J. Austral. Math. Soc. Ser. B 35(1994), 348–365

STABILITY ANALYSIS FROM FOURTH ORDER EVOLUTION EQUATION FOR SMALL BUT FINITE AMPLITUDE INTERFACIAL WAVES IN THE PRESENCE OF A BASIC CURRENT SHEAR

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(Received 15 April 1992; revised 23 July 1992)

Abstract

A fourth-order nonlinear evolution equation is derived for a wave propagating at the interface of two superposed fluids of infinite depths in the presence of a basic current shear. On the basis of this equation a stability analysis is made for a uniform wave train. Discussions are given for both an air-water interface and a Boussinesq approximation. Significant deviations are noticed from the results obtained from the third-order evolution equation, which is the nonlinear Schrödinger equation. In the Boussinesq approximation, it has been possible to compare the present results with the exact numerical analysis of Pullin and Grimshaw [12], and they are found to agree rather favourably.

1. Introduction

Three-dimensional stability of finite-amplitude water waves on the surface of deep water has been studied numerically by McLean *et al.* [10]. This study reveals that there are two distinct types of instabilities for gravity waves of finite amplitude in deep water. One is predominantly two-dimensional and is related to all known results (for example Benjamin-Feir instability) for special cases, and this has been designated as type-I instability. The other designated as type-II instability is predominantly three-dimensional and becomes dominant when wave steepness is sufficiently large.

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https://doi.org/10.1017/S0334270000009346 Published online by Cambridge University Press

Yuen [14] has made an extension of the above-mentioned paper to the case of interfacial waves with a current jump. The type-I instability in the particular case of long-wavelength perturbation and small wave steepness can be investigated analytically from the nonlinear evolution equation, which consists of a nonlinear Schrödinger equation coupled to an equation for the wave-induced mean flow. Such an analytical study for the stability of interfacial wave has been made by Grimshaw and Pullin [5]. The corresponding numerical stability analysis for finite wave length of perturbation and finite wave steepness has been carried out in a subsequent paper by Pullin and Grimshaw [11]. Very recently, Pullin and Grimshaw [12] have made an extension of the above two papers for interfacial gravity waves propagating on a basic current shear, in which both analytical and numerical results are presented. Analytical results are for long-wavelength modulational instability of small-amplitude waves. This has been done starting from a third-order nonlinear evolution equation for two space dimensions (i.e. one dimension in propagation space), which is a nonlinear Schrödinger equation. The results are presented for two superposed fluids of finite depths. The case of infinitely deep fluids on both sides of the interface has been considered in detail. Specific examples of air-water interface and Boussinesq approximation have been discussed.

It has been seen that the results obtained from the lowest order nonlinear evolution equation for a deep-water wave, which is the nonlinear Schrödinger equation, does not agree with the exact calculations of Longuet-Higgins [8, 9] and with the experimental results of Benjamin and Feir [1]. By taking perturbation analysis one step further to $O(\varepsilon^4)$, where ε is the order of wave steepness, Dysthe [4] achieved a considerable improvement on the results. The fourth-order effect gives surprising improvement compared to ordinary nonlinear Schrödinger effects in many respects, and some of these points have been elaborated by Janssen [7].

Derivation of a fourth-order nonlinear evolution equation for deep-water surface gravity waves including different effects and stability analysis made from them were done by several authors (Staissnie [13], Hogan [6], Dhar and Das [3]). Regarding the stability of finite-amplitude uniform wave trains, the dominant new effect that comes first in the fourth-order analysis, is the wave-induced mean flow. The stability results obtained from the fourth-order nonlinear evolution equation for wave steepness less than 0.25 are in good agreement with Longuet-Higgins [9] and Benjamin and Feir [1].

On the other hand Brinch-Nielsen and Jonsson [2] have done a stability analysis for a three-dimensional Stokes wave on water of finite depth, starting from a fourth-order evolution equation. Their study reveals that in the expression for the growth rate of instability, no extra term arises which is responsible for the fourth-order effect, and this is contrary to the findings of Dysthe [4] for infinite-depth. So it is expected that for stability analysis of infinite depth fluids, the fourth-order non-linear evolution equation is a good starting point.

Keeping this in mind, in the present paper we make a stability analysis of interfacial gravity waves propagating on a basic current shear, for the particular case of infinitely-deep fluids on both sides of the interface, starting from a fourth-order nonlinear evolution equation. As has already been mentioned, this problem has been considered by Pullin and Grimshaw [12]. They have made both an exact numerical and an analytical study. The analytical study has been made from the lowest-order nonlinear evolution equation. The present fourth-order analysis shows appreciable deviation from the third-order analysis. In the Boussinesq approximation, the maximum growth rate of instability has been compared with the exact numerical calculations of Pullin and Grimshaw [12] in Figure 2(b). From this it is observed that the present fourth-order analysis gives better results (i.e. is closer to the exact numerical results) than that given by the nonlinear Schrödinger equation.

A two-dimensional (i.e. one dimension in propagation space) fourth-order nonlinear evolution equation is derived for an interfacial gravity wave propagating on a basic current shear for the case of infinitely deep fluids on both sides of the interface. From this nonlinear evolution equation, a nonlinear dispersion relation is determined, and an expression for the maximum growth rate of instability is obtained. Graphs are plotted showing the maximum growth rate of instability against wave steepness for both an air-water interface and a Boussinesq approximation. It is observed that in the fourth-order analysis, the maximum growth rate of instability first increases with the increase of wave steepness and then it decreases, while in the third-order analysis, the growth rate increases steadily with the increase of wave steepness. Stable and unstable regions in η_o-λ space (η_o is wave steepness and λ is wave number of perturbation) are shown in the figures for both air-water interface and Boussinesq approximation.

