ON A SET OF NORMAL SUBGROUPS

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1. The commutator $[a, b]$ of two elements a and b in a group G satisfies the identity

$$ab = ba[a, b].$$

The subgroups we study are contained in the commutator subgroup G', which is the subgroup generated by all the commutators.

The group G is covered by a well-known set of normal subgroups, namely the normal closures $\{g\}^G$ of the cyclic subgroups $\{g\}$ in G. In a similar way one may associate a subgroup $K(g)$ with each element g, by defining $K(g)$ to be the subgroup generated by the commutators $[g, x]$ as x takes all values in G. These subgroups generate G' (but do not cover G' in general), and are normal in G in consequence of the identical relation

$$\text{(A)} \qquad [g, x]^y = [g, y]^{-1}[g, xy]$$

holding for all g, x and y in G. (By a^b we mean $b^{-1}ab$.) It is easy to see that

$$\{g\}^G = \{g, K(g)\}.$$

The subgroups $K(g)$ appear in a number of situations. For instance, it is shown in Theorem 3 of [**1**] that if every $K(g)$ in G is abelian, then the commutator subgroup G'' of G' lies in the centre of G and has exponent 2. Again, every $K(g)$ is finite if and only if every element of G has just a finite number of conjugates. One part of this statement is clear, and to prove the other part suppose that every element of G has only a finite number of conjugates. Then any subgroup $K(g)$ is generated by a finite set of commutators of the form $[g, x]$ for certain elements x; each $[g, x]$ has finite order by Theorem 5.1 in [**2**]. These facts, and the condition on conjugates in G, and use of Corollary 5.21 of [**2**], show that each $K(g)$ is finite. We further note that, because of Theorem 3.1 of [**3**], each $K(g)$ is boundedly finite if and only if G' is finite.

In § 2 we consider groups G in which each $K(g)$ contains elements of the form $[g, x]$ only. This with minimal condition on the $K(g)$ appears to be a strong restriction on G, which will be shown to be a ZA group. An unusual feature of this result is that conditions on G' give a conclusion on the structure of G, not just of G'. In § 3 we turn to groups G in which each $K(g)$ is cyclic. As it can be shown that G' is then locally cyclic, it is worth considering groups with each $K(g)$ locally cyclic. We show that again G' is locally cyclic.

2. The subgroup $K(g)$ contains only commutators of the form $[g, x]$ if and only if the equations

$$\text{(B)} \qquad [g, x][g, y] = [g, z_1],$$

$$\text{(C)} \qquad [g, x]^{-1} = [g, z_2]$$

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can be solved for z_1 and z_2, the elements x and y being arbitrary. An equivalent condition is, clearly, that

$$\text{(D)} \qquad [g, y]^{-1}[g, xy] = [g, z]$$

should be soluble for z. By (A) and (D) another equivalent condition is that

$$\text{(E)} \qquad [g, x]^y = [g, z]$$

should be soluble for z.

A further condition may be obtained when it is noted that solubility of (B) is sufficient for solubility of (C). For if there is an element z_0 such that

$$[g, x^{-1}][g, x^{-1}] = [g, z_0],$$

then we have successively

$$g^{-1}xgx^{-1}g^{-1}xgx^{-1} = g^{-1}z_0^{-1}gz_0,$$

$$x^{-1}g^{-1}xg = g^{-1}x^{-1}z_0^{-1}gz_0x,$$

$$[g, x]^{-1} = [g, z_0x];$$

thus $z_2 = z_0x$ is a solution of (C).

Next we suppose that every $K(g)$ in G satisfies such a condition, and in addition we impose a minimal condition.

