-exponential decay of $\nu = (1/2)|\zeta|^2$. The exponential decay of α follows from the following lemma, whose proof can be proved again by the three-interval method in a much easier way and so omitted.

LEMMA 11.20. *Suppose the complex-valued functions α and ν defined on $[0,\infty) \times S^1$ satisfy*

$$\begin{cases} \overline{\partial}\alpha = \nu \\ \|\nu\|_{L^2(S^1)} + \|\nabla\nu\|_{L^2(S^1)} \leqslant Ce^{-\delta\tau} \quad \text{for some constants } C, \delta > 0 \\ \lim_{\tau\to+\infty} \alpha = 0 \end{cases}$$

then $\|\alpha\|_{L^2(S^1)} \leqslant \overline{C}e^{-\delta\tau}$ for some constant $\overline{C}$.

11.6 C^0 exponential convergence

Now we prove C^0-exponential convergence of $w(\tau,\cdot)$ to some Reeb orbit as $\tau \to \infty$ from the L^2-exponential estimates presented in previous sections.

PROPOSITION 11.21. *Under Hypothesis 9.1, for any contact instanton w with vanishing charge, there exists a unique Reeb orbit $z(\cdot) = \gamma(T\cdot) : S^1 \to M$ with period $\mathcal{T} > 0$, such that*

$$\|d(w(\tau,\cdot), z(\cdot))\|_{C^0(S^1)} \to 0,$$

as $\tau \to +\infty$, where d denotes the distance on M defined by the triad metric.

Proof. We start with the following lemma

LEMMA 11.22. *Let $t \in S^1$ be given. Then for any given $\epsilon > 0$, there exists sufficiently large $\tau_1 > 0$ such that*

$$d(w(\tau,t), w(\tau',t)) < \epsilon$$

for all τ, $\tau' \geqslant \tau_1$.

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Proof. Suppose to the contrary that there exist some $t_0 \in S^1$ and some constant $\epsilon > 0$, sequences $\tau_k \to \infty$, $p_k > 0$ such that

$$d(w(\tau_{k+p_k}, t_0), w(\tau_k, t_0)) \geqslant \epsilon. \tag{11.34}$$

Then combining this with the continuity of w in t, there exists some $l > 0$ small such that

$$d(w(\tau_{k+p_k}, t), w(\tau_k, t)) \geqslant \frac{\epsilon}{2}, \qquad |t - t_0| \leqslant l.$$

Therefore,

$$\begin{aligned}
&\int_{S^1} d(w(\tau_{k+p_k}, t), w(\tau_k, t))\, dt \\
&\quad = \int_{|t-t_0|\leqslant l} d(w(\tau_{k+p_k}, t), w(\tau_k, t))\, dt + \int_{|t-t_0|>l} d(w(\tau_{k+p_k}, t), w(\tau_k, t))\, dt \\
&\quad \geqslant \int_{|t-t_0|\leqslant l} d(w(\tau_{k+p_k}, t), w(\tau_k, t))\, dt \geqslant \epsilon l.
\end{aligned}$$

On the other hand, we compute

$$\begin{aligned}
&\int_{S^1} d(w(\tau_{k+p_k}, t), w(\tau_k, t))\, dt \\
&\quad \leqslant \int_{S^1} \int_{\tau_k}^{\tau_{k+p_k}} \left|\frac{\partial w}{\partial s}(s,t)\right| ds\, dt = \int_{\tau_k}^{\tau_{k+p_k}} \int_{S^1} \left|\frac{\partial w}{\partial s}(s,t)\right| dt\, ds \\
&\quad \leqslant \int_{\tau_k}^{\tau_{k+p_k}} \left(\int_{S^1} \left|\frac{\partial w}{\partial s}(s,t)\right|^2 dt\right)^{1/2} ds \\
&\quad \leqslant \int_{\tau_k}^{\tau_{k+p_k}} Ce^{-\delta s}\, ds = \frac{C}{\delta}(1 - e^{-(\tau_{k+p_k} - \tau_k)})e^{-\tau_k} \leqslant \frac{C}{\delta} e^{-\tau_k}.
\end{aligned}$$

When τ_k sufficiently large, this inequality gives rise to a contradiction to (11.34). Hence the proof. □

Now using the subsequence convergence from Theorem 9.4, we can pick a subsequence $\{\tau_k\}$ and a closed Reeb orbit (γ, T) such that

$$w(\tau_k, t) \to z(t) := \gamma(Tt), \quad k \to \infty$$

uniformly in t. Then the above lemma immediately implies $w(\tau, t)$ uniformly converges to $z(t)$ for any $t \in S^1$.

