J. Austral. Math. Soc. (Series A) **46** (1989), 402–414

L^p-IMPROVING MEASURES ON COMPACT NON-ABELIAN GROUPS

KATHRYN E. HARE

(Received 25 May 1987)

Communicated by J. F. Price

Abstract

A Borel measure μ on a compact group G is called L^p-improving if the operator T_μ: $L^2(G) \to L^2(G)$, defined by $T_\mu(f) = \mu * f$, maps into $L^p(G)$ for some $p > 2$. We characterize L^p-improving measures on compact non-abelian groups by the eigenspaces of the operator T_μ if T_μ is normal, and otherwise by the eigenspaces of $|T_\mu|$. This result is a generalization of our recent characterization of L^p-improving measures on compact abelian groups.

Two examples of Riesz product-like measures are constructed. In contrast with the abelian case one of these is not L^p-improving, while the other is a non-trivial example of an L^p-improving measure.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 43 A 05, 43 A 46.

0. Introduction

A Borel measure μ on a compact group G is said to be L^p-*improving* if for some $1 < p < \infty$ and $\varepsilon > 0$, $\mu * L^p \subsetneq L^{p+\varepsilon}$. Since $\mu * L^1 \subseteq L^1$ and $\mu * L^\infty \subseteq L^\infty$, an application of the Riesz-Thorin interpolation theorem [16, page 179] shows that if μ is L^p-improving then for every $1 < p < \infty$ there exists $\varepsilon > 0$ such that $\mu * L^p \subseteq L^{p+\varepsilon}$. Examples of L^p-improving measures include L^q functions for $q > 1$, the Cantor-Lebesgue measure on the circle [9], and many Riesz products on compact, abelian groups [12].

Stein [15] posed the problem of characterizing L^p-improving measures in terms of their "size". The purpose of this paper is to provide an answer to

This research was partially supported by NSERC Grant A9333.

https://doi.org/10.1017/S1446788700030895 Published online by Cambridge University Press

this question for compact, non-abelian groups G. For the abelian case this was accomplished in [6] where the following theorem was proved.

THEOREM 0. *Let μ be a Borel measure on a compact abelian group G with $\|\mu\| \leq 1$. Let $\hat{G}$ denote the dual group of G. For $\varepsilon > 0$, let $E(\varepsilon) = \{\chi \in \hat{G}: |\hat{\mu}(\chi)| \geq \varepsilon\}$. Let $L^2_{E(\varepsilon)} = \{f \in L^2(G): \operatorname{supp} \hat{f} \subseteq E(\varepsilon)\}$. The following are equivalent.*

(1) *μ is L^p-improving.*

(2) *For some $\alpha \geq 1$ and $p > 2$,*

$$\sup\left\{\frac{\|f\|_p}{\|f\|_2}: f \in L^2_{E(\varepsilon)}(G)\right\} = O(\varepsilon^{-\alpha}).$$

(3) *There is a constant c so that for every $2 < p < \infty$,*

$$\sup\left\{\frac{\|f\|_p}{\|f\|_2}: f \in L^2_{E(\varepsilon)}(G)\right\} = O\left(\frac{1}{\varepsilon} p^{-c \log \varepsilon}\right).$$

Let $T_\mu: L^2(G) \to L^2(G)$ be defined by $T_\mu(f) = \mu * f$. We will show that similar equivalences to those above may be obtained for a normal measure μ on a compact non-abelian group G provided the space $L^2_{E(\varepsilon)}$ is replaced by the subspace of $L^2(G)$ which is the direct sum of the eigenspaces corresponding to eigenvalues λ of T_μ with $|\lambda| \geq \varepsilon$. (If μ is not normal we consider instead the eigenspaces of $|T_\mu|$.) As the eigenvalues of a measure on a compact abelian group are the complex numbers $\{\hat{\mu}(\chi)\}_{\chi \in \hat{G}}$, it is clear that our result, Theorem 1.1, generalizes Theorem 0.

In the second section we will construct two Riesz product-like measures on specific compact non-abelian groups. Theorem 1.1 will be used to show that one of these is a non-trivial example of an L^p-improving measure, while the other, in contrast with the abelian case, is not L^p-improving.

1. Characterization of L^p-improving measures

Let $\mu \in M(G)$ and define $\mu^* \in M(G)$ by $\mu^*(E) = \overline{\mu(E^{-1})}$ for every Borel measurable set $E \subseteq G$. If $T_\mu: L^2(G) \to L^2(G)$ is the operator given by convolution on the left with μ, that is, $T_\mu(f) = \mu * f$, then its adjoint is T_{μ^*}. An operator T is a normal operator if T commutes with T^*. Thus T_μ is a normal operator if and only if $\mu^* * \mu = \mu * \mu^*$, and in this case we will also refer to μ as normal. We will be using the spectral theory for normal bounded operators on a Hilbert space as found in [13].