THEOREM 1. *Let each subgroup $K(g)$ of the group G consist of commutators of the form $[g, x]$, and let G be such that the minimal condition holds for the subgroups $K(g)$. Then a non-trivial element of each subgroup $\{g\}^G$ lies in the centre of G, provided that $g \neq 1$.*

Proof. If g is an arbitrary element of G we may suppose that $K(g)$ is not the trivial subgroup 1, for otherwise g is in the centre of G and the theorem holds. In $K(g)$ we choose a minimal non-trivial subgroup of the form $K(g_0)$—this exists because of the minimal condition—and we choose an element $h \neq 1$ in $K(g_0)$. As we have $K(h) \subseteq K(g_0)$, we see that $K(h) = K(g_0)$ or $K(h) = 1$. In the former case $h^{-1} \in K(g_0) = K(h)$, so by hypothesis there is an element x in G for which

$$h^{-1} = [h, x] = h^{-1}x^{-1}hx,$$

implying that $h = 1$, a contradiction. Therefore we must have $K(h) = 1$, that is, h is central in G. As $h \in K(g_0) \subseteq K(g) \subseteq \{g\}^G$, the theorem follows.

COROLLARY 1. *Under the hypotheses of Theorem 1, G is a ZA group.*

Proof. By a ZA group is meant a group with an ascending central series which eventually exhausts the group. As Theorem 1 shows that G has a non-trivial centre, the corollary follows once it is verified that the properties required in Theorem 1 persist in homomorphic images of G. This is elementary.

COROLLARY 2. *A group in which each $K(g)$ consists of elements $[g, x]$, and in which every element has only a finite number of conjugates, is ZA.*

Proof. We remarked earlier that this finiteness condition on conjugates is equivalent to each $K(g)$ being finite. Application of Corollary 1 completes the proof.

In particular, finite groups with the condition of Theorem 1 on the $K(g)$ are nilpotent. However, it is not difficult to see that the class of nilpotency is arbitrary.

We are in a position to show that neither of the following conditions on a group G implies the other:

(i) G' consists of commutators;
(ii) for each g in G, $K(g)$ consists of the commutators $[g, x]$ as x varies in G.

Though many finite non-nilpotent groups satisfy (i), no such group satisfies (ii), by a remark above. For examples, we refer to Ore's paper [**4**], where it is established that the alternating groups of finite degree greater than or equal to 5 satisfy (i). On the other hand, it is clear that any group that is nilpotent of class 2 satisfies (ii), and it seems to be well-known that such a group need not satisfy (i). We present a supporting example as no record of one can be readily found.

The example G_1 is simply the free nilpotent group of class 2 on 4 generators a_1, a_2, a_3, a_4; if $c_{ij} = [a_i, a_j]$ for $1 \leqq i < j \leqq 4$, the relations in G_1 are

$$[c_{ij}, a_k] = 1$$

for $1 \leqq i < j \leqq 4$ and $1 \leqq k \leqq 4$, and their consequences. Each element of G_1 has a unique representation in the form

$$a_1^{\alpha_1} a_2^{\alpha_2} a_3^{\alpha_3} a_4^{\alpha_4} \prod c_{ij}^{\alpha_{ij}},$$

where the product is taken over all i and j with $1 \leqq i < j \leqq 4$. So an arbitrary commutator may be written as

$$[a_1^{\alpha_1} a_2^{\alpha_2} a_3^{\alpha_3} a_4^{\alpha_4}, \quad a_1^{\beta_1} a_2^{\beta_2} a_3^{\beta_3} a_4^{\beta_4}],$$

which may be simplified by use of the defining relations to

$$\prod_{1 \leqq i < j \leqq 4} c_{ij}^{\delta_{ij}},$$

where $\delta_{ij} = \alpha_i \beta_j - \alpha_j \beta_i$. It may be verified directly that the δ_{ij} satisfy

$$\delta_{12}\delta_{34} - \delta_{13}\delta_{24} + \delta_{14}\delta_{23} = 0.$$

If $c_{13}c_{24}$ is the commutator of two elements of G_1, the uniqueness of the representation shows that we must have $\delta_{12} = \delta_{34} = \delta_{14} = \delta_{23} = 0$, $\delta_{13} = \delta_{24} = 1$. Since these δ_{ij} do not satisfy the above identity, we have a contradiction; so $c_{13}c_{24}$ is not a commutator, and G_1 satisfies (ii) but not (i). We note without proof that finite groups with similar properties may be found by taking factor groups of G_1.