It remains to show that this convergence is uniform in t. Suppose to the contrary that there exist some $\epsilon > 0$ and some sequence (τ_k, t_k) such that

$$d(w(\tau_k, t_k), z(t_k)) \geqslant 2\epsilon.$$

Since $t_k \in S^1$, we can further take a subsequence, still denoted by t_k, such that $t_k \to t_0 \in S^1$. We can take k so large that $d(z(t_k), z(t_0)) \leqslant (1/2)\epsilon$. We also note

$$d(w(\tau, t_k), w(\tau, t_0)) \leqslant \int_{t_0}^{t_k} \left| \frac{\partial w}{\partial t}(\tau, s) \right| ds \leqslant (t_k - t_0)\|dw\|_{C^0},$$

by which we can make the distance less than $(1/2)\epsilon$ by taking k sufficiently large.

Combing these, we derive

$$\begin{aligned} d(w(\tau_k, t_0), z(t_0)) &\geqslant d(w(\tau_k, t_k), z(t_k)) - d(w(\tau_k, t_k), w(\tau_k, t_0)) \\ &\quad - d(z(t_k), z(t_0)) \\ &\geqslant 2\epsilon - \tfrac{1}{2}\epsilon - \tfrac{1}{2}\epsilon = \epsilon \end{aligned}$$

for all sufficiently large k's. This gives rise to contradiction to the pointwise convergence $w(\tau_k, t_k) \to z(t_0)$, which finishes the proof of uniform convergence for $t \in S^1$ and hence completes the proof. ∎

Then the following C^0-exponential convergence immediately follows.

PROPOSITION 11.23. *There exist some constants $C > 0$, $\delta > 0$ and τ_0 large such that for any $\tau > \tau_0$,*

$$\|d(w(\tau, \cdot), z(\cdot))\|_{C^0(S^1)} \leqslant C\, e^{-\delta\tau}.$$

Proof. For any $\tau < \tau_+$, similarly as in the previous proof,

$$d(w(\tau, t), w(\tau_+, t)) \leqslant \int_{\tau}^{\tau_+} \left| \frac{\partial w}{\partial \tau}(s, t) \right| ds \leqslant \frac{C}{\delta} e^{-\delta\tau}.$$

Take $\tau_+ \to +\infty$ and using the C^0 convergence of w part, that is, Proposition 11.21, we get

$$d(w(\tau, t), z(t)) \leqslant \frac{C}{\delta} e^{-\delta\tau}.$$

This proves the first inequality. ∎

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11.7 C^∞-exponential decay of $dw - X_\lambda(w)\, d\tau$

We recall the coordinate expression of $w = (u, e)$ under the identification of a tubular neighborhood of Q with a neighborhood of the zero section of the normal bundle of Q. So far, we have established the following:

- $W^{1,2}$-exponential decay of the normal component e;
- L^2-exponential decay of the derivative du of the base component u;
- C^0-exponential convergence of $w(\tau, \cdot) \to z(\cdot)$ as $\tau \to \infty$ for some closed Reeb orbit z.

Now we are ready to complete the proof of C^∞-exponential convergence $w(\tau, \cdot) \to z$ by establishing the C^∞-exponential decay of $dw - X_\lambda(w)\, dt$. The proof of the latter decay is now in order which will be carried out by the bootstrapping arguments applied to the system (11.4).

Combining the above three, we have obtained L^2-exponential estimates of the full derivative dw. As already used in Section 11.5, we consider the equation

$$\overline{\partial}^\lambda_J w = 0, \qquad d(w^* \lambda \circ j) = 0$$

where $\lambda = f\lambda_E$, under Hypothesis 9.7. By the bootstrapping argument using the local uniform *a priori* estimates on the cylindrical region (see [OW2] for the details), we obtain higher-order $W^{k,2}$-exponential decays of the term

$$\frac{\partial w}{\partial t} - \mathcal{T} X_\lambda(z), \qquad \frac{\partial w}{\partial \tau}$$

for all $k \geqslant 0$, where $w(\tau, \cdot)$ converges to z as $\tau \to \infty$ in C^0 sense. This, combined with the Sobolev's embedding, then completes proof of C^∞-convergence of $w(\tau, \cdot) \to z$ as $\tau \to \infty$.