For a complex number λ and an operator $T: L^2 \to L^2$, let $E(T, \lambda) = \{f \in L^2(G): Tf = \lambda f\}$, and for $\varepsilon \geq 0$ let $F(T, \varepsilon) = \bigoplus_{|\lambda| \geq \varepsilon} E(T, \lambda)$. When the operator T is understood we will write $E(\lambda)$ and $F(\varepsilon)$.

https://doi.org/10.1017/S1446788700030895 Published online by Cambridge University Press

As usual, $\hat{G}$ will denote the dual object of the compact group G, that is, the maximal set of irreducible, inequivalent unitary representations of G. For $\rho \in \hat{G}$, d_ρ is the dimension of ρ, and for $f \in L^2(G)$, $\hat{f}(\rho)$ is the $d_\rho \times d_\rho$ matrix defined by $\hat{f}(\rho) = \int f(x)\rho(x^{-1})\,dx$, where dx is the normalized Haar measure on G. Any $f \in L^2(G)$ can be expressed as $\sum_{\rho\in\hat{G}} d_\rho \operatorname{Tr} \hat{f}(\rho)\rho(x)$, and $\|f\|_2^2 = \sum_{\rho\in\hat{G}} d_\rho \operatorname{Tr} \hat{f}(\rho)\hat{f}(\rho)^*$. A standard reference for representations of compact groups is [7].

THEOREM 1.1. *Let μ be a normal Borel measure on a compact group G, $\|\mu\|_{M(G)} \le 1$. The following are equivalent.*

(1) *μ is L^p-improving.*

(2) *There are constants $p > 2$ and $\alpha \ge 1$ such that*

$$\sup\left\{\frac{\|f\|_p}{\|f\|_2} : f \in F(T_\mu, \varepsilon)\right\} = O(\varepsilon^{-\alpha}).$$

(3) *There is a constant c such that for all $2 < p < \infty$,,*

$$\sup\left\{\frac{\|f\|_p}{\|f\|_2} : f \in F(T_\mu, \varepsilon)\right\} = O\left(\frac{1}{\varepsilon}p^{-c\log\varepsilon}\right).$$

REMARK 1. Recall that $E \subset \hat{G}$ (G abelian or non-abelian) is called a $\Lambda(p)$ *set* for some $p > 2$, if there is a constant c such that $\|f\|_p \le c\|f\|_2$ whenever $f \in L^2(G)$ and $\hat{f}(\rho) = 0$ for $\rho \notin E$. The least such constant c is called the $\Lambda(p)$ constant of E. If for each $\varepsilon > 0$ there is a set $X(\varepsilon) \subseteq \hat{G}$ such that $F(\varepsilon) = \{f \in L^2(G) : \hat{f}(\rho) = 0 \text{ if } \rho \notin X(\varepsilon)\}$, then (3) is equivalent to saying $X(\varepsilon)$ is a $\Lambda(p)$ set for all $2 < p < \infty$, with $\Lambda(p)$ constant $O(\frac{1}{\varepsilon}p^{-c\log\varepsilon})$. This is the terminology used in [6]. If $\mu * f = f * \mu$ for all $f \in L^2(G)$ then $F(\varepsilon)$ is a closed two-sided ideal and hence is of the form described above.

REMARK 2. Although we have defined L^p-improving measures in terms of convolution on the left, our results apply equally to measures acting by convolution on the right. Indeed, for $\mu \in M(G)$ let T^μ denote the operator on $L^2(G)$ defined by $T^\mu(f) = f * \mu$. Since $\|f * \mu\|_p = \|\mu^* * f^*\|_p$ and $\|f\|_2 = \|f^*\|_2$, $T^\mu : L^2 \to L^p$ for some $p > 2$ if and only if $T_{\mu^*} : L^2 \to L^p$. By duality $T_{\mu^*} : L^2 \to L^p$ if and only if $T_\mu : L^{p'} \to L^2$, hence $T^\mu : L^2 \to L^p$ for some $p > 2$ if and only if μ is L^p-improving.

PROOF. ($1 \Rightarrow 2$) Since μ is L^p-improving there exists some $p > 2$ and a constant k such that $\|\mu * f\|_p \le k\|f\|_2$ for all $f \in L^2(G)$.

Suppose $f \in F(\varepsilon)$, say $f = \sum_{|\lambda|\ge\varepsilon} f_\lambda$, with $f_\lambda \in E(\lambda)$. Since T_μ is normal, $\langle f_\lambda, f_\beta\rangle = 0$ if $\lambda \ne \beta$, and thus $\|f\|_2^2 = \sum_{|\lambda|\ge\varepsilon} \|f_\lambda\|_2^2$. Let $g = \sum_{|\lambda|\ge\varepsilon} \frac{1}{\lambda} f_\lambda$.

https://doi.org/10.1017/S1446788700030895 Published online by Cambridge University Press

Observe that

$$\|g\|_2^2 = \sum_{|\lambda|\geq\varepsilon} \frac{1}{|\lambda|^2}\|f_\lambda\|_2^2 \leq \frac{1}{\varepsilon^2}\|f\|_2^2,$$

and therefore $g \in L^2(G)$. Also,

$$T_\mu g = \sum_{|\lambda|\geq\varepsilon} \frac{1}{\lambda} T_\mu(f_\lambda) = \sum_{|\lambda|\geq\varepsilon} \frac{1}{\lambda}(\lambda f_\lambda) = f,$$

and thus,

$$\|f\|_p = \|\mu * g\|_p \leq k\|g\|_2 \leq \frac{k}{\varepsilon}\|f\|_2.$$

To prove ($1 \Rightarrow 3$) we need the following lemma, which is valid for measures which are not necessarily normal.