We now construct a group G_2 with the purpose of showing that the minimal condition cannot be omitted from the hypotheses of Theorem 1 and its corollaries. Let U be a multiplicative group isomorphic to the additive group of rationals, with u in U corresponding to the rational 1; thus u^r corresponds to the rational r, and

$$u^{r_1} u^{r_2} = u^{r_1 + r_2}$$

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for any rationals r_1 and r_2. The unit element of U is u^0, which will be written as 1. Now U has an automorphism α_0 such that $x\alpha_0 = x^{-1}$ for all $x \in U$, and automorphisms α_n such that $x\alpha_n = x^{p_n}$ for all $x \in U$, where p_n is the nth *odd* prime. These automorphisms $\alpha_0, \alpha_1, \alpha_2, \ldots$ generate an abelian group A of automorphisms of U. The example G_2 is the splitting extension of U by A, and we suppose that the element a_i of G_2 corresponds to the automorphism α_i of A.

The proof that $K(g)$ consists of elements $[g, x]$ is in two parts.

(i) Let $[g, u] \neq 1$, say $u^g = u^\rho$ for some rational $\rho \neq 1$. Clearly $K(g) \subseteq U$, and we can in fact solve the equation

$$[g, x] = u^\sigma$$

for x, where σ is any rational. It is easy to verify that $x = u^{\sigma/(1-\rho)}$ is a solution. Thus $K(g)$ has the required property.

(ii) Let $[g, u] = 1$. We may assume that g is not central in G, for then $K(g) = 1$ and the result is trivial. Thus $g = au^\tau$, where a is central in G and $\tau \neq 0$, and $K(g)$ is generated by elements of the form $u^{\phi\tau}$, where ϕ is rational with even numerator and odd denominator. Consequently all elements of $K(g)$ have the same form as $u^{\phi\tau}$, and we have to solve an equation of the form

$$[g, x] = u^{\phi\tau},$$

or equivalently

$$(u^\tau)^x = u^{(\phi+1)\tau},$$

for x. The form of ϕ shows that $\phi+1$ is the quotient of two odd integers and, in particular, that $\phi+1$ is non-zero:

$$\phi+1 = (-1)^\varepsilon p_{i_1}^{\varepsilon_1} p_{i_2}^{\varepsilon_2} \ldots p_{i_s}^{\varepsilon_s},$$

where $\varepsilon = 0$ or 1, the ε_1 are non-zero integers, and $s \geqq 0$. Then for x we take the element $a_0^\varepsilon a_{i_1}^{\varepsilon_1} a_{i_2}^{\varepsilon_2} \ldots a_{i_s}^{\varepsilon_s}$.

In either case $K(g)$ contains no elements other than the $[g, x]$. But G_2 is not a ZA group as its subgroup $\{u, a_0\}$ is certainly not ZA.

3. In this section we discuss groups with each $K(g)$ cyclic, after a digression on similar conditions for $\{g\}^G$.

More precisely, we start by proving the equivalence of the three following conditions on the group G:

(i) every subgroup is normal;
(ii) every $\{g\}^G$ is cyclic;
(iii) every $\{g\}^G$ is locally cyclic.

Clearly (i) implies (ii) and (ii) implies (iii), leaving us to show that (iii) implies (i). Consider the subgroup $\{g, g^x\}$, where x is an arbitrary element of the group G, which satisfies (iii). As $\{g, g^x\}$is cyclic, being a finitely generated subgroup of $\{g\}^G$, we have

$$g = h^\alpha, \quad g^x = h^\beta$$

for some h and some coprime α and β; thus

$$(h^\alpha)^x = h^\beta, \qquad x^{h^\alpha} = xh^{\alpha-\beta}.$$

Now $\{x, xh^{\alpha-\beta}\}$ is cyclic and so abelian. Hence we have

$$h^{\alpha(\alpha-\beta)} = (h^{\alpha(\alpha-\beta)})^x = h^{\beta(\alpha-\beta)}, \qquad h^{(\alpha-\beta)^2} = 1.$$

If h has infinite order, then $\alpha = \beta$, that is $[g, x] = 1$. If h has finite order, then the numbers α, β and $(\alpha-\beta)^2$ are coprime in pairs; so g^x lies in $\{g\}$. In either case $\{g\}$ is normal in G, which at once gives (i).