2. Basic equations

We take the $y = 0$ plane as the interface between two superposed inviscid fluids in the undisturbed state. Each fluid has a basic current, which has uniform vorticity Ω_1 and Ω_2 respectively corresponding to a basic horizontal current in the x-direction, $-\Omega_1 y$ and $-\Omega_2 y$ respectively. We take $y = \eta(x,t)$ as the equation of the common interface at any time t in the perturbed state. Let φ^*, $\varphi^{*\prime}$, Ψ^*, $\Psi^{*\prime}$, η^*, (x^*, y^*), t^*, Ω_1^*, Ω_2^*, γ^* be the dimensionless quantities denoting respectively perturbed velocity potentials of the lower and upper fluids, perturbed stream function of the lower and upper fluids, surface elevation of the common interface, space co-ordinates, time, current shear of the upper and lower fluids, and ratio of densities of the upper and lower fluids. These dimensionless quantities are related to the corresponding dimensional quantities denoted without a star by the following relations

$$\begin{aligned}
\varphi^* &= \left(\frac{k^3}{8g}\right)^{1/2}\varphi, & \varphi^{*\prime} &= \left(\frac{k^3}{8g}\right)^{1/2}\varphi', \\
\Psi^* &= \left(\frac{k^3}{8g}\right)^{1/2}\Psi, & \Psi^{*\prime} &= \left(\frac{k^3}{8g}\right)^{1/2}\Psi', \\
\eta^* &= \left(\frac{k_o}{2}\right)\eta, & (x^*, y^*) &= \left(\frac{k_o}{2}x, \frac{k_o}{2}y\right), \\
t^* &= \left(\frac{k_o g}{2}\right)^{1/2} t, & \Omega_1^* &= \left(\frac{2}{k_o g}\right)^{1/2}\Omega_1, \\
\Omega_2^* &= \left(\frac{2}{k_o g}\right)^{1/2}\Omega_2, & \gamma^* &= \rho_1/\rho_2,
\end{aligned} \tag{1}$$

where k_o is some characteristic wavenumber. In future, all these dimensionless quantities will be written with their stars dropped.

The perturbed velocity potentials φ, φ' and stream functions Ψ, Ψ' of the lower and upper fluids satisfy the following two-dimensional Laplace's equation

$$\nabla^2\varphi = 0, \qquad \nabla^2\Psi = 0, \qquad \text{in } -\infty < y < \eta, \tag{2}$$

$$\nabla^2\varphi' = 0, \qquad \nabla^2\Psi' = 0, \qquad \text{in } \eta < y < \infty. \tag{3}$$

The boundary conditions to be satisfied at the interface are

$$\frac{\partial\varphi}{\partial y} - \frac{\partial\eta}{\partial t} = \left(\frac{\partial\varphi}{\partial x} - \Omega_2\eta\right)\frac{\partial\eta}{\partial x}, \qquad \text{when } y = \eta, \tag{4}$$

$$\frac{\partial\varphi'}{\partial y} - \frac{\partial\eta}{\partial t} = \left(\frac{\partial\varphi'}{\partial x} - \Omega_1\eta\right)\frac{\partial\eta}{\partial x}, \qquad \text{when } y = \eta, \tag{5}$$

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right] + \Omega_2\Psi - \Omega_2 y\frac{\partial\varphi}{\partial x} + \eta$$

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$$= \gamma \left[\frac{\partial \varphi'}{\partial t} + \frac{1}{2} \left\{ \left(\frac{\partial \varphi'}{\partial x} \right)^2 + \left(\frac{\partial \varphi'}{\partial y} \right)^2 \right\} + \Omega_1 \Psi' - \Omega_1 y \frac{\partial \varphi'}{\partial x} + \eta \right],$$

$$\text{when } y = \eta. \quad (6)$$

We also have

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \Psi}{\partial y}, \qquad \frac{\partial \varphi}{\partial y} = -\frac{\partial \Psi}{\partial x}, \qquad \frac{\partial \varphi'}{\partial x} = \frac{\partial \Psi'}{\partial y}, \qquad \frac{\partial \varphi'}{\partial y} = -\frac{\partial \Psi'}{\partial x}. \quad (7)$$

Finally φ, φ', Ψ, Ψ' should satisfy the following conditions at infinity

$$\varphi, \Psi \to 0 \quad \text{as} \quad y \to -\infty; \qquad \varphi', \Psi' \to 0 \quad \text{as} \quad y \to \infty. \quad (8)$$

We look for solutions of the above equations in the form

$$G = G_o + \sum_{n=1}^{\infty} \left[G_n \exp in(kx - \omega t) + G_n^* \exp -in(kx - \omega t) \right], \quad (9)$$

where G stands for φ, φ', Ψ, Ψ', η and a star denotes the complex conjugate. Here it is assumed that φ_o, φ'_o, φ_n, φ'_n, Ψ_o, Ψ'_o, Ψ_n, Ψ'_n and their complex conjugates are functions of x_1, y, t_1, where $x_1 = \varepsilon x$, $t_1 = \varepsilon t$, while η_o, η_n, η_n^* are functions of x_1, t_1 only. Here ε is a slowness parameter, and ω, k satisfy the following linear dispersion relation:

$$\lambda(\omega, k) \equiv (1 + \gamma)\omega^2 + (\gamma \Omega_1 - \Omega_2)\omega - (1 - \gamma)k = 0. \quad (10)$$