§12. Exponential decay: general Morse–Bott case

In this section, we consider the general case of the Morse–Bott submanifold. For this one, it is enough to consider the normalized contact triad $(F, f\lambda_F, J)$ where J is adapted to the zero section Q.

Write $w = (u, s) = (u, \mu, e)$, where $\mu \in u^* JT\mathcal{N}$ and $e \in u^* E$. By the calculations in Section 7, and with similar calculation of Section 11, the e part can be dealt with exactly the same as in the prequantization case, whose details are skipped here.

After the e-part is taken care of, for the (u, μ) part, we derive

$$\begin{pmatrix} \pi_\theta \dfrac{\partial u}{\partial \tau} \\ \nabla_\tau \mu \end{pmatrix} + J \begin{pmatrix} \pi_\theta \dfrac{\partial u}{\partial t} \\ \nabla_t \mu \end{pmatrix} = L,$$

where $|L| \leqslant Ce^{-\delta\tau}$ similarly as for the prequantization case.

Then we apply the three-interval argument whose details are similar to the prequantization case and so are omitted. We only need to establish as in the prequantization case for the limiting $(\overline{\zeta}_\infty, \overline{\mu}_\infty)$ is not in kernel of B_∞.

If $(\overline{\zeta}_\infty, \overline{\mu}_\infty)$ is in the kernel of B_∞, then by the Morse–Bott condition, we have $\overline{\mu}_\infty = 0$. With the same procedure for introducing the center of mass, we can use the same argument to prove that $\overline{\zeta}_\infty$ must vanish if it is contained in the kernel of B_∞. This will then prove the following proposition.

Proposition 12.1. *For any $k = 0, 1, \ldots,$ there exists some constant $C_k > 0$ and $\delta_k > 0$*

$$\left|\nabla^k \left(\pi \frac{\partial u}{\partial \tau}\right)\right| < C_k\, e^{-\delta_k \tau}, \qquad |\nabla^k \mu| < C_k\, e^{-\delta_k \tau}$$

for each $k \geqslant 0$.

§13. The case of asymptotically cylindrical symplectic manifolds

In this section, we explain how we can apply the three-interval method and our tensorial scheme to noncompact symplectic manifolds with *asymptotically cylindrical ends*. Here we use Bao's precise definition [Ba] of the asymptotically cylindrical ends but restricted to the case where the asymptotical manifold is a contact manifold (V, ξ). In this section, we denote a contact manifold by V, instead of M which is what we used in the previous sections, to make comparison of our definition with Bao's transparent.

Let (V, ξ) be a closed contact manifold of dimension $2n + 1$ and let J be an almost complex structure on $W = [0, \infty) \times V$. We denote

$$\mathbf{R} := J \frac{\partial}{\partial r} \tag{13.1}$$

a smooth vector field on W, and let $\xi \subset TW$ be a subbundle defined by

$$\xi_{(r,v)} = JT_{(r,v)}(\{r\} \times V) \cap T_{(r,v)}(\{r\} \times V). \tag{13.2}$$

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Then we have splitting

$$TW = \mathbb{R}\left\{\frac{\partial}{\partial r}\right\} \oplus \mathbb{R}\{\mathbf{R}\} \oplus \xi_{(r,v)} \tag{13.3}$$

and denote by $i : \mathbb{R}\{\partial/\partial r\} \oplus \mathbb{R}\{\mathbf{R}\} \to \mathbb{R}\{\partial/\partial r\} \oplus \mathbb{R}\{\mathbf{R}\}$ the almost complex structure

$$i\frac{\partial}{\partial r} = \mathbf{R}, \qquad i\mathbf{R} = -\frac{\partial}{\partial r}.$$

We denote by λ and σ the dual 1-forms of $\partial/\partial r$ and $\mathbf{R}$ such that $\lambda|_\xi = 0 = \sigma|_\xi$. In particular,

$$\lambda(\mathbf{R}) = 1 = \sigma\left(\frac{\partial}{\partial r}\right), \qquad \lambda\left(\frac{\partial}{\partial r}\right) = 0 = \sigma(\mathbf{R}).$$

We denote by $T_s : [0, \infty) \times V \to [-s, \infty) \times V$ the translation $T_s(r, v) = (r + s, v)$ and call a tensor on W is translational invariant if it is invariant under the translation.