The theorem of Dedekind and Zassenhaus describes completely the groups satisfying (i); see [**5**, pp. 159–161]. Such a group, if non-abelian, is the direct product of a quaternion group, an abelian group of exponent two, and an abelian group with every element of odd order.

When we impose the condition that every $K(g)$ is cyclic or locally cyclic (see Theorems 2 and 3), we cannot hope to determine more than the structure of G'.

In preparation for both these theorems we prove now that G' is abelian when every $K(g)$ in G is locally cyclic. If $c = [a, b]$ and $d = [a', b']$ are arbitrary commutators in G, then the subgroup

$$\{d, d^{a^{-1}}, d^{a^{-1}b^{-1}}, d^{a^{-1}b^{-1}a}, d^{[a,b]}\}$$

of $K(a')$ is cyclic, with generator h say. Hence we have

$$d = h^\alpha, \quad d^{a^{-1}} = h^\beta, \quad d^{a^{-1}b^{-1}} = h^\gamma, \quad d^{a^{-1}b^{-1}a} = h^\delta, \quad d^{[a,b]} = h^\varepsilon.$$

It follows that

$$h^{\alpha\gamma} = d^\gamma = (h^{\beta\gamma})^a = (d^{a^{-1}b^{-1}a})^\beta = h^{\beta\delta} = (d^{a^{-1}})^\delta = (h^{\gamma\delta})^b = (d^{[a,b]})^\gamma = h^{\varepsilon\gamma}, \qquad h^{\gamma(\alpha-\varepsilon)} = 1.$$

Successive transformations by b, ba, a and ab give

$$h^{\beta(\alpha-\varepsilon)} = h^{\alpha(\alpha-\varepsilon)} = h^{\delta(\alpha-\varepsilon)} = h^{\varepsilon(\alpha-\varepsilon)} = 1.$$

Since

$$h \in \{h^\alpha, h^\beta, h^\gamma, h^\delta, h^\varepsilon\},$$

we have

$$h^{\alpha-\varepsilon} = 1,$$

that is, $d = d^{[a,b]}$. Therefore, as arbitrary commutators c and d in G commute, we conclude that G' is abelian.

We state once and for all the fact that if every $K(g)$ in G is locally cyclic, or cyclic, then the same property is to be found in all subgroups and factor groups of G.

It is convenient to consider first the case in which G' is finite.

THEOREM 2. *Let G be a group with finite commutator subgroup. Then G' is cyclic if and only if $K(g)$ is cyclic for each g in G.*

Proof. When every $K(g)$ is cyclic, we use induction on the order of G' to establish that G' is cyclic. Suppose that the abelian group G' has two distinct non-trivial Sylow subgroups S_p and S_q. Each of these is characteristic in G' and so normal in G. By the induction hypothesis, G/S_p has its commutator subgroup G'/S_p cyclic, and similarly G'/S_q is cyclic. Therefore G' is cyclic.

Next suppose that G' is a non-trivial p-group for some prime p. In this case G' contains a subgroup N, of order p, which is normal in G, because we may take N to be the subgroup of order p in any non-trivial $K(g)$. We have G'/N cyclic. If G' is non-cyclic, then it is the direct product of cyclic subgroups of orders p and p^n respectively, where $n \geqq 1$, and we show that this case is impossible.