Lemma 1. *Let $\mu \in M(G)$, $\|\mu\| \leq 1$. Suppose $p > 2$ and $\|\mu * f\|_p \leq c\|f\|_2$ whenever $f \in L^2(G)$. For a non-negative integer n let*

$$p(n) = \frac{p^{n+1}}{2^n}, \qquad s(n) = \sum_{j=0}^{n} \left(\frac{2}{p}\right)^j$$

and μ^n denote the nth convolution power of μ. Then $\|\mu^{n+1} * f\|_{p(n)} \leq c^{s(n)}\|f\|_2$ for all $f \in L^2(G)$.

Proof. The proof is the same as for the abelian case [6].

We will continue to use the notation $p(n)$, $s(n)$ and μ^n defined in the lemma.

($1 \Rightarrow 3$). Assume $\|\mu * f\|_p \leq c\|f\|_2$ whenever $f \in L^2(G)$. Notice that if $f \in E(\lambda)$ then $\mu^{n+1} * f = \lambda^{n+1} f$, thus $F(T_\mu, \varepsilon) \subseteq F(T_{\mu^{n+1}}, \varepsilon^{n+1})$. The proof of $1 \Rightarrow 2$, together with the lemma, shows that

$$\begin{aligned}\sup\left\{\frac{\|f\|_{p(n)}}{\|f\|_2} : f \in F(\varepsilon)\right\} &\leq \sup\left\{\frac{\|f\|_{p(n)}}{\|f\|_2} : f \in F(T_{\mu^{n+1}}, \varepsilon^{n+1})\right\} \\ &\leq c^{s(n)}\varepsilon^{-(n+1)}.\end{aligned}$$

Since $p(n) \to \infty$ as $n \to \infty$, and $\|f\|_q \leq \|f\|_p$ if $q < p$, elementary arguments now complete the proof (cf. [6]).

($3 \Rightarrow 2$) is clear.

One of the differences in the proof of ($2 \Rightarrow 1$) from the abelian case is the necessity of showing that the eigenspaces of T_μ span $L^2(G)$. For abelian groups this fact is obvious. For the non-abelian case we were unable to find a proof in the literature, and so we provide one below. Actually we prove a more general result which we will make use of later.

We recall the definition of a multiplier. The operator $T: L^2(G) \to L^2(G)$ is called a *multiplier* if, for all $\rho \in \hat{G}$, there is a $d_\rho \times d_\rho$ matrix A_ρ with $\widehat{Tf}(\rho) = A_\rho \hat{f}(\rho)$. We will denote A_ρ by $\hat{T}(\rho)$.

LEMMA 2. *Let T be a normal operator on $L^2(G)$ which is a multiplier. Then $L^2(G) = \bigoplus_{\lambda\in\mathbf{C}} E(T,\lambda)$.*

REMARK. If $f \in L^2(G)$ and $f = \sum f_\lambda$ where $f_\lambda \in E(T,\lambda)$, then at most countably many of the functions f_λ are non-zero.

PROOF. Fix $\rho \in \hat{G}$. Since G is compact and T is normal, $\hat{T}(\rho)$ is a finite dimensional normal matrix and thus has a complete set of eigenvectors $\{e_i\}_{i=1}^{d_\rho}$ corresponding to the eigenvalues $\{\lambda_i\}_{i=1}^{d_\rho}$. Let E_{ij} be the matrix with e_i its jth column, and zero elsewhere. Observe that $T(\operatorname{Tr} E_{ij}\rho(x)) = \operatorname{Tr}\hat{T}(\rho)E_{ij}\rho(x) = \lambda_i \operatorname{Tr} E_{ij}\rho(x)$, thus $\operatorname{Tr} E_{ij}\rho(x) \in \bigoplus_{\lambda\in\mathbf{C}} E(T,\lambda)$. Let $f \in L^2(G)$. Since there exist scalars c_{ij} such that $\hat{f}(\rho) = \sum_{i,j=1}^{d_\rho} c_{ij}E_{ij}$, $\operatorname{Tr}\hat{f}(\rho)\rho(x) \in \bigoplus_{\lambda\in\mathbf{C}} E(T,\lambda)$. By linearity and closure it follows that $f \in \bigoplus_{\lambda\in\mathbf{C}} E(T,\lambda)$ and hence $L^2(G) = \bigoplus_{\lambda\in\mathbf{C}} E(T,\lambda)$.

LEMMA 3. *Suppose $L^2(G) = A \oplus A^\perp$ and assume that for some $p > 2$ and constant c, $\sup\{\|f\|_p/\|f\|_2 \colon f \in A\} \le c$. Let $1/p + 1/p' = 1$. If P is the projection onto A with kernel $A^\perp$, then, whenever $f \in L^2(G)$, $\|Pf\|_2 \le c\|f\|_{p'}$.*

PROOF. The hypothesis implies that the identity map $\mathrm{Id}\colon A \subseteq L^2(G) \to L^p(G)$ is bounded with norm at most c. Its adjoint, the quotient map $Q\colon L^{p'} \to L^2/A^\perp$, has the same norm. But

$$\begin{aligned}\|f\|_{L^2/A^\perp} &= \inf\{\|g\|_2 \colon g - f \in A^\perp\}\\ &= \inf\{\|g\|_2 \colon P(g-f) = 0\} = \|Pf\|_2.\end{aligned}$$

LEMMA 4. *Let $\mu \in M(G)$, not necessarily normal, and suppose $(\mu^* * \mu)^n$ is L^p-improving. Then μ is L^p-improving.*

PROOF. Let $|T_\mu|$ be the positive square root of $T_{\mu^*}T_\mu = |T_\mu|^2$. For $\operatorname{Re} z \ge 0$ we will let $S_z = \int_{\sigma(|T_\mu|^2)} \lambda^{zn}\, dE$, where E is the spectral measure of the normal operator $|T_\mu|^2$ and $\sigma(|T_\mu|^2)$ is the spectrum of $|T_\mu|^2$. Clearly, $z \mapsto S_z$ is continuous if $\operatorname{Re} z \ge 0$ and analytic if $\operatorname{Re} z > 0$.