We now suppose that the first harmonic linear wave whose nonlinear evolution equation we are going to study, has as its wave number k_o, the characteristic wave number. So we have $k = 1$ in (10) and the linear dispersion relation determining ω becomes

$$(1 + \gamma)\omega^2 - \Omega\omega - (1 - \gamma) = 0, \quad (11)$$

where $\Omega = \Omega_2 - \gamma \Omega_1$. This equation gives two values of ω, given by

$$\omega_\pm = \frac{1}{2} \left[\Omega \pm \sqrt{\Omega^2 + 4(1 - \gamma^2)} \right] \cdot (1 + \gamma)^{-1}, \quad (12)$$

which correspond to two modes, and we designate these two modes as positive and negative modes. The positive mode moves along the positive direction of x-axis with a frequency equal to

$$\frac{1}{2} \left[\sqrt{\Omega^2 + 4(1 - \gamma^2)} + \Omega \right] \cdot (1 + \gamma)^{-1},$$

while the negative mode moves along the negative direction of x-axis with a frequency equal to

$$\frac{1}{2}\left[\sqrt{\Omega^2+4(1-\gamma^2)}-\Omega\right]\cdot(1+\gamma)^{-1}.$$

If Ω is replaced by $-\Omega$ the frequency of the positive mode becomes equal to the frequency of the negative mode. So the results for the negative mode can be obtained from those for the positive mode by replacing Ω by $-\Omega$. Therefore we have made a nonlinear analysis for positive mode only.

3. Evolution equation

By a standard procedure as outlined in Appendix A, we find that $\eta = \eta_{11}+\varepsilon\eta_{12}$ satisfies the fourth-order evolution equation

$$i\frac{\partial\eta}{\partial\tau}-\beta_1\frac{\partial^2\eta}{\partial\xi^2}+i\beta_2\frac{\partial^3\eta}{\partial\xi^3}=\Lambda_1\eta^2\eta^*+i\Lambda_2\eta\eta^*\frac{\partial\eta}{\partial\xi}+i\Lambda_3\eta^2\frac{\partial\eta^*}{\partial\xi}+\Lambda_4\eta H\frac{\partial}{\partial\xi}(\eta\eta^*), \tag{13}$$

where the coefficients β_1, β_2, Λ_1, Λ_2, Λ_3, Λ_4 are given in Appendix B, and where

$$\xi=\varepsilon(x-c_g t),\qquad \tau=\varepsilon^2 t,\qquad c_g=\frac{d\omega}{dk} \tag{14}$$

and H is the Hilbert transform given by

$$H(\bar{\Psi})=\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\bar{\Psi}(\xi')}{\xi'-\xi}d\xi'. \tag{15}$$

4. Stability analysis

The equation (13) admits a Stokes wave-solution

$$\eta=\frac{1}{2}\eta_o\exp(i\Delta\omega\tau), \tag{16}$$

where η_o is a real constant and the nonlinear frequency shift $\Delta\omega$ is given by

$$\Delta\omega=-\frac{1}{4}\eta_o^2\Lambda_1. \tag{17}$$

To study modulational instability of this uniform wave train, we introduce the following perturbation in the uniform solution

$$\eta = \frac{1}{2}\eta_o(1+\eta')\exp i(\theta' + \Delta\omega\tau), \tag{18}$$

where η' and θ' are small perturbations in amplitude and phase respectively, and are real.

Substituting (18) in the evolution equation (13) and then following the procedure as given in Appendix C, we arrive at the following dispersion relation, that includes the effect of weak nonlinearity. Here it is assumed that the space-time dependence of η', θ' is of the form $\exp i(\lambda\xi - \nu'\tau)$.

$$\bar{P}_1 = -\frac{1}{4}\eta_o^2\Lambda_2\lambda \pm \left[\bar{P}_2\left\{\bar{P}_2 - \frac{\eta_0^2}{2}\Lambda_1\left(1 - \frac{\Lambda_4|\lambda|}{\Lambda_1}\right)\right\}\right]^{1/2}, \tag{19}$$

where

$$\bar{P}_1 = \nu - c_g\lambda - \beta_2\lambda^3, \qquad \bar{P}_2 = \beta_1\lambda^2 \quad \text{and} \quad \nu = \nu' + c_g\lambda. \tag{20}$$

From (19) it follows that for instability we must have

$$\lambda^2 < \eta_o^2\Lambda_1\left[1 - \frac{\Lambda_4}{\Lambda_1}|\lambda|\right]/2\beta_1, \tag{21}$$

and if this condition is met, then the maximum growth rate γ_M is given by

$$\gamma_M = \frac{\eta_o^2\Lambda_1}{4}\left[1 - \frac{\eta_o\Lambda_4}{2\sqrt{\beta_1\Lambda_1}}\right]. \tag{22}$$

For $\gamma = 0$, $\Omega_1 = \Omega_2 = 0$, (22) reduces to (3.10) of Dysthe [4]. At marginal stability we have

$$\bar{P}_2\left[\bar{P}_2 - \frac{1}{2}\eta_o^2\Lambda_1\left(1 - \frac{\Lambda_4|\lambda|}{\Lambda_1}\right)\right] = 0 \tag{23}$$

and this gives the following expression for λ at marginal stability.

$$\lambda = \eta_o\left[1 - \frac{\eta_o\Lambda_4}{\sqrt{8\Lambda_1\beta_1}}\right]\cdot\sqrt{\frac{\Lambda_1}{2\beta_1}}. \tag{24}$$

For the positive mode in the case of the air-water interface, the maximum growth rate of instability γ_M given by (22) has been plotted in Figure 1(a)

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against wave-steepness η_o, for some different values of Ω_1 with $\Omega_2 = 0$. The same growth rate for some different values of Ω_2 with $\Omega_1 = 0$ has been plotted in Figure 1(b). From these figures, it is seen that γ_M first increases with the increase of η_o and then its value decreases, while the growth rate obtained from the third-order evolution equation increases steadily with the increase of η_o.