The following definition is the special case of the one in [Ba] restricted to the contact-type asymptotical boundary.

Definition 13.1. (Asymptotically cylindrical (W, ω, J) [Ba]) The almost complex structure is called C^ℓ-*asymptotically cylindrical* if there exists a 2-form ω on W such that the pair (J, ω) satisfies the following:

(AC1) $\partial/\partial r \rfloor \omega = 0 = \mathbf{R} \rfloor \omega$;
(AC2) $\omega|_\xi(v, J\, v) \geqslant 0$ and equality holds if and only if $v = 0$;
(AC3) there exists a smooth translational invariant almost complex structure J_∞ on $\mathbb{R} \times V$ and constants $R_\ell > 0$ and C_ℓ, $\delta_\ell > 0$

$$\|(J - J_\infty)|_{[r,\infty)\times V}\|_{C^\ell} \leqslant C_\ell e^{-\delta_\ell r}$$

for all $r \geqslant R_\ell$. Here the norm is computed in terms of the translational invariant metric g_∞ and a translational invariant connection;
(AC4) there exists a smooth translational invariant closed 2-form ω_∞ on $\mathbb{R} \times V$ such that

$$\|(\omega - \omega_\infty)|_{[r,\infty)\times V}\|_{C^\ell} \leqslant C_\ell e^{-\delta_\ell r}$$

for all $r \geqslant R_\ell$;
(AC5) $(J_\infty, \omega_\infty)$ satisfies (AC1) and (AC2);

(AC6) $\mathbf{R}_\infty \rfloor d\lambda_\infty = 0$, where $\mathbf{R}_\infty := \lim_{s\to\infty} T_s^* \mathbf{R}$, $\lambda_\infty := \lim_{s\to\infty} T_s^* \lambda$ where both limit exist on $\mathbb{R} \times V$ by (AC3);

(AC7) $\mathbf{R}_\infty(r, v) = J_\infty(\partial/\partial r) \in T_{(r,v)}(\{r\} \times V)$.

For the purpose of current paper, we restrict ourselves to the case when λ_∞ is a contact form of a contact manifold (V, ξ) and $\mathbf{R}$ the translational invariant vector field induced by the Reeb vector field on V associated to the contact form λ_∞ of (V, ξ). More precisely, we have

$$\mathbf{R}(r, v) = (0, X_{\lambda_\infty}(v))$$

with respect to the canonical splitting $T_{(r,v)}W = \mathbb{R} \oplus T_v V$. Furthermore, we also assume that $(V, \lambda_\infty, J_\infty)$ is a contact triad.

Now suppose that $Q \subset V$ is a Morse–Bott submanifold of closed Reeb orbits of λ_∞ and that $\widetilde{u} : [0, \infty) \times S^1 \to W$ is a $\widetilde{J}$-holomorphic curve for which the Subsequence theorem given in [Ba, Section 3.2] holds. We also assume that J_∞ is adapted to Q in the sense of Definition 8.2. Let $\tau_k \to \infty$ be a sequence such that $a(\tau_k, t) \to \infty$ and $w(\tau_k, t) \to z$ uniformly as $k \to \infty$ where z is a closed Reeb orbit whose image is contained in Q. By the local uniform elliptic estimates, we may assume that the same uniform convergence holds on the intervals

$$[\tau_k, \tau_k + 3] \times S^1$$

as $k \to \infty$. On these intervals, we can write the equation $\overline{\partial}_{\widetilde{J}} \widetilde{u} = 0$ as

$$\overline{\partial}_{J_\infty} \widetilde{u} \left(\frac{\partial}{\partial \tau}\right) = (\widetilde{J} - J_\infty) \frac{\partial \widetilde{u}}{\partial t}.$$

We can write the endomorphism $(\widetilde{J} - J_\infty)(r, \Theta) =: M(r, \Theta)$ where $(r, \Theta) \in \mathbb{R} \times V$ so that

$$|\nabla^k M(r, \Theta)| \leqslant C_k e^{-\delta r} \tag{13.4}$$

for all $r \geqslant R_0$. Therefore, $u = (a, w)$ with $a = r \circ \widetilde{u}$, $w = \Theta \circ \widetilde{u}$ satisfies

$$\overline{\partial}_{J_\infty} \widetilde{u} \left(\frac{\partial}{\partial \tau}\right) = M(a, w) \left(\frac{\partial \widetilde{u}}{\partial t}\right).$$