Since $\sigma(|T_\mu|^2)$ is a subset of $\mathbf{R}$, (indeed a subset of $\mathbf{R}^+$), it follows from spectral theory that $S_{1+iy}^* S_{1+iy} = |T_\mu|^{4n}$, $S_{iy}^* S_{iy} = I$ and $S_{(1/2n)} = |T_\mu|$.

Suppose a constant c and $p' < 2$ are chosen so that $\|(\mu^* * \mu)^n * f\|_2^2 \le c\|f\|_{p'}^2$ whenever $f \in L^2(G)$. Then, for any $f \in L^2(G)$,

$$\begin{aligned}\|S_{1+iy}f\|_2^2 &= \langle S_{1+iy}^* S_{1+iy} f, f\rangle = \|\,|T_\mu|^{2n} f\|_2^2\\ &= \|(u^* * \mu)^n * f\|_2^2 \le c\|f\|_{p'}^2.\end{aligned}$$

Similarly, $\|S_{iy}f\|_2^2 = \|f\|_2^2$.

https://doi.org/10.1017/S1446788700030895 Published online by Cambridge University Press

By Stein's analytic interpolation theorem [16, page 205] we may conclude that for some $q < 2$ and constant k, $\|S_{(1/2n)}f\|_2^2 \le k\|f\|_q^2$ whenever $f \in L^2(G)$. But

$$\|S_{(1/2n)}f\|_2^2 = \langle |T_\mu|^2 f, f\rangle = \|\mu * f\|_2^2,$$

so μ is L^p-improving.

Proof of Theorem 1.1, continued. $(2 \Rightarrow 1)$ Let $E_n = \bigoplus_{2^{-n}<|\lambda|\le 2^{-n+1}} E(\lambda)$ for $n = 1, 2, 3, \dots$, and $E_0 = E(0)$. If $\|\mu\| \le 1$ and λ is an eigenvalue of μ, then clearly $|\lambda| \le 1$. Thus Lemma 2 shows that $L^2(G) = \bigoplus_{n=0}^{\infty} E_n$.

Let $f \in L^2(G)$, say $f = \sum_{n=0}^{\infty} f_n$, with $f_n \in E_n$. By the assumption of (2) there are constants $p > 2$, $\alpha \ge 1$ and c so that whenever $n \ge 1$,

$$\sup\left\{\frac{\|f\|_p}{\|f\|_2} : f \in E_n\right\} \le 2^{n\alpha}c.$$

The spaces $\{E_n\}_{n=0}^{\infty}$ are orthogonal, and thus by Lemma 3 $\|f_n\|_2 \le 2^{n\alpha}c\|f\|_{p'}$ whenever $f \in L^2(G)$ and $n \ge 1$.

Let N be a positive integer and $g \in E(\lambda)$. Then $(\mu^* * \mu)^N * g = |\lambda|^{2N} g \in E(\lambda)$ and hence the spaces $(\mu^* * \mu)^N * E_n$ are orthogonal. Since $(\mu^* * \mu)^N * f_0 = 0$ it follows that

$$\begin{aligned}\|(\mu^* * \mu)^N * f\|_2^2 &= \sum_{n=1}^{\infty} \|(\mu^* * \mu)^N * f_n\|_2^2 \\ &\le \sum_{n=1}^{\infty} \frac{1}{2^{4N(n-1)}}\|f_n\|_2^2 \le \sum_{n=1}^{\infty} \frac{1}{2^{4N(n-1)}} 2^{2n\alpha}c^2\|f\|_{p'}^2 \\ &\le c'\|f\|_{p'}^2\end{aligned}$$

for some constant c', provided N is sufficiently large. Thus $(\mu^* * \mu)^N$ is L^p-improving for N sufficiently large and hence, by Lemma 4, so is μ.

Corollary 1.2. *Let $\mu \in M(G)$, $\|\mu\|_{M(G)} \le 1$. Let $|T_\mu|$ denote the operator on $L^2(G)$ which is the positive square root of $T_{\mu^*}T_\mu$. The following are equivalent.*

(1) *μ is L^p-improving.*

(2) *There are constants $p > 2$ and $\alpha \ge 1$ so that*

$$\sup\left\{\frac{\|f\|_p}{\|f\|_2} : f \in F(|T_\mu|, \varepsilon)\right\} = O(\varepsilon^{-\alpha}).$$

(3) *There is a constant c so that for all $2 < p < \infty$*

$$\sup\left\{\frac{\|f\|_p}{\|f\|_2} : f \in F(|T_\mu|, \varepsilon)\right\} = O\left(\frac{1}{\varepsilon}p^{-c\log\varepsilon}\right).$$

https://doi.org/10.1017/S1446788700030895 Published online by Cambridge University Press

PROOF. We remark that the operator $S: L^2(G) \to L^2(G)$ defined by $\widehat{Sf}(\rho) = |\hat{\mu}(\rho)|\hat{f}(\rho)$, for $|\hat{\mu}(\rho)|$ the positive square root of $\hat{\mu}(\rho)^*\hat{\mu}(\rho)$, is a positive operator whose square is $T_{\mu^*}T_\mu$. Thus $S = |T_\mu|$ and so $|T_\mu|$ is a normal operator on $L^2(G)$ which is a multiplier. Certainly $||\mu * f||_2 = || \, |T_\mu| f||_2$, hence there exists some $p > 2$ such that $\mu: L^{p'} \to L^2$ if and only if $|T_\mu|: L^{p'} \to L^2$ if and only if (by duality), $|T_\mu|: L^2 \to L^p$.