For the positive mode and for a Boussinesq approximation ($\gamma \cong 1$), the growth rate γ_M has been plotted in Figure 2(a) against wave-steepness η_o, for some different values of Ω_1 with $\Omega_2 = 0$. To compare our results with the exact numerical calculations of Pullin and Grimshaw [12] in the case of the Boussinesq approximation, we have shown in Figure 2(b) the plots of the ratios of maximum growth rate of instability and the square of wave steepness γ_M/η_o^2 as obtained from the lowest-order evolution equation, fourth-order evolution equation, and from the exact numerical analysis, against Ω_1 with $\Omega_2 = 0$, for some different values of wave-steepness η_o. This figure shows fairly good agreement between the results obtained from the present fourth-order analysis and the results obtained from the exact numerical calculations of Pullin and Grimshaw [12].

For the positive mode in the case of an air-water interface, the wave number λ at marginal stability given by (24) has been plotted in Figure 3(a), against wave-steepness η_o for some different values of Ω_1 with $\Omega_2 = 0$, and hence this gives the stable-unstable region in the $\lambda\eta_o$-plane. The same λ for some different values of Ω_2 with $\Omega_1 = 0$ has been plotted in Figure 3(b). From these two figures, it is seen that for any particular value of the pair (η_o, λ) there exists a critical value Ω_{1c} of Ω_1 when $\Omega_2 = 0$, and a critical value Ω_{2c} of Ω_2 when $\Omega_1 = 0$, such that there is instability when $|\Omega_1| > |\Omega_{1c}|$, $(\Omega_2 = 0)$ and when $\Omega_2 > \Omega_{2c}$, $(\Omega_1 = 0)$.

For the positive mode in the Boussinesq approximation ($\gamma \cong 1$), the wave number λ at marginal stability given by (24) has been plotted in Figure 4 for some different values of Ω_1 with $\Omega_2 = 0$. From this figure it is seen that for any particular value of the pair (η_o, λ), there exists a critical value Ω_{1c} of Ω_1 when $\Omega_2 = 0$ such that there is instability when $|\Omega_1| < |\Omega_{1c}|$, $(\Omega_2 = 0)$.

5. Conclusion

In this paper we have studied analytically, starting from a fourth-order nonlinear evolution equation, the stability of a uniform wave train propagating at the interface of two fluids extending to infinity on both sides of the interface

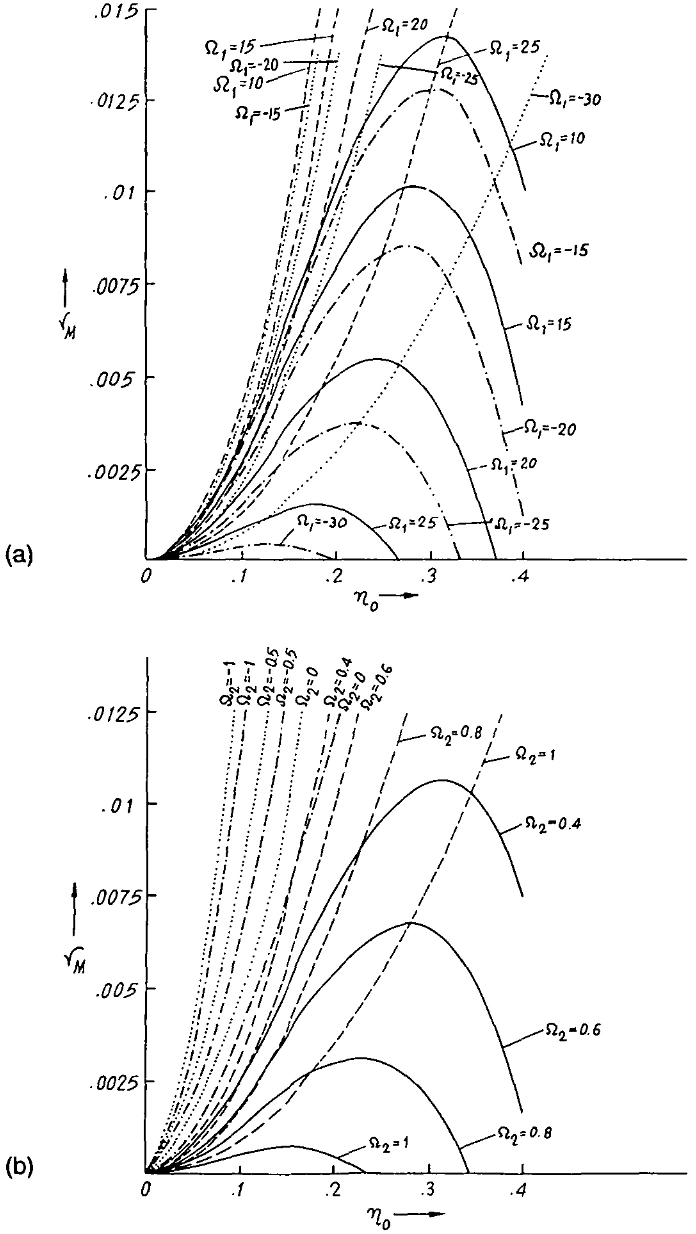

FIGURE 1. One-dimensional modulational instability, $\gamma = 0.0012$.
(a) $\Omega_2 = 0$. Maximum growth rate γ_M against wave steepness η_o.
(b) $\Omega_1 = 0$. Maximum growth rate γ_M against wave steepness η_o.
Ω_i positive: (dashed line), Third-order result, (solid line) Fourth-order result;
Ω_i negative: (dotted line), Third-order result, (chain-dotted line), Fourth-order result ($i = 1,2$).