Decomposing $\overline{\partial}_{J_\infty} \widetilde{u}$ and $\partial \widetilde{u}/\partial t$ with respect to the decomposition

$$TW = \mathbb{R} \oplus TV = \mathbb{R} \cdot \frac{\partial}{\partial r} \oplus \mathbb{R} \cdot X_{\lambda_\infty} \oplus \xi$$

we have derived

$$\overline{\partial}^{\pi_\xi} w\left(\frac{\partial}{\partial\tau}\right) = \pi_\xi\left(M(a,w)\left(\frac{\partial\widetilde{u}}{\partial t}\right)\right) \tag{13.5}$$

$$(dw^*\circ j - da)\left(\frac{\partial}{\partial\tau}\right) = \pi_{\mathbb{C}}\left(M(a,w)\left(\frac{\partial\widetilde{u}}{\partial t}\right)\right) \tag{13.6}$$

where π_ξ is the projection to ξ with respect to the contact form λ_∞ and $\pi_{\mathbb{C}}$ is the projection to $\mathbb{R}\cdot\partial/\partial r \oplus \mathbb{R}\cdot X_{\lambda_\infty}$ with respect to the cylindrical $(W, \omega_\infty, J_\infty)$. Then we obtain from (13.4)

$$|\overline{\partial}^{\pi_\xi} w| \leqslant Ce^{-\delta a}$$

as $a \to \infty$. By the subsequence convergence theorem assumption and local *a priori* estimates on $\widetilde{u}$, we have immediately obtained the following

$$|\nabla''_\tau e(\tau,t)| \leqslant C\, e^{\delta_1\tau}, \qquad |\nabla''_\tau \xi_{\mathcal{F}}(\tau,t)| \leqslant C\, e^{\delta_1\tau}, \qquad |\nabla''_\tau \xi_G(\tau,t)| \leqslant C\, e^{\delta_1\tau}$$

where $w = \exp_Z(\xi_G + \xi_{\mathcal{F}} + e)$ is the decomposition similarly as before. Now we can apply exactly the same proof as the one given in the previous section to establish the exponential decay property of dw.

For the component a, we can use (13.6) and the argument used in [OW2] and obtain the necessary exponential property as before.

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Appendix A. Proof of Proposition 8.3

In this appendix, we prove contractibility of the set of Q-adapted CR-almost complex structures postponed from the proof of Proposition 8.3.

We first notice that for any $d\lambda$-compatible CR-almost complex structure J, $(TQ \cap JTQ) \cap T\mathcal{F} = \{0\}$: this is because for any $v \in (TQ \cap JTQ) \cap T\mathcal{F}$,

$$|v|^2 = d\lambda(v, Jv) = 0,$$

since $Jv \in TQ$ and $v \in T\mathcal{F} = \ker \omega_Q$. Therefore, $(TQ \cap JTQ)$ and $T\mathcal{F}$ are linearly independent.

We now give the following lemma.

LEMMA A.1. *J satisfies the condition $JTQ \subset TQ + JT\mathcal{N}$ if and only if it satisfies $TQ = (TQ \cap JTQ) \oplus T\mathcal{F}$.*

Proof. It is obvious to see that $TQ = (TQ \cap JTQ) \oplus T\mathcal{F}$ indicates J is Q-adapted.

It remains to prove the other direction. For this, we only need to prove that $TQ \subset (TQ \cap JTQ) + T\mathcal{F}$ by the discussion right in front of the statement of the lemma.

Let $v \in TQ$. By the definition of the adapted condition, $Jv \in TQ + JT\mathcal{N}$. Therefore, we can write

$$Jv = w + Ju,$$

for some $w \in TQ$ and $u \in T\mathcal{N}$. Then it follows that $v = -Jw + u$, noting that $Jw \in TQ \cap JTQ$, we derive $v \in (TQ \cap JTQ) + T\mathcal{F}$ and so we have finished the proof. $\square$

This lemma shows that any Q-adapted J naturally defines a splitting

$$T\mathcal{F} \oplus G_J = TQ, \qquad G_J := TQ \cap JTQ. \tag{A.1}$$

We also note that such J preserves the subbundle $TQ + JT\mathcal{F} \subset TM$ and so defines an invariant splitting

$$TM = TQ \oplus JT\mathcal{F} \oplus E_J; \quad E_J = (TQ \oplus JT\mathcal{F})^{\perp_{g_J}}. \tag{A.2}$$

Conversely, for given splittings (A.1), (A.2), we can always choose Q-adapted J so that $TQ \cap JTQ = G$ but the choice of such J is not unique.

