LETTERS TO THE EDITOR

Dear Editor,

A solvable model for a finite-capacity queueing system

Giorno et al. [3] have recently derived explicit results for the N-capacity system with birth and death rates $\lambda_n = \lambda(N-n)$ $(n = 0, 1, \cdots, N-1)$ and $\mu_n = \mu n$ $(n = 1, 2, \cdots, N)$ respectively. An equivalent birth–death process, though not of course an equivalent queue, is obtained by considering N independent $M/M/1/1$ queues each of which has $\lambda_0 = \lambda$, $\mu_1 = \mu$. The equivalent process provides immediate insight for the transient solution, the moments and the stationary binomial distribution. A similar comparison is made in Conolly [1] of $M/M/\infty$ and a single-server queue, with $\lambda_n = \lambda$ and $\mu_n = \mu n$.

The interarrival-time probability density function (p.d.f.), $a_n(t)$, can also be derived using a direct probability argument for the system of N queues. For an $M/M/1/1$ queue the waiting time until a busy queue next returns to the busy state is the sum of independent random variables with p.d.f.'s $\mu \exp(-\mu t)$ and $\lambda \exp(-\lambda t)$ respectively. When $\lambda \neq \mu$ this is a generalised Erlang (or hyperexponential) distribution. $a_n(t)$ is thus a p.d.f. for the minimum of n independent generalised Erlangs and $N-n$ independent exponentials.

The above alternative argument for deriving $a_n(t)$ works only for the specific (λ_n, μ_n) used in the binomial model considered in [3]. For general rates $a_n(t)$ is from (2.2) of [3] the p.d.f. of a finite-mixture of generalised Erlangs. $a_n(t)$ can thus be expressed as a general exponential mixture density. This result is perhaps best demonstrated using Laplace transforms.

Let

$$p_n = \frac{\lambda_n}{\nu_n} \qquad (n = 0, 1, 2, \cdots, N-1)$$

$$q_n = \frac{\mu_n}{\nu_n} \qquad (n = 1, 2, 3, \cdots, N) \text{ (where } p_0 = q_N = 1)$$

be the conditional probabilities of a move to the right or left. Also let $f_n(t) = \nu_n \exp(-\nu_n t)$ be the p.d.f. of the dwell time in state n, with Laplace

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transform $f_n(s)$. If $\alpha_n(s)$ is the Laplace transform of $a_n(t)$, then (again using equations as in [3]) (2.1) has a transform version

$$\alpha_0(s) = f_0(s)$$

$$\alpha_n(s) = p_n f_n(s) + q_n p_{n-1} f_n(s) f_{n-1}(s) + \cdots + q_n q_{n-1} \cdots q_1 f_n(s) f_{n-1}(s) \cdots f_0(s)$$

$$(n = 1, 2, \cdots, N-1)$$

$$\alpha_N(s) = f_N(s) a_{N-1}(s).$$

Since $f_n(s) = \nu_n/(\nu_n + s)$, $\alpha_n(s)$ can be expressed as a linear combination of $f_n(s), f_{n-1}(s) \cdots f_0(s)$,

$$\alpha_n(s) = \sum_{j=0}^{n} \theta_{n-j} f_{n-j}(s)$$

where the weights θ_{n-j} can be computed for any specified rates (λ_n, μ_n). A generalisation of (2.9) follows, namely

$$E[T_n^k] = \sum_{j=0}^{n} \frac{\theta_{n-j}}{\nu_{n-j}^k}$$

(assuming the ν_{n-j} are distinct). Modification is required in the case of non-distinct ν_{n-j}. The effective interarrival time in (3.2) can also be written as a general exponential mixture.

An N-capacity queue is a finite random walk (with exponential dwell times and state dependent transition probabilities). Results concerning service intervals can thus be deduced from the reversed process with $\lambda_n^* = \mu_{N-n}$, $\mu_n^* = \lambda_{N-n}$. The non-congestion condition from (3.9c) is, for any rates (λ_n, μ_n),

$$\rho = \frac{1 - \bar{p}_0}{1 - \bar{p}_N} < 1.$$

This condition means that the product of the arrival rates has to be less than the product of the service rates, which is informative about possible forms of stationary distribution. See Keilson [4], p. 72 for discussion.

The finite nature of the system also allows the mean (and variance) of B, the busy period, given in (3.13) to be calculated very quickly for any rates. B is the time taken to go from $N-1$ to N for the reversed process. Again using standard results from [4], p. 61,

$$E[B] = \frac{1 - \bar{p}_N}{\lambda_{N-1} \bar{p}_{N-1}}$$

(where $\bar{p}_j$ and λ_j are the values for the reversed process). The variance can also

be calculated in this way. In the binomial case even the above argument is not required. For the N-queue system the state space can be redefined as $F \equiv$ all queues free and the complementary event $\bar{F}$. The system is then an alternating renewal process between the states F and $\bar{F}$, with the dwell time in F being exponential with mean $1/N\lambda$. Assume, without loss of generality, that the system begins in F at $t = 0$, then standard renewal theory arguments (Cox [2], p. 83) give the explicit form of the Laplace transform of the busy period. If only the mean is required, from [2], p. 84,

$$\bar{p}_0 = \frac{1}{(1+\rho)^N} = \frac{1/N\lambda}{E[B] + 1/N\lambda} \; .$$

Although the queuing systems differ, an heuristic argument for the mean waiting time in (3.13) is at least possible using the analogy between systems. Input and output rates are the same for both systems. For the N-queue system the waiting time before starting service is 0 and the mean time spent in the system is $1/\mu$. The value of $E[w] = 1/\mu$ could thus perhaps be anticipated because of the identical birth and death processes.

The above observations are offered very much in the spirit of the comparisons used in [1]. Alternative probabalistic arguments are given which appear to be useful, as well as being of interest in their own right.

References

[1] CONOLLY, B. W. (1975) *Queueing Systems.* Ellis Horwood, Chichester.
[2] COX, D. R. (1962) *Renewal Theory.* Methuen, London.
[3] GIORNO, V., NEGRI, C. AND NOBILE, A. G. (1985) A solvable model for a finite-capacity queueing system. *J. Appl. Prob.* **22**, 903–911.
[4] KEILSON, J. (1979) *Markov Chain Models — Rarity and Exponentiality.* Applied Mathematical Sciences Series **28**, Springer-Verlag, Berlin.

University of St Andrews   Yours sincerely,

                                                                          J. NEWTON

Dear Editor,

Comments on a paper by V. Giorno, C. Negri and A. G. Nobile [1]

The model described in [1] appears to be a new interpretation of a model which is well known in queueing theory. Some of the definitions in the paper