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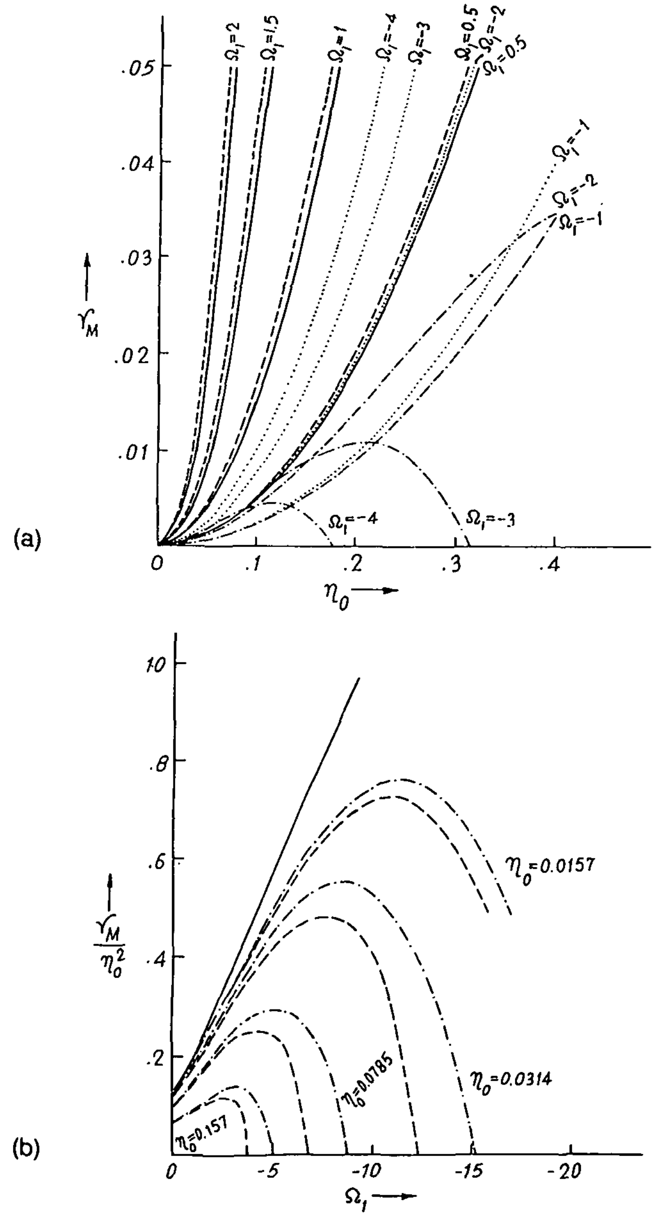
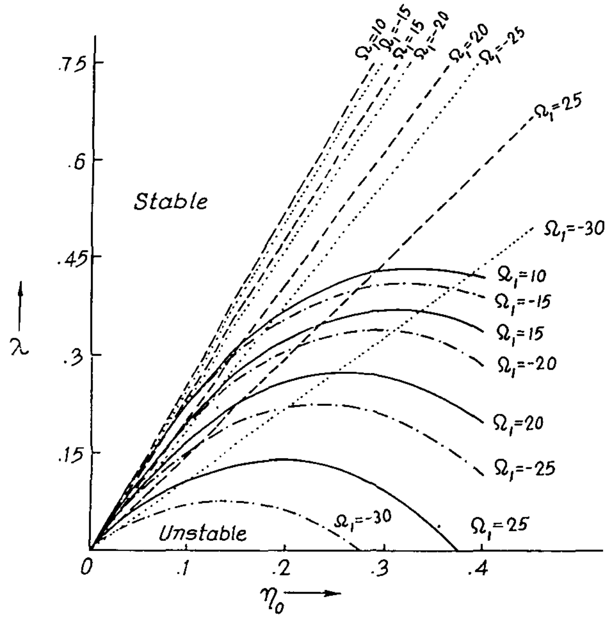


FIGURE 2. One-dimensional modulational stability, $\gamma \cong 1$, $\Omega_2 = 0$.
(a) Maximum growth rate γ_M against wave steepness η_o.
Ω_i positive: (dashed line), Third-order result, (solid line) Fourth-order result;
Ω_i negative: (dotted line), Third-order result, (chain-dotted line), Fourth-order result (i = 1,2).
(b) γ_M/η_o^2 against Ω_1. (solid line) Third-order result, (chain-dotted line) Fourth-order result, (dashed line) Numerical result.

(a)



(b)

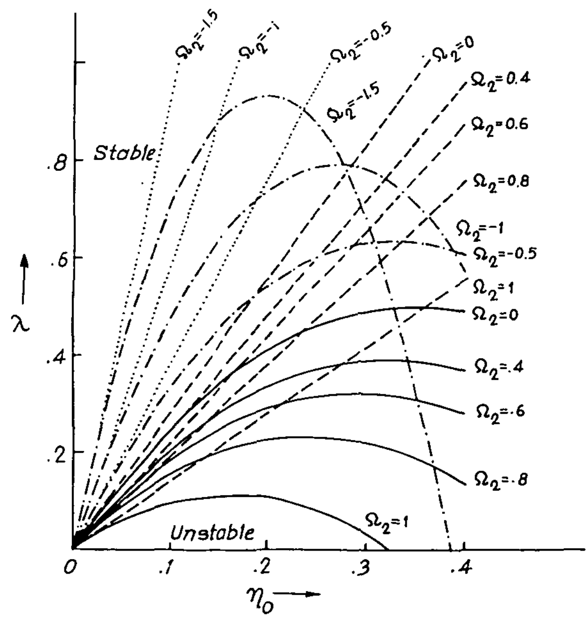


FIGURE 3. One-dimensional modulational instability, $\gamma = 0.0012$.
(a) $\Omega_2 = 0$. Perturbed wave number λ at marginal stability against wave steepness η_o.
(b) $\Omega_1 = 0$. Perturbed wave number λ at marginal stability against wave steepness η_o.
Ω_i positive: (dashed line), Third-order result, (solid line) Fourth-order result;
Ω_i negative: (dotted line), Third-order result, (chain-dotted line), Fourth-order result ($i = 1,2$).