These comments, together with the method of proof of Theorem 1.1 ($1 \Rightarrow 2$) yield ($1 \Rightarrow 2$) of the corollary.

If $||\mu * f||_p \leq c||f||_2$ for all $f \in L^2(G)$ then the same inequality holds with μ replaced by $\mu^* * \mu$. Lemma 1 shows that

$$||(\mu^* * \mu)^{n+1} * f||_{p(n)} = || \, |T_\mu|^{2(n+1)} f||_{p(n)} \leq c^{s(n)} ||f||_2$$

where $p(n)$ and $s(n)$ are as defined in the lemma. Thus ($1 \Rightarrow 3$) can be proved in the same manner as it was in Theorem 1.1.

Certainly ($3 \Rightarrow 2$) is clear.

Since $|T_\mu|$ is a normal multiplier and Lemma 2 applies to normal multipliers, similar arguments to those used in the Theorem for the proof of ($2 \Rightarrow 1$) show that assumption (2) of the corollary yields that $|T_\mu|^{2N} = (\mu^* * \mu)^N$ is L^p-improving for N sufficiently large. Lemma 4 now shows that μ is L^p-improving.

2. Examples

In this section we will consider two families of examples of measures which correspond in some way to Riesz products. Theorem 1.1 will be used to show that each of the measures in one family is L^p-improving, while in contrast to the abelian case, none of the measures in the other family is L^p-improving.

2.1. A non-L^p-improving measure. Let $G_n = SU(2)$, $G = \prod_{n=1}^{\infty} G_n$. For each $n = 1, 2, \ldots$ let u_n be an irreducible representation of G of degree n (cf. [7, Section 29]). Let π_n be the projection of G onto G_n and let $\sigma_n = u_n \circ \pi_n$. Let $E = \{\sigma_n\}_{n=1}^{\infty}$. In [10] it is shown that E is a central Sidon set, hence there exists a measure μ which commutes with all measures, (such a measure is called central) such that $\hat{\mu}(\sigma_n) = I_n/2$ for $\sigma_n \in E$, I_n being the $n \times n$ identity matrix.

Now μ is a normal measure and

$$F\left(T_\mu, \tfrac{1}{2}\right) \supseteq \{f \in L^2(G): \operatorname{supp} \hat{f} \subseteq E\}.$$

It is shown in [10] that E is not a $\Lambda(4)$ set so Theorem 1.1(3) (see particularly the remark after Theorem 1.1) is not satisfied. Consequently, no such measure μ is L^p-improving.

The set E is a central Sidon set because it satisfies an independence property defined in [10], which independent characters in the dual group of a compact abelian group also satisfy. A measure μ on a compact abelian group which satisfies $\hat{\mu}(\chi_i) = \frac{1}{2}$ for all χ_i in a set of independent characters is the Riesz product $\mu = \prod[1+\frac{1}{2}(\chi_i(x)+\chi_i^{-1}(x))]$. In [12] it is shown that this measure is L^p-improving. Thus our first example is in contrast with the abelian case.

More generally, if G is a compact, non-abelian group whose dual object has no infinite sets which are $\Lambda(p)$ for all $2 < p < \infty$, then any central measure μ which is L^p-improving must satisfy $\hat{\mu}(\rho) = c_\rho I_{d_\rho}$ with $c_\rho \to 0$. This follows from Theorem 1.1(3) since, if μ is L^p-improving, $\{\rho\colon |c_\rho| \geq \varepsilon\}$ is a $\Lambda(p)$ set for all $\varepsilon > 0$ and for all $2 < p < \infty$.

Of course not all central measures μ satisfying $\|\hat{\mu}(\rho)\|_\infty \to 0$ are L^p-improving (for any compact infinite group G). Not even all central L^1 functions can be L^p-improving, for if so then a Baire category theorem argument as given in [5, Lemma 1.5] and the closed graph theorem show that there is some $1 < p < 2$ and constant C such that

$$\|f * g\|_2 \leq C\|f\|_1\|g\|_p$$

for all $g \in L^p$ and all central L^1 functions f. By taking $f = f_\alpha$ where $\{f_\alpha\}$ is a central bounded approximate identity for $L^1(G)$ we obtain a contradiction.

2.2. An L^p-improving Riesz product. An application of Young's inequality shows that any $L^q(G)$ function, $q > 1$, is L^p-improving. Such measures have the property that $\|\hat{f}(\rho)\|_\infty \to 0$. We construct here a family of examples of L^p-improving measures μ with $\limsup \|\hat{\mu}(\rho)\|_\infty > 0$.