We must have $n = 1$; for if $n > 1$, and if H denotes the subgroup of G' generated by the pth powers of all its elements, then $H \supset 1$ and G'/H is cyclic by the induction hypothesis, which is impossible. Therefore G' is the direct product of subgroups $K(a_1)$ and $K(a_3)$, with

$$[a_1, a_2] = c_{12} \neq 1, \qquad [a_3, a_4] = c_{34} \neq 1,$$

say. Clearly

$$K(a_1) = K(a_2) = \{c_{12}\}, \qquad K(a_3) = K(a_4) = \{c_{34}\},$$

$$[a_2, a_3] \in K(a_2) \cap K(a_3) = 1, \qquad [a_1, a_4] \in K(a_1) \cap K(a_4) = 1.$$

Consider $K(a_1 a_3)$. We have

$$(a_1 a_3)^{a_2} = a_1 c_{12} a_3 = a_1 a_3 c_{12}^{a_3}, \qquad (a_1 a_3)^{a_4} = a_1 a_3 c_{34},$$

and therefore

$$K(a_1 a_3) \supseteq \{c_{12}, c_{34}\}.$$

This contradicts the fact that $K(a_1 a_3)$ is cyclic, and completes the proof of Theorem 2.

A lemma of a technical nature precedes Theorem 3.

LEMMA. *A finitely generated group* $\{c_1, c_2, \ldots, c_n\}$ *is cyclic if and only if* $\{c_i, c_j\}$ *is cyclic for all* i *and* j *with* $1 \leqq i < j \leqq n$.

Proof. We establish the less trivial part of the lemma in several stages. If each $\{c_i, c_j\}$ is cyclic, then $[c_i, c_j] = 1$ and the group A generated by $c_1, c_2, \ldots, c_n$ is abelian. Suppose that A is a p-group. The fact that $\{c_i, c_j\}$ is a cyclic p-group shows that $c_i \in \{c_j\}$ or $c_j \in \{c_i\}$; so either c_i or c_j can be omitted from the given system of generators. An obvious induction on n shows that A is cyclic.

Suppose next that A is periodic. For an arbitrary prime p we choose elements c_{ip} which generate the Sylow p-subgroup of $\{c_i\}$, for $1 \leqq i \leqq n$. The generators $c_{1p}, c_{2p}, \ldots, c_{np}$ of the Sylow p-subgroup of A inherit the property of the generators of A; therefore each Sylow p-subgroup of A is cyclic. It follows that A is cyclic.

There remains the case in which A is infinite, though the periodic subgroup of A is finite as A is finitely generated. If this subgroup has order m, we consider the group A/M, where M is generated by the mth powers of the elements of A. Now A/M is finite, and its generators $c_1 M, c_2 M, \ldots, c_n M$ inherit the property of the generators of A; so A/M must be cyclic. Its order is at most m. As A has elements of infinite order and its periodic part has order m, we must have $m = 1$.

Therefore we consider the factor group A/S of the torsion-free group A, where S is generated by all squares in A. It is finite, and so cyclic. But A, having the same number of generators as A/S, is then cyclic.

This completes the proof of the lemma.

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THEOREM 3. *The commutator subgroup of the group G is locally cyclic if and only if $K(g)$ is locally cyclic for each g in G.*

Proof. It is enough to show that if each $K(g)$ is locally cyclic, then so is G', which was shown above to be abelian. That is, we wish to show that any finite set of elements of G' generates a cyclic subgroup, or (equivalently) that any finite set of commutators generates a cyclic subgroup. The lemma reduces this problem to that of showing that any pair of commutators generates a cyclic subgroup.

Let $[a_1, a_2]$ and $[a_3, a_4]$ be any two commutators. As we work in the subgroup $\{a_1, a_2, a_3, a_4\}$ only, we shall take this to be G. If $c_{ij} = [a_i, a_j]$ for $1 \leqq i \leqq 4$ and $1 \leqq j \leqq 4$, our aim is to show that $\{c_{12}, c_{34}\}$ is cyclic. We suppose that $c_{12} \neq 1$ and $c_{34} \neq 1$.