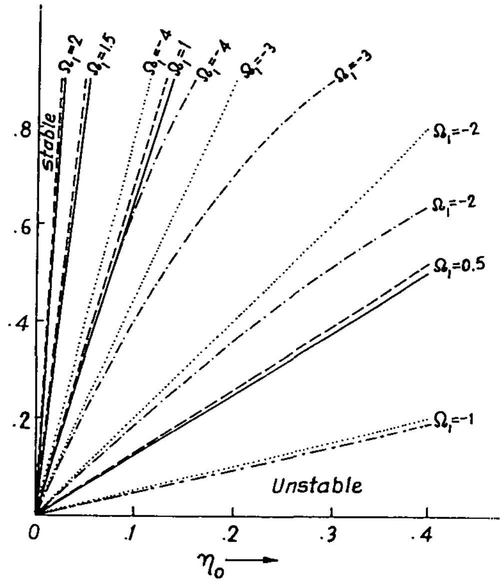


FIGURE 4. One-dimensional modulational instability $\gamma \cong 1$, $\Omega_2 = 0$. Perturbed wave number λ at marginal stability against wave steepness η_o.
Ω_i positive: (dashed line), Third-order result, (solid line) Fourth-order result;
Ω_i negative: (dotted line), Third-order result, (chain-dotted line), Fourth-order result (i = 1,2).

in the presence of a basic current shear. The reason for starting from a fourth-order nonlinear evolution equation is that the investigations of Dysthe [4] and Brinch-Nielsen and Jonsson [2] reveal that for infinite-depth fluids, the fourth-order nonlinear evolution equation gives results consistent with the exact results of Longuet-Higgins [8, 9] and experimental results of Benjamin and Feir [1] for wave steepness up to 0.25. In the case of the Boussinesq approximation, it has been possible to compare our results with the exact results of Pullin and Grimshaw [12], and we find that the results from the fourth-order nonlinear evolution equation give much better results than those given by the third-order (lowest order) evolution equation, which is the nonlinear Schrödinger equation. All the present results show marked deviation from the results obtained from the third-order evolution equation.

Appendix A

On substituting the expansions (9) in (2) and (3), and then equating coefficients of $\exp in(x - \omega t)$, $(n = 1, 2)$ we get the following equations

$$\begin{aligned} \frac{d^2\varphi_n}{dy^2} - \Delta_n^2\varphi_n &= 0, \\ \frac{d^2\varphi'_n}{dy^2} - \Delta_n^2\varphi'_n &= 0, \end{aligned} \tag{A.1}$$

$$\begin{aligned} \frac{d^2\Psi_n}{dy^2} - \Delta_n^2\Psi_n &= 0, \\ \frac{d^2\Psi'_n}{dy^2} - \Delta_n^2\Psi'_n &= 0, \end{aligned} \tag{A.2}$$

where Δ_n is an operator given by

$$\Delta_n = n - i\varepsilon\frac{\partial}{\partial x_1}, \qquad (n = 1, 2). \tag{A.3}$$

The solution of these equations satisfying the boundary conditions (8) can be put in the form

$$\varphi_n = \exp(y\Delta_n)A_n, \qquad \varphi'_n = \exp(-y\Delta_n)A'_n, \tag{A.4}$$

$$\Psi_n = \exp(y\Delta_n)B_n, \qquad \Psi'_n = \exp(-y\Delta_n)B'_n, \tag{A.5}$$

where A_n, B_n, A'_n, B'_n, $(n = 1,\ 2)$ are functions of x_1, t_1.

For the sake of convenience we take the Fourier transform of (2) and (3) for $n = 0$. The solution of these transformed equations becomes

$$\bar{\varphi}_o = \bar{A}_o \exp(|\bar{k}_x|y), \qquad \bar{\varphi}'_o = \bar{A}'_o \exp(-|\bar{k}_x|y), \tag{A.6}$$

$$\bar{\Psi}_o = \bar{B}_o \exp(|\bar{k}_x|y), \qquad \bar{\Psi}'_o = \bar{B}'_o \exp(-|\bar{k}_x|y), \tag{A.7}$$

where $\bar{\varphi}_o$, $\bar{\varphi}'_o$, $\bar{\Psi}_o$, $\bar{\Psi}'_o$, are Fourier transforms of φ_o, φ'_o, Ψ_o, Ψ'_o respectively defined by

$$(\bar{\varphi}_o, \bar{\varphi}'_o, \bar{\Psi}_o, \bar{\Psi}'_o) = \int\int_{-\infty}^{\infty} (\varphi_o, \varphi'_o, \Psi_o, \Psi'_o) \exp i(\bar{k}_x x_1 - \bar{\omega} t_1)\, dx_1\, dt_1 \tag{A.8}$$

and $\bar{A}_o$, $\bar{A}'_o$, $\bar{B}_o$, $\bar{B}'_o$ are functions of $\bar{k}_x$, $\bar{\omega}$.