Let U_n be the group of unitary matrices of degree d_n, and let $G = \bigoplus U_n$. The projection $\pi_n\colon G \to U_n$ defines an irreducible representation of G of degree d_n. It follows that if $\otimes$ denotes the tensor product, then

$$\{\pi_{i_1}^{\varepsilon_1} \otimes \cdots \otimes \pi_{i_k}^{\varepsilon_k} \colon i_1 < i_2 < \cdots < i_k,\ \varepsilon_i = \pm 1 \text{ for } i = 1, \ldots, k\}$$

are non-equivalent irreducible representations of G. Observe that $\pi_i^{-1} \sim \bar{\pi}_i$ since $\pi_i(x)$ is unitary for every $x \in G$ and thus $\operatorname{Tr}\pi_i^{-1}(x) = \overline{\operatorname{Tr}\pi_i(x)}$.

Let $f_N(x) = \prod_{i=1}^N[1 + \frac{d_i}{2}\operatorname{Tr}[A_i\pi_i(x) + \bar{A}_i\bar{\pi}_i(x)]]$ and suppose

$$\max_i\{\operatorname{Tr}|\bar{A}_i|, \operatorname{Tr}|A_i|\} \leq 1/d_i.$$

Since $\operatorname{Tr}(A_i\pi_i(x)+\bar{A}_i\bar{\pi}_i(x))$ is real valued and $|\operatorname{Tr}(A_i\pi_i(x)+\bar{A}_i\bar{\pi}_i(x))| \leq 2/d_i$, f_N is non-negative. Clearly $\hat{f}_N(1) = 1$. It follows that $\{f_N\}_{N=1}^\infty$ converges

https://doi.org/10.1017/S1446788700030895 Published online by Cambridge University Press

weak* to a measure μ where

$$\hat{\mu}(\rho) = \begin{cases} 1 & \text{if } \rho = 1, \\ \frac{1}{2^k} A_{i_1}^{\varepsilon_1} \otimes \cdots \otimes A_{i_k}^{\varepsilon_k} & \text{if } \rho = \pi_{i_1}^{\varepsilon_1} \otimes \cdots \otimes \pi_{i_k}^{\varepsilon_k},\ \varepsilon_i = \pm 1, \\ & i_1 < i_2 < \cdots < i_k, \\ 0 & \text{else.} \end{cases}$$

Here we write $A_{i_j}^{-1}$ to denote $\overline{A_{i_j}}$.

We will call such a measure μ a *Riesz product* as this is clearly a generalization of the notion of Riesz products on abelian groups.

THEOREM 2.1. *Let U_n be the group of unitary matrices of dimension d_n and let $G = \bigoplus_{n=1}^{\infty} U_n$. Suppose $\sup_n d_n < \infty$. Then any Riesz product on G is L^p-improving.*

REMARK. If $\sup d_i$ is infinite and μ is the Riesz product defined by $\hat{\mu}(\pi_i) = I/(2d_i)$, then $\limsup \|\hat{\mu}(\rho)\| = 0$. If $\sup d_i \le d < \infty$ and μ is defined in the same manner, then $\limsup \|\hat{\mu}(\rho)\| \ge 1/(2d)$. This is the interesting case.

The proof of Theorem 2.1 follows from Theorem 1.1 and

THEOREM 2.2. *Let U_n be the group of unitary matrices of dimension d_n and let $G = \bigoplus_{n=1}^{\infty} U_n$. Suppose $\sup_n d_n < \infty$. Let k be any positive integer and let*

$$X_k = \{\pi_{i_1}^{\varepsilon_1} \otimes \cdots \otimes \pi_{i_k}^{\varepsilon_k} : i_1 < i_2 < \cdots < i_k,\ \varepsilon_i = \pm 1,\ i = 1, \ldots, k\}.$$

Then, for any integer $s \ge 2$, X_k is a $\Lambda(2s)$ set with $\Lambda(2s)$ constant at most $A^k s^{k/2}$ for some constant A independent of k and s. The set $\bigcup_{k=1}^{n} X_k$ is a $\Lambda(2s)$ set with constant at most $(2A)^n s^{n/2}$.

We will first prove Theorem 2.1 assuming Theorem 2.2, and then we will prove Theorem 2.2.

PROOF OF THEOREM 2.1. As was shown in the proof of Corollary 1.2, $|T_\mu|^\wedge(\rho) = |\hat{\mu}(\rho)|$, thus, since $|A_{i_1}^{\varepsilon_1} \otimes \cdots \otimes A_{i_k}^{\varepsilon_k}| = |A_{i_1}^{\varepsilon_1}| \otimes \cdots \otimes |A_{i_k}^{\varepsilon_k}|$,

$$|T_\mu| = \prod \left(1 + \frac{d_i}{2} \operatorname{Tr}(|A_i| \pi_i(x) + |\bar{A}_i| \bar{\pi}_i(x))\right).$$

Suppose $|T_\mu| f = \lambda f$ with $\lambda \ge 1/2^k$. Because the eigenvalues of $|A_i|$ and $|\bar{A}_i|$ are non-negative and at most one, (indeed, they are at most $1/d_i$), $\operatorname{supp} \hat{f} \subseteq \bigcup_1^k X_j$. Thus

$$\sup \left\{ \frac{\|f\|_8}{\|f\|_2} : f \in F\left(|T_\mu|, \frac{1}{2^k}\right) \right\} \le (2A)^k 2^k \le 2^{k\alpha},$$

if α is chosen so that $4A \le 2^\alpha$. Since (2) of Corollary 1.2 is fulfilled, μ is L^p-improving.