First we discuss the case when $c_{13} = c_{24} = 1$. Then

$$(a_1a_4)^{a_2} = a_1c_{12}a_4 = a_1a_4c_{12}^{a_4}, \qquad (a_1a_4)^{a_3} = a_1a_4c_{34}^{-1};$$

as c_{12} and c_{34} therefore lie in the locally cyclic group $K(a_1a_4)$, we see that $\{c_{12}, c_{34}\}$ is cyclic. We may, and shall, assume from here onwards that $c_{13} \neq 1$.

Next, suppose that c_{12} has finite order. Then c_{13} also has finite order, as $\{c_{12}, c_{13}\}$ is a cyclic subgroup of $K(a_1)$. Similarly c_{34} has finite order, and indeed each subgroup $K(a_i)$ for $1 \leqq i \leqq 4$ is periodic, as it contains a non-trivial element of finite order. Now an arbitrary commutator c in G may be written as

$$[x_1x_2 \dots x_r, y_1y_2 \dots y_s],$$

where each x_i and each y_i is one of $a_i^{\pm 1}$ for $1 \leqq i \leqq 4$, and the well-known identical relations

$$[xy, z] = [x, z]^y[y, z], \qquad [x^{-1}, y] = [y, x]^{x^{-1}}$$

may be used to expand c as the product of certain conjugates of the commutators c_{ij}. This proves that $G' \subseteq K(a_1)K(a_2)K(a_3)$; hence G' is periodic. Because each c_{ij} has finite order it generates a characteristic subgroup of $K(a_i)$, and so a normal subgroup of G. It follows that c_{ij} has a finite number of conjugates. As G' is finitely generated and periodic, G' is finite. By Theorem 2, G' is cyclic, and so is its subgroup $\{c_{12}, c_{34}\}$. We shall, therefore, in future suppose that c_{12} has infinite order, which clearly implies that c_{34} has infinite order.

In this case we investigate the structure of G' and its embedding in G. If d is a generator of the cyclic subgroup $\{c_{12}, c_{13}\}$ of $K(a_1)$, we have

$$c_{12} = d^{\alpha}, \qquad c_{13} = d^{\beta}$$

for some $\alpha \neq 0$ and $\beta \neq 0$; so $c_{12}^{\beta} = c_{13}^{\alpha}$. Similar consideration of $K(a_3)$ gives $c_{13}^{\beta'} = c_{34}^{\alpha'}$ for some $\alpha' \neq 0$ and some $\beta' \neq 0$. These results combine to give

$$c_{12}^{\gamma_{12}} = c_{34}^{\gamma_{34}}, \tag{$*$}$$

where $\gamma_{12} = \beta\beta' \neq 0$, $\gamma_{34} = \alpha\alpha' \neq 0$. It is easy to see that a relation of the same sort holds when c_{34} is replaced by any non-trivial commutator among $c_{13}, c_{14}, c_{23}, c_{24}$, because $K(a_1)$ and $K(a_2)$ are locally cyclic.

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Now let x be an arbitrary element of G, and let d_{12} be a generator of the cyclic subgroup $\{c_{12}, c_{12}^x\}$ of $K(a_1)$, so that

$$c_{12} = d_{12}^{\lambda}, \quad c_{12}^x = d_{12}^{\mu},$$

where λ and μ are coprime; thus

$$(d_{12}^{\lambda})^x = d_{12}^{\mu}.$$

On raising both sides to the power γ_{12} and using (*), we find that

$$(c_{34}^{\lambda\gamma_{34}})^x = c_{34}^{\mu\gamma_{34}}.$$

Let d_{34} generate the cyclic subgroup $\{c_{34}, c_{34}^x\}$, so that

$$c_{34} = d_{34}^{\theta}, \qquad c_{34}^x = d_{34}^{\omega}, \qquad (d_{34}^{\theta})^x = d_{34}^{\omega},$$

where θ and ω are coprime. This last relation gives

$$(c_{34}^{\theta\gamma_{34}})^x = c_{34}^{\omega\gamma_{34}}.$$

We therefore have

$$c_{34}^{\mu\theta\gamma_{34}} = (c_{34}^{\lambda\theta\gamma_{34}})^x = c_{34}^{\lambda\omega\gamma_{34}}, \qquad c_{34}^{(\mu\theta-\lambda\omega)\gamma_{34}} = 1.$$