It is easy to see that the set of such splittings forms a contractible manifold (see [OP, Lemma 4.1] for a proof). We also note that the 2-form $d\lambda$ induces nondegenerate (fiberwise) bilinear 2-forms on G and E which we denote by ω_G and ω_E. Now we denote by $\mathcal{J}_{G,E}(\lambda; Q)$ the subset of $\mathcal{J}(\lambda; Q)$ consisting of $J \in \mathcal{J}(\lambda; Q)$ that satisfy (A.1), (A.2). Then $\mathcal{J}(\lambda; Q)$ forms a fibration

$$\mathcal{J}(\lambda; Q) = \bigcup_{G,E} \mathcal{J}_{G,E}(\lambda; Q).$$

Therefore, it is enough to prove that $\mathcal{J}_{G,E}(\lambda; Q)$ is contractible for each fixed G, E.

We denote each $J : TM \to TM$ as a block 4×4 matrix in terms of the splitting

$$TM = T\mathcal{F} \oplus G \oplus JT\mathcal{F} \oplus E.$$

Then one can easily check that the Q-adaptedness of J implies J must have the form

$$\begin{pmatrix} 0 & 0 & \mathrm{Id} & 0 \\ 0 & J_G & 0 & 0 \\ -\mathrm{Id} & 0 & 0 & 0 \\ 0 & B & 0 & J_E \end{pmatrix}$$

where $J_G : G \to G$ is ω_G-compatible and $J_E : E \to E$ is ω_E-compatible, and B satisfies the relation $BJ_G = 0$ which in turn implies $B = 0$. Since each set of such J_G's or of such J_E's is contractible, it follows that $\mathcal{J}_{G,E}(\lambda; Q)$ is contractible. This finishes the proof of contractibility of $\mathcal{J}(\lambda; Q)$.

Appendix B. Proof of Theorem 9.4

In this appendix, we provide the proof of Theorem 9.4 borrowing the exposition from [OW2].

For a given contact instanton $w : [0, \infty) \times S^1 \to M$, we define maps $w_s : [-s, \infty) \times S^1 \to M$ by $w_s(\tau, t) = w(\tau + s, t)$. For any compact set $K \subset \mathbb{R}$, there exists sufficiently large s_0 such that for every $s \geqslant s_0$, $K \subset [-s, \infty)$. For such $s \geqslant s_0$, we also get an $[s_0, \infty)$-family of maps by defining $w_s^K := w_s|_{K\times S^1} : K \times S^1 \to M$.

The asymptotic behavior of w at infinity can be understood by studying the limit of the sequence of maps $\{w_s^K : K \times S^1 \to M\}_{s\in[s_0,\infty)}$, for any compact set $K \subset \mathbb{R}$.

First of all, it is easy to check that under Hypothesis 9.1, the family $\{w_s^K : K \times S^1 \to M\}_{s\in[s_0,\infty)}$ satisfies the following:

(1) $\overline{\partial}^\pi w_s^K = 0$, $d((w_s^K)^*\lambda \circ j) = 0$, for every $s \in [s_0, \infty)$;
(2) $\lim_{s\to\infty} \|d^\pi w_s^K\|_{L^2(K\times S^1)} = 0$;
(3) $\|dw_s^K\|_{C^0(K\times S^1)} \leqslant \|dw\|_{C^0([0,\infty)\times S^1)} < \infty$.

From (1) and (3) together with the compactness of the target manifold M (which provides the uniform $L^2(K \times S^1)$ bound) and the coercive estimate for contact-instanton equation derived in [OW2, Theorem 5.7], we obtain

$$\|w_s^K\|_{W^{3,2}(K\times S^1)} \leqslant C_{K;(3,2)} < \infty,$$

for some constant $C_{K;(3,2)}$ independent of s. Then it follows from the compactness of the embedding of $W^{3,2}(K \times S^1)$ into $C^{1,\alpha}(K \times S^1)$, with $0 < \alpha < 1$, that the set $\{w_s^K : K \times S^1 \to M\}_{s\in[s_0,\infty)}$ is sequentially compact. Therefore, for any sequence $s_k \to \infty$, there exists a subsequence, still denoted by s_k, that converges to a map $w_\infty^K \in C^{1,\alpha}(K \times S^1, M)$ in $C^{1,\alpha}(K \times S^1, M)$ as $k \to \infty$.