On substituting the expansions (9) in the Taylor-expanded form of equations (4)-(6) about $y = 0$ and then equating coefficients of $\exp in(x - \omega t)$ for $n =$

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0, 1, 2 on both sides, we get three sets of equations, in each of which we substitute the solutions for φ_n, φ_n', Ψ_n, Ψ_n' given by (A.4)-(A.7). For the sake of convenience, we take the Fourier transform of the set of equations corresponding to $n = 0$. The set of equations corresponding to $n = 1, 2, 0$ has been designated respectively as the first, second and third set. To solve these three sets of equations, we make the following perturbation expansion of the quantities A_n, A_n', B_n, B_n', η_n, $(n = 0, 1, 2)$.

$$E_1 = \sum_{n=1}^{\infty} \varepsilon^n E_{1n}, \qquad E_m = \sum_{n=2}^{\infty} \varepsilon^n E_{mn}, \quad (m = 0, 2), \tag{A.9}$$

where E_j stands for A_j, A_j', B_j, B_j', η_j, $(j = 0, 1, 2)$.

On substituting the expansions (A.9) in the above three sets of equations, and then equating various powers of ε on both sides we get a sequence of equations.

From the first-order (i.e. lowest-order) and second-order equations corresponding to the first set of equations resulting from (4) and (5), we get solutions for A_{11}, A_{11}' and A_{12}, A_{12}' respectively. Next, from the first-order and second-order equations corresponding to the second set of equations resulting from (4)-(6), we get solutions for A_{22}, A_{22}', η_{22} and A_{23}, A_{23}', η_{23} respectively. Finally, from the first-order equations corresponding to the third set of equations resulting from (4)-(6), we get the solutions for A_{02}, A_{02}', η_{02}, and from the second-order equations corresponding to the third set of equations resulting from (6), we get solution for η_{03}.

The equation corresponding to (6) of the first set of equations, which has not been used in getting the above perturbation solutions, can be put in the following convenient form after eliminating A_1, A_1' B_1, B_1'.

$$\lambda(\omega', k')\eta_1 = i(\Omega_2 - \omega')a_1 - i\gamma(\Omega_1 + \omega')b_1 + k'c_1 - \gamma k'd_1, \tag{A.10}$$

where

$$\omega' = \omega + i\varepsilon\frac{\partial}{\partial t_1}, \qquad k' = 1 - i\varepsilon\frac{\partial}{\partial x_1} \tag{A.11}$$

and a_1, b_1, c_1, d_1 are contributions from nonlinear terms.

We keep terms up to ε^4 in (A.10) and then substitute solutions for various perturbed quantities appearing on its right hand side, and finally using the transformations,

$$\xi = x_1 - c_g t_1, \qquad \tau = \varepsilon t_1, \tag{A.12}$$

and writing $\eta = \eta_{11} + \varepsilon\eta_{12}$ we arrive at the fourth-order nonlinear evolution equation (13).

Appendix B

$$\beta_1 = -\frac{1}{2}\frac{dc_g}{dk}, \qquad \beta_2 = \frac{1}{2}\cdot c_g \lambda_{\omega\omega}\lambda_\omega^{-1}\frac{dc_g}{dk},$$

$$\Lambda_1 = \delta_1/\lambda_\omega, \qquad \Lambda_2 = \left(\delta_4 - c_g\delta_2 + 2\delta_1 c_g \lambda_\omega^{-1}\lambda_{\omega\omega}\right)/\lambda_\omega,$$

$$\Lambda_3 = \left(\delta_5 - c_g\delta_3 + \delta_1 c_g \lambda_\omega^{-1}\lambda_{\omega\omega}\right)/\lambda_\omega, \quad \Lambda_4 = \delta_6/\lambda_\omega,$$

where δ_i's are given by

$$\begin{aligned}
\delta_1 = \Bigg\{ & 2\omega^2(1+\gamma) - (\gamma\Omega_1^2 + \Omega_2^2) \\
& + \frac{1}{2}(1+\gamma)^{-1}\omega^{-2}\left[2\omega^2(1-\gamma) - 4\omega(\gamma\Omega_1 + \Omega_2) - (\gamma\Omega_1^2 - \Omega_2^2)\right]^2 \\
& - \left[2\omega(\gamma\Omega_1 + \Omega_2) + \gamma\Omega_1^2 - \Omega_2^2\right]^2 \Bigg\}\left[1 - \gamma - c_g(\gamma\Omega_1 - \Omega_2)\right]^{-1},
\end{aligned}$$

$$\begin{aligned}
\delta_2 = {} & 2\omega^{-2}(1+\gamma)^{-2}\left[2\omega^2(1-\gamma) - 4\omega(\gamma\Omega_1 + \Omega_2) - \gamma\Omega_1^2 + \Omega_2^2\right] \\
& \qquad \cdot\left[\omega(1-\gamma) - (\gamma\Omega_1 + \Omega_2)\right] \\
& - \frac{1}{2}\omega^{-4}(1+\gamma)^{-2}\left[4\omega(1+\gamma) + \gamma\Omega_1 - \Omega_2\right] \\
& \qquad \cdot\left[2\omega^2(1-\gamma) - 4\omega(\gamma\Omega_1 + \Omega_2) - (\gamma\Omega_1^2 - \Omega_2^2)\right]^2 \\
& + 2\omega^{-2}(1+\gamma)^{-1}\left[(1-\gamma)\omega - (\gamma\Omega_1 + \Omega_2)\right] \\
& \qquad \cdot\left[2\omega^2 - 4\omega(\gamma\Omega_1 + \Omega_2) - (\gamma\Omega_1^2 - \Omega_2^2)\right] \\
& + 4\omega(1+\gamma) + 2(\gamma\Omega_1 + \Omega_2)\left[2\omega(\gamma\Omega_1 + \Omega_2) + \gamma\Omega_1^2 - \Omega_2^2)\right] \\
& \qquad \cdot\left[1 - \gamma - c_g(\gamma\Omega_1 - \Omega_2)\right]^{-1} \\
& + \left[\gamma(\omega + \Omega_1) + (\Omega_2 - \omega)\right]\cdot\left[2\omega(\gamma\Omega_1 + \Omega_2) + \gamma\Omega_1^2 - \Omega_2^2\right]
\end{aligned}$$