It follows, since c_{34} has infinite order, that

$$\mu\theta - \lambda\omega = 0;$$

and because λ and μ are coprime, and θ and ω are coprime, we have $\lambda = \theta$, $\mu = \omega$.

Therefore we have

$$c_{34} = d_{34}^{\lambda}, \qquad c_{34}^x = d_{34}^{\mu},$$

where x, λ and μ have the meanings explained above. A similar argument will show that

$$c_{ij} = d_{ij}^{\lambda}, \qquad c_{ij}^x = d_{ij}^{\mu}$$

for a suitable element d_{ij}, where $1 \leqq i < j \leqq 4$. When $x = a_k$, we shall write λ and μ as λ_k and μ_k respectively, for $1 \leqq k \leqq 4$.

There is one case in which (*) at once shows that $\{c_{12}, c_{34}\}$ is cyclic, namely when this subgroup is torsion-free. If $|\gamma_{12}|$ is taken to be minimal, and if γ_{12} and γ_{34} are then coprime, it follows that $\{c_{12}, c_{34}\}$ is cyclic; for we can find integers δ and ε for which

$$\delta\gamma_{12} + \varepsilon\gamma_{34} = 1,$$

and we have

$$\{c_{12}, c_{34}\} = \{c_{12}^{\varepsilon} c_{34}^{\delta}\}.$$

Therefore we shall assume that $\{c_{12}, c_{43}\}$ is infinite but not torsion-free, and it will be convenient to assume that its periodic subgroup is a p-group for some prime p. For the elements in G' of finite order prime to any fixed p form a characteristic subgroup of G' and so a normal subgroup N of G; we may replace G by G/N without disturbing any of our assumptions about c_{12} and c_{34}. What we shall show is that $\{c_{12}, c_{34}\}$ contains in fact no element of order p, and so is torsion-free.

Consider the case in which c_{12} is central in G. Then $\lambda_k = \mu_k = 1$ for $1 \leqq k \leqq 4$; so c_{34} is central in G. This ensures that the subgroup M generated by the mth powers of the elements of $\{c_{12}, c_{34}\}$ is normal, m being the order of the periodic subgroup of $\{c_{12}, c_{34}\}$. Thus $c_{12}M$ and $c_{34}M$ have finite orders exceeding 1; so the subgroup $\{c_{12}M, c_{34}M\}$ of G/M is cyclic, and its order is at most m. Therefore $m = 1$, and then $\{c_{12}, c_{34}\}$ is cyclic, as we wished to prove.

For the rest of the proof of Theorem 3 we assume that neither c_{12} nor c_{34} is central in G, that is that, in the notation introduced above, some $\mu_k - \lambda_k$ is non-zero. We have that $[c_{12}, a_k]^{\lambda_k}$ belongs to $K(c_{12})$ and is equal to

$$d_{12}^{(-\lambda_k+\mu_k)\lambda_k} = c_{12}^{\mu_k-\lambda_k},$$

for a suitable element d_{12}; this shows that c_{12} has finite order modulo $K(c_{12})$. Consequently, the relation (∗) shows that c_{34} has finite order modulo $K(c_{12})$. A similar argument shows that c_{12} and c_{34} have finite orders modulo $K(c_{34})$. If we put $K = K(c_{12}) \cap K(c_{34})$, we see that $c_{12}K$ and $c_{34}K$ are elements of finite order in G/K.