Combined with (2), we derive the convergence

$$dw_{s_k}^K \to dw_\infty^K \qquad \text{and} \qquad dw_\infty^K = (w_\infty^K)^*\lambda \otimes X_\lambda.$$

Finally by taking (1) into consideration, we also derive that both $(w_\infty^K)^*\lambda$ and $(w_\infty^K)^*\lambda \circ j$ are harmonic 1-forms.

Recall that these limiting maps w_∞^K have common extension $w_\infty : \mathbb{R} \times S^1 \to M$ by the nature of the diagonal argument which takes a sequence of compact sets K in the way one including another and exhausting full $\mathbb{R}$ as $k \to \infty$. Then w_∞ is $C^{1,\alpha}$ (actually C^∞) and satisfies

$$\|dw_\infty\|_{C^0(\mathbb{R}\times S^1)} \leqslant \|dw\|_{C^0([0,\infty)\times S^1)} < \infty$$

and $dw_\infty = (w_\infty)^*\lambda \otimes X_\lambda$. We also note that both $(w_\infty)^*\lambda$ and $(w_\infty)^*\lambda \circ j$ are bounded harmonic one-forms on $\mathbb{R} \times S^1$. Therefore, they must be written into the forms

$$(w_\infty)^*\lambda = a\, d\tau + b\, dt, \qquad (w_\infty)^*\lambda \circ j = b\, d\tau - a\, dt,$$

where a, b are some constants. Now we show that such a and b are actually related to $\mathcal{T}$ and $\mathcal{Q}$ as follows

Lemma B.1.

$$a = -\mathcal{Q}, \qquad b = \mathcal{T}.$$

Proof. Take an arbitrary point $r \in K$. Using the $C^{1,\alpha}$-convergence of some sequence $w_{s_k}|_{\{r\}\times S^1}$ to $w_\infty|_{\{r\}\times S^1}$, we derive

$$\begin{aligned} b = \int_{\{r\}\times S^1} (w_\infty|_{\{r\}\times S^1})^*\lambda &= \int_{\{r\}\times S^1} \lim_{k\to\infty} (w_{s_k}|_{\{r\}\times S^1})^*\lambda \\ &= \lim_{k\to\infty} \int_{\{r\}\times S^1} (w_{s_k}|_{\{r\}\times S^1})^*\lambda \\ &= \lim_{k\to\infty} \int_{\{r+s_k\}\times S^1} (w|_{\{r+s_k\}\times S^1})^*\lambda. \end{aligned}$$

On the other hand, recalling $w^*d\lambda = (1/2)|d^\pi w|^2$ and applying Stokes' formula and finiteness of the π-energy on $[0,\infty) \times S^1$, the latter becomes

$$\lim_{k\to\infty} \left(T - \frac{1}{2} \int_{[r+s_k,\infty)\times S^1} |d^\pi w|^2 \right) = \mathcal{T} - \lim_{k\to\infty} \frac{1}{2} \int_{[r+s_k,\infty)\times S^1} |d^\pi w|^2 = \mathcal{T}$$

which proves $a = \mathcal{T}$. On the other hand, using the closeness of $w^*\lambda \circ j$ and Stokes' formula, we easily compute

$$\begin{aligned} -a = \int_{\{r\}\times S^1} (w_\infty|_{\{r\}\times S^1})^* \lambda \circ j &= \int_{\{r\}\times S^1} \lim_{k\to\infty} (w_{s_k}|_{\{r\}\times S^1})^* \lambda \circ j \\ &= \lim_{k\to\infty} \int_{\{r\}\times S^1} (w_{s_k}|_{\{r\}\times S^1})^* \lambda \circ j \\ &= \lim_{k\to\infty} \int_{\{r+s_k\}\times S^1} (w|_{\{r+s_k\}\times S^1})^* \lambda \circ j = \mathcal{Q}. \end{aligned}$$

Here in our derivation, we used Remark 9.3. This proves the lemma. □

By the connectedness of $[0,\infty)\times S^1$, the image of w_∞ is contained in a single leaf of the Reeb foliation. If $\gamma : \mathbb{R} \to M$ is a parameterization of the leaf so that it satisfies $\dot\gamma = X_\lambda(\gamma)$, then we can write $w_\infty(\tau,t) = \gamma(s(\tau,t))$, where $s : \mathbb{R}\times S^1 \to \mathbb{R}$ and $s = -\mathcal{Q}\,\tau + \mathcal{T}\,t + c_0$ since $ds = -\mathcal{Q}\,d\tau + \mathcal{T}\,dt$, where c_0 is some constant.