The proof of Theorem 2.2 is very similar to the proof that $\{\pm 3^{j_1} \pm \cdots \pm 3^{j_k} : j_1 < j_2 < \cdots < j_k\}$ is a $\Lambda(2s)$ set in $\mathbb{Z}$ with $\Lambda(2s)$ constant $A^k s^{k/2}$ [1]. We will state the necessary lemmas and sketch the proofs primarily indicating the differences which arise in the non-abelian case. Our standing assumption is that G satisfies the hypothesis of Theorem 2.2. We will let $d = \sup d_i$.

Proof of Theorem 2.2. First we remark that the final statement of Theorem 2.2 is immediate from the first part, since elementary calculations show that the union of N $\Lambda(p)$ sets with $\Lambda(p)$ constants at most c, is a $\Lambda(p)$ set with constant at most $2^N c$.

For a positive integer s, π_i^s will denote π_i tensor product with itself s times. The notation $\langle \gamma, \beta \rangle$ will denote $\int_G \operatorname{Tr} \gamma(x) \overline{\operatorname{Tr} \beta(x)}\, dx$.

Lemma 1. *Let $\{s_i\}_{i=1}^n$ and $\{t_i\}_{i=1}^m$ be sets of positive integers. Let $\gamma = \pi_{i_1}^{s_1} \otimes \cdots \otimes \pi_{i_n}^{s_n}$ and $\beta = \pi_{j_1}^{t_1} \otimes \cdots \otimes \pi_{j_m}^{t_m}$, $i_1 < i_2 < \cdots < i_n$, $j_1 < j_2 < \cdots < j_m$. Then $\langle \gamma, \beta \rangle = 0$, (that is, γ and β have no equivalent subrepresentations), unless $m = n$, and upon some reordering, $i_k = j_k$ and $s_k = t_k$ for all $k = 1, \ldots, n$.*

Proof. This can essentially be found in [4].

Lemma 2. *Let $E_k = \{\pi_{i_1} \otimes \cdots \otimes \pi_{i_k} : i_1 < i_2 < \cdots < i_k\}$. Then E_k is a $\Lambda(2s)$ set with constant at most $s^{k/2} d^{5k/2}$.*

Remark. As in the abelian case, E_k is a Sidon set if and only if $k = 1$ ([3], [4]).

Proof. Lemma 1 shows that for any representation γ,

$$\left| \left\{ \rho_1 \otimes \cdots \otimes \rho_s : \rho_i \in E_k,\ \left\langle \bigotimes_{i=1}^{s} \rho_i, \gamma \right\rangle \neq 0 \right\} \right| \le \frac{(sk)!}{(k!)^s} \le s^{ks}.$$

This result, together with the fact that $\sup d_i = d < \infty$ shows that we may apply [2, Section 3] to obtain

$$\|f\|_{2s} \le d^{3k/2} s^{k/2} \|f\|_2$$

for all central polynomials $f \in L^2_{E_k}(G)$. It follows from [11, Lemma 6] that for any $f \in L^2_{E_k}(G)$, say $f = \sum_{\rho \in E_k} d_\rho \operatorname{Tr} A_\rho \rho(x)$,

$$\|f\|_{2s} \le s^{k/2} d^{3k/2} \left(\sum_{\rho \in E_k} \|d_\rho \operatorname{Tr} A_\rho \rho(x)\|_{2s}^2 \right)^{1/2}.$$

Again using the fact that $\sup d_i = d$, one can easily see that for $\rho \in E_k$,

$$\|d_\rho \operatorname{Tr} A_\rho \rho(x)\|_{2s} \le d^k \|d_\rho \operatorname{Tr} A_\rho \rho(x)\|_2.$$

Combining these observations we have $\|f\|_{2s} \le s^{k/2} d^{5k/2} \|f\|_2$ whenever $f \in L^2_{E_k}(G)$ which establishes the lemma.

Lemma 3. *There exists a constant A so that for each $k \in \mathbf{Z}^+$ and for every $\omega = (\omega_i) \in G$, there is a measure $\mu_\omega \in M(G)$ satisfying $\|\mu_\omega\| \le (4d^3A)^k$ and $\hat{\mu}_\omega(\bigotimes_{i\in J} \pi_i^{\varepsilon_i}) = \bigotimes_{i\in J} \omega_i$ whenever $|J| = k$, $\varepsilon_i = \pm 1$.*

Proof. The method of proof of [8, 5.11] can be used to construct, for given $k \in \mathbf{Z}^+$ and $\omega \in G$, d_m dimensional matrices z_m^l, $l = 1, \dots, k$, such that each entry of z_m^l is real and at most $1/(2d^3)$ in absolute value, and for each set of indices J, $|J| = k$,

$$\bigotimes_{m\in J} \omega_m = (4d^3)^k \sum_{l=0}^{k} c_l \bigotimes_{m\in J} z_m^l.$$

Let $\mu_\omega^l = \prod_m [1 + d_m \operatorname{Tr}[z_m^l(\pi_m(x) + \bar{\pi}_m(x))]]$. The choice of z_m^l ensures that μ_ω^l is a Riesz product and $\|\mu_\omega^l\| \le 1$. Certainly $\widehat{\mu_\omega^l}(\bigotimes_{i\in J} \pi_i^{\varepsilon_i}) = \bigotimes_{i\in J} z_i^l$. Let $\mu_\omega = \sum_{l=0}^k (4d^3)^k c_l \mu_\omega^l$.