An earlier result now indicates that $\{c_{12}K, c_{34}K\}$ is a cyclic subgroup of G/K, say $\{cK\}$, where

$$c_{12}K = (cK)^{\alpha}, \quad c_{34}K = (cK)^{\beta},$$

and α and β are coprime. The relations

$$(c_{12}K)^{\beta} = (c_{34}K)^{\alpha}, \qquad c_{12}^{\beta}c_{34}^{-\alpha} \in K$$

follow, and here we assume without losing generality that α and p are coprime. Then we have

$$c_{12}^{\beta}c_{34}^{-\alpha} \in K(c_{12}).$$

Next we examine a typical generator of $K(c_{12})$. This has the form $[c_{12}, x]$, or (because G' is abelian) $[c_{12}, a]$, where $a = a_1^{n_1}a_2^{n_2}a_3^{n_3}a_4^{n_4}$ for certain integers n_k. If we suppose for the moment that each n_k is positive, then there is an element d such that

$$c_{12} = d^{\lambda}, \quad \lambda = \lambda_1^{n_1}\lambda_2^{n_2}\lambda_3^{n_3}\lambda_4^{n_4}; \qquad c_{12}^{a} = d^{\mu}, \quad \mu = \mu_1^{n_1}\mu_2^{n_2}\mu_3^{n_3}\mu_4^{n_4}.$$

It follows from this that

$$[c_{12}, x] = [c_{12}, a] = d^{\mu-\lambda}, \qquad [c_{12}, x]^{\lambda} = c_{12}{}^{\mu-\lambda}.$$

Relations of the same sort can be found when some n_k are negative. For instance, when n_1 is negative and the rest are positive the last relation holds provided we take λ to be $\mu_1^{n_1}\lambda_2^{n_2}\lambda_3^{n_3}\lambda_4^{n_4}$, and μ to be $\lambda_1^{n_1}\mu_2^{n_2}\mu_3^{n_3}\mu_4^{n_4}$.

We note that no λ_k or μ_k is divisible by p. For we may assume that $\{c_{12}, c_{34}\}$ contains an element c of order p, and for this element we have

$$(c^{\lambda_k})^{a_k} = c^{\mu_k},$$

where λ_k and μ_k are coprime. If μ_k, for instance, was divisible by p, then the transformation of G by a_k would not be an automorphism. Therefore $[c_{12}, x]^{\lambda}$ lies in $\{c_{12}\}$, where λ and p are coprime. This with the earlier relation $c_{12}^{\beta}c_{34}^{-\alpha} \in K(c_{12})$ shows that

$$c_{12}^{\beta\omega}c_{34}^{-\alpha\omega} \in \{c_{12}\}$$

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for some number ω prime to p, and so

$$c_{34}^{\omega_1} = c_{12}^{\omega_2},$$

where ω_1 is prime to p.

But the relation of this form with $|\omega_1|$ minimal, and the fact that elements of finite order in $\{c_{12}, c_{34}\}$ have p-power order, show (as explained earlier) that $\{c_{12}, c_{34}\}$ is cyclic.

This completes the proof of Theorem 3.

COROLLARY. *The commutator subgroup of the group G is locally cyclic if and only if $[g, x]$ and $[g, y]$ generate a cyclic subgroup, where g, x and y are arbitrary elements of G.*

Proof. The lemma shows that our hypothesis implies that every $K(g)$ is locally cyclic. Application of Theorem 3 completes the proof.

Finally we describe a group G_3 in which G_3' is an arbitrary locally cyclic group while every $K(g)$ is cyclic. Let F be the free nilpotent group of class two on two generators, take a countable infinity of copies of F, and let P be the restricted directed product of all these groups. Thus the centre of P, which is also P', is a free abelian group of countably infinite rank. Now to any given locally cyclic group L there corresponds a subgroup N of P such that L is isomorphic to P'/N; the example G_3 is defined to be P/N. The proof of the fact asserted about $K(g)$ is easy, and is omitted.

In particular a non-cyclic group, for instance the additive rationals, may be taken for L. This shows that Theorem 2 does not always hold when G' is infinite.

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