This implies that if $\mathcal{T} \neq 0$ γ defines a closed Reeb orbit of period T. On the other hand, if $T = 0$ but $\mathcal{Q} \neq 0$, we can only conclude that γ is some Reeb trajectory parameterized by $\tau \in \mathbb{R}$.

Remark B.2. Of course, if both $\mathcal{T}$ and $\mathcal{Q}$ vanish, w_∞ is a constant map. In [Oh2], it is shown that such a puncture is a removable singularity under the finiteness of a suitably defined Hofer-type energy.

Appendix C. Sobolev's inequality for the sections of $\mathcal{E}_1 \to \mathbb{R}$

In this section, we give the proof of (10.20) for the sections of the bundle $\mathcal{E}_1 \to \mathbb{R}$ whose fiber is a Hilbert space possibly with infinite dimension.

As in the main text, we assume $\mathcal{E}_2 \subset \mathcal{E}_1$ a pair of Hilbert bundles that satisfies all the properties and is equipped with a compatible connection ∇. We denote by Π_s^τ the parallel transport from the fiber $\mathcal{E}_s$ to $\mathcal{E}_\tau$.

Proposition C.1. *Let $I \subset \mathbb{R}$ be a closed interval and $\zeta : I \to \mathcal{E}_2$ be a smooth section. Then there exists $C_3 = C_3(I) > 0$ depending only on the length $|I|$ of the interval but independent of ζ such that*

$$\|\zeta(\tau)\|_{L^\infty(I,\mathcal{E}_1)} \leqslant C_3 \|\zeta\|_{W^{1,2}(I,\mathcal{E}_1)}.$$

Proof. Thanks to (10.18), there must be a point $\tau_0 \in I_k$ such that

$$|\zeta(\tau_0)|_{\mathcal{E}_1,\tau} \leqslant \frac{1}{\sqrt{|I|}} \|\zeta\|_{L^2(I,\mathcal{E}_1)} \tag{C.1}$$

where $|I|$ is the length of the interval I. Then for any $\tau \in I$, we write

$$\zeta(\tau) - \Pi_{\tau_0}^{\tau}\zeta(\tau_0) = \int_{\tau_0}^{\tau} \Pi_s^{\tau}\nabla_s\zeta(s)\, ds.$$

Therefore, we obtain

$$|\zeta(\tau)|_{\mathcal{E}_1,\tau} \leqslant |\zeta(\tau_0)|_{\mathcal{E}_1,\tau_0} + \int_{\tau_0}^{\tau} |\nabla_s\zeta(s)|_{\mathcal{E}_1,s}\, ds.$$

Applying the Hölder's inequality, we derive

$$\begin{aligned}\int_{\tau_0}^{\tau} |\nabla_s\zeta(s)|_{\mathcal{E}_1,\tau}\, ds &\leqslant \sqrt{|I|}\sqrt{\int_{\tau_0}^{\tau} |\nabla_s\zeta(s)|^2_{\mathcal{E}_1,\tau}\, ds} \\ &\leqslant \sqrt{|I|}\sqrt{\int_I |\nabla_s\zeta(s)|^2_{\mathcal{E}_1,\tau}\, ds} \leqslant \sqrt{|I|}\, \|\nabla_\tau\zeta\|_{L^2(I,\mathcal{E}_1)}\end{aligned}$$

since τ_0, $\tau \in I$. Combining the two, we have obtained

$$|\zeta(\tau)|_{\mathcal{E}_1,\tau} \leqslant \frac{1}{\sqrt{|I|}}\, \|\zeta\|_{L^2(I,\mathcal{E}_1)} + \sqrt{|I|}\, \|\nabla_\tau\zeta\|_{L^2(I,\mathcal{E}_1)}$$

for all $\tau \in I$. By setting $C_3 = 2\max\{\sqrt{|I|}, 1/\sqrt{|I|}\}$, we have finished the proof. ∎

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