One can see from [8, 5.11] that $\sum_{l=0}^k |c_l| \le A^k$ for some A independent of k. Thus $\|\mu_\omega\| \le (4d^3A)^k$ and $\hat{\mu}_\omega(\bigotimes_{i\in J} \pi_i^{\varepsilon_i}) = \sum_{l=0}^k (4d^3)^k c_l \bigotimes_{i\in J} z_i^l = \bigotimes_{i\in J} \omega_i$.

Proof of Theorem 2.2, continued. We now proceed essentially as in [8, 5.12]. Let $f \in L^2(G)$, with $\hat{f}(\rho) = 0$ if $\rho \notin X_k$ and suppose $\hat{f}(\rho) \ne 0$ for only finitely many $\rho \in \hat{G}$. For each $\omega \in G$ let $\mu_\omega \in M(G)$ be as in Lemma 3. Define F on $G \times G$ by $F(\omega, \omega') = \mu_\omega * f(\omega')$.

If we let $\psi_\rho = \bigotimes_{i\in J} \pi_i$ when $\rho = \bigotimes_{i\in J} \pi_i^{\varepsilon_i}$ then, since $\hat{\mu}_\omega(\rho) = \bigotimes_{i\in J} \omega_i = \psi_\rho(\omega)$,

$$\begin{aligned} \mu_\omega * f(\omega') &= \sum_{\rho\in X_k} d_\rho \operatorname{Tr} \hat{\mu}_\omega(\rho) \hat{f}(\rho) \rho(\omega') \\ &= \sum_{\rho\in X_k} d_\rho \operatorname{Tr} \psi_\rho(\omega) \hat{f}(\rho) \rho(\omega') \\ &= \sum_{\rho\in X_k} d_\rho \operatorname{Tr}(\hat{f}(\rho)\rho(\omega')) \psi_\rho(\omega). \end{aligned}$$

Let $F_{\omega'}: G \to \mathbf{C}$ be given by $F_{\omega'}(\omega) = F(\omega, \omega')$.

The display above shows that $\hat{F}_{\omega'}(\rho) = 0$ if $\rho \notin E_k$ and

$$\begin{aligned}
d_\gamma \operatorname{Tr} |\hat{F}_{\omega'}(\gamma)|^2 &= d_\gamma \operatorname{Tr} \left(\sum_{\psi_\rho = \gamma} \hat{f}(\rho)\rho(\omega') \right) \left(\sum_{\psi_\rho = \gamma} \hat{f}(\rho)\rho(\omega') \right)^* \\
&\leq 2^{2k} d_\gamma \sum_{\psi_\rho = \gamma} \operatorname{Tr} \hat{f}(\rho)\rho(\omega')\rho(w')^* \hat{f}(\rho)^* \\
&= 2^{2k} d_\rho \sum_{\psi_\rho = \gamma} \operatorname{Tr} \hat{f}(\rho)\hat{f}(\rho)^*.
\end{aligned}$$

Thus $||F_{\omega'}||_2 \leq 2^k ||f||_2$. The proof is completed as in the abelian case [8, 5.12].

References

[1] A. Bonami, 'Étude des coefficients de Fourier des fonctions de $L^p(G)$', *Ann. Inst. Fourier (Grenoble)* **20** (1970), 335–402.

[2] G. Benke, 'On the hypergroup structure of central $\Lambda(p)$ sets', *Pacific J. Math.* **50** (1974), 19–27.

[3] D. Cartwright and J. McMullen, 'A structural criterion for the existence of infinite Sidon sets', *Pacific J. Math.* **96** (1981), 301–317.

[4] A. Figa-Talamanca and D. Rider, 'A theorem of Littlewood and lacunary series for compact groups', *Pacific J. Math.* **16** (1966), 505–514.

[5] C. Graham, K. Hare and D. Ritter, 'The size of L^p-improving measures', *J. Funct. Anal.*, to appear.

[6] K. E. Hare, 'A characterization of L^p-improving measures', *Proc. Amer. Math. Soc.* **102** (1988), 295–299.

[7] E. Hewitt and K. Ross, *Abstract harmonic analysis*, Vol. II (Springer-Verlag, Berlin-Heidelberg-New York, 1979).

[8] J. Lopez and K. Ross, *Sidon sets*, (Lecture Notes in Pure and Applied Mathematics, 13, Marcel Dekker, New York, 1975).

[9] D. Oberlin, 'A convolution property of the Cantor-Lebesgue measure', *Colloq. Math.* **67** (1982), 113–117.

[10] W. Parker, 'Central Sidon and central $\Lambda(p)$ sets', *J. Austral. Math. Soc.* **14** (1972), 62–74.

[11] D. Rider, 'Central lacunary sets', *Monatsh. Math.* **76** (1972), 328–338.

[12] D. Ritter, 'Most Riesz product measures are L^p-improving', *Proc. Amer. Math. Soc.* **97** (1986), 291–295.

[13] W. Rudin, *Functional analysis*, (McGraw-Hill, New York, 1973).

[14] W. Rudin, 'Trigonometric series with gaps', *J. Math. Mech.* **9** (1960), 203–227.

[15] E. M. Stein, 'Harmonic analysis on R^n', *Studies in harmonic analysis*, pp. 97–135 (M.A.A. Studies Series 13, J. M. Ash ed., 1976).
[16] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces* (Princeton University Press, Princeton, N.J., 1971).

Department of Pure Mathematics
University of Waterloo
Waterloo, Ontario N2L 3G1
Canada