$$\cdot\Big[1-\gamma-c_g(\gamma\Omega_1-\Omega_2)\Big]^{-1},$$

$$\delta_3=\Big[\gamma(\omega+\Omega_1)+(\Omega_2-\omega)\Big]\Big[2\omega(\gamma\Omega_1+\Omega_2)+\gamma\Omega_1^2-\Omega_2^2)\Big]$$
$$\cdot\Big[1-\gamma-c_g(\gamma\Omega_1-\Omega_2)\Big]^{-1},$$

$$\delta_4=-\omega^{-2}(1+\gamma)^{-1}\Big[2\omega^2(1-\gamma)-4\omega(\gamma\Omega_1+\Omega_2)-(\gamma\Omega_1^2-\Omega_2^2)\Big]^2$$
$$-4\omega^2(1+\gamma)+3(\gamma\Omega_1^2+\Omega_2^2)+\omega(\gamma\Omega_1-\Omega_2)$$
$$-\frac{1}{2}(1-\gamma)(1+\gamma)^{-2}\omega^{-4}$$
$$\cdot\Big[2\omega^2(1-\gamma)-4\omega^2(\gamma\Omega_1+\Omega_2)-(\gamma\Omega_1^2-\Omega_2^2)\Big]^2$$
$$-\gamma_1,$$

$$\delta_5=\Big[\omega(\gamma\Omega_1-\Omega_2)+(\gamma\Omega_1^2+\Omega_2^2)\Big]$$
$$-\frac{1}{2}\Big[2\omega^2(1-\gamma)-4\omega(\gamma\Omega_1+\Omega_2)-(\gamma\Omega_1^2-\Omega_2^2)\Big]^2\omega^{-2}(1+\gamma)^{-1}-\gamma_1,$$

$$\delta_6=\Big\{\gamma(2\omega+\Omega_1)\Big[(1-\gamma)(2\omega+\Omega_1)\Omega_2c_g(4\omega+\Omega_1-\Omega_2)\Big]$$
$$+(2\omega-\Omega_2)\Big[(1-\gamma)(2\omega-\Omega_2)-(4\omega+\Omega_1-\Omega_2)\gamma\Omega_1c_g\Big]\Big\}$$
$$\cdot\Big[1-\gamma-c_g(\gamma\Omega_1-\Omega_2)\Big]^{-1}.$$

γ_1 appearing in the expressions for δ_4 and δ_5 is given by

$$\gamma_1=\Big[2\omega(\gamma\Omega_1+\Omega_2)+(\gamma\Omega_1^2-\Omega_2^2)\Big]^2\Big[1-\gamma-c_g(\gamma\Omega_1-\Omega_2)\Big]^{-1}$$
$$+\omega\Big\{\gamma\Big[(1-\gamma)(2\omega+\Omega_1)+(4\omega+\Omega_1-\Omega_2)\Omega_2c_g\Big]$$
$$+(1-\gamma)(2\omega-\Omega_2)-(4\omega+\Omega_1-\Omega_2)\gamma\Omega_1c_g\Big\}$$

$$\cdot\left[1-\gamma-c_g(\gamma\Omega_1-\Omega_2)\right]^{-1}$$

$$+\omega(\gamma\Omega_1+\Omega_2)c_g\cdot\Big\{(1-\gamma)(2\omega-\Omega_2)-(4\omega+\Omega_1-\Omega_2)\gamma\Omega_1 c_g$$

$$-\gamma\left[(1-\gamma)(2\omega+\Omega_1)+(4\omega+\Omega_1-\Omega_2)\Omega_2 c_g\right\}$$

$$\cdot\left[1-\gamma-c_g(\gamma\Omega_1-\Omega_2)\right]^{-2}.$$

Appendix C

Inserting (18) in the evolution equation (13), and linearising with respect to η' and θ', we get the following equations:

$$P_1\eta' + P_2\theta' - \frac{1}{4}\eta_o^2(\Lambda_2+\Lambda_3)\frac{\partial\eta'}{\partial\xi} = 0, \tag{C.1}$$

$$P_1\theta' - P_2\eta' + \frac{1}{2}\Lambda_1\eta_o^2\eta' + \frac{1}{4}\eta_o^2(\Lambda_3-\Lambda_2)\frac{\partial\theta'}{\partial\xi} + \frac{\Lambda_4\eta_o^2}{2\pi}P\int_{-\infty}^{\infty}\frac{d\xi'}{\xi'-\xi}\frac{\partial\eta'}{\partial\xi'} = 0, \tag{C.2}$$

where

$$P_1 = \frac{\partial}{\partial\tau} + \beta_2\frac{\partial^3}{\partial\xi^3}, \qquad P_2 = -\beta_1\frac{\partial^2}{\partial\xi^2}. \tag{C.3}$$

Now if we suppose that τ-dependence of η' and θ' is of the form exp($-i\nu'\tau$), then (C.1), (C.2) remain the same as before but P_1 now stands for

$$P_1 = -i\nu' + \beta_2\frac{\partial^3}{\partial\xi^3}. \tag{C.4}$$

Next, taking the Fourier transform of (C.1), (C.2), with respect to ξ defined by

$$(\bar{\theta}', \bar{\eta}') = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\theta'(\xi), \eta'(\xi)\right]\exp(-i\lambda\xi)d\xi, \tag{C.5}$$

we get two linear algebraic equations for $\bar{\theta}'$ and $\bar{\eta}'$. The condition for the existence of a nontrivial solution of these two equations gives the dispersion relation (19).

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