EXTREME POINTS IN SPACES OF ANALYTIC FUNCTIONS

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1. Introduction and statement of results. Our aim in this paper is to obtain some theorems concerning spaces of analytic functions on a finite open Riemann surface R which extend known results for the disc $\Delta = \{|z| < 1\}$. Suppose that R has a smooth boundary bR consisting of t closed curves, and that the interior genus of R is s. The first Betti number of R is then $r = 2s + t - 1$.

Let $H^\infty(R)$ be the algebra of bounded analytic functions on R, with the uniform norm $||f||_\infty = \sup_R|f|$, and let $A(R)$ be the subalgebra of $H^\infty(R)$ consisting of functions which have continuous extensions to bR. For a fixed point $q \in R$, let $d\mu$ be the harmonic measure for q on bR. $H^1(d\mu)$ will denote the space of analytic functions f on R such that $|f|$ has a harmonic majorant. The norm of f in H^1 is defined by $||f|| = u(q)$, where u is the least harmonic majorant of f. A function $f \in H^1(d\mu)$ is determined by its boundary values, which exist almost everywhere and will also be denoted by f. Also,

$$||f|| = \int_{bR} |f| \, d\mu.$$

de Leeuw and Rudin (**2**) characterized the extreme points of the unit ball of $H^1(\Delta)$ as the outer functions of norm 1. From this it easily follows that f is in the closure of the set of extreme points of the unit ball of $H^1(\Delta)$ if and only if $||f|| = 1$ and f has no zeros in the open disc. Our analogue of this result is the following.

THEOREM 1. *The closure of the set of extreme points of the unit ball of $H^1(d\mu)$ consists of all functions $f \in H^1(d\mu)$ such that $||f|| = 1$ and the number of zeros of f, counting multiplicity, does not exceed $r/2$.*

It turns out that the nature of the zeros of extreme points depends on the existence of certain meromorphic functions on R (cf. Lemma 4). Once this correspondence is established, it is easy to obtain the following result.

THEOREM 2. *There exists a point on R which is not the zero of an extreme point of the unit ball of $H^1(d\mu)$ if and only if R is conformally equivalent to a parallel slit domain with slits along the real axis.*

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In this case, we calculate explicitly the extreme points in Theorem 6. The classification of the extreme points of the unit balls of some other function spaces associated with R turns out to be the same as for the disc (**4**).

THEOREM 3. *A function $f \in H^\infty(R)$ is an extreme point of the unit ball of $H^\infty(R)$ if and only if $||f||_\infty = 1$ and $\int \log[1 - |f|]\, d\mu = -\infty$.*

THEOREM 4. *A function $f \in A(R)$ is an extreme point of the unit ball of $A(R)$ if and only if $||f||_\infty = 1$ and $\int \log[1 - |f|]\, d\mu = -\infty$.*

2. Extreme points in $H^\infty(R)$ and $A(R)$. Theorems 3 and 4 are probably well known; the proofs proceed exactly as in (**4**, pp. 138–139), except that an additional argument is needed to show that the analytic functions constructed there can be chosen to be single-valued. This can be done by recourse to Theorem 4 of (**1**). We will prove a lemma which also does this, and which has some other applications.

Let $\gamma_1, \ldots, \gamma_r$ be curves on R with the fixed base point q, such that $\gamma_1, \ldots, \gamma_r$ is a basis for the homology of R. If u is any continuous function on bR, it has a continuous harmonic extension, also denoted by u, to R. The functional $u \to \int_{\gamma_j} {}^*du$ is a real continuous functional on $C(bR)$ which is orthogonal to $A(R)$. The measure on bR which represents this functional is the boundary value of an analytic differential ω_j, so that $\int_{\gamma_j} {}^*du = \int_{bR} u\,\omega_j$. The Schottky differentials $\omega_1, \ldots, \omega_r$ form a basis for the space of real measures orthogonal to $A(R)$. Since they are real along bR, they extend analytically across bR and form a basis for the space of analytic differentials on the doubled surface of R (**1**). Note that $\int_{bR} u\omega_j = \int_{\gamma_j} {}^*du$ is the period of the harmonic conjugate of u around γ_j.

If u is a bounded measurable function on bR, u can be extended harmonically to R in the sense that there is a unique bounded harmonic function on R which has non-tangential boundary values coinciding with u almost everywhere.

LEMMA 1. *Let E be a measurable subset of bR such that $\mu(E) > 0$, and let $c_1, \ldots, c_r$ be real numbers. Then there is a function $u \in L^\infty(d\mu)$ such that $u \leqq 0$, u is zero on $bR - E$, and the period of the harmonic conjugate ${}^*u$ of the harmonic extension of u around γ_j is congruent to c_j modulo 2π. If, in addition, E is open, then u can be chosen to be C^∞ on bR.*

Proof. Let U be the set of all real functions $u \in L^\infty(d\mu)$ such that $u \leqq 0$ and u is zero on $bR - E$. If E is open, we also assume that the functions in U are C^∞. In any event, U is a convex cone. The map $\Phi(u) = [\int u\omega_1, \ldots, \int u\omega_r]$ is a linear transformation of U into $\mathbf{R}^r$ (r-dimensional Euclidean space), so $\Phi(U)$ is a convex cone in $\mathbf{R}^r$.

Suppose $\Phi(U)$ has no interior point. Then $\Phi(U)$ is contained in a hyperplane, say

$$\sum_{j=1}^{r} a_j x_j = 0$$

for all $[x_1, \ldots, x_r] \in \Phi(U)$, where $0 \neq [a_1, \ldots, a_r] \in \mathbf{R}^r$. Then

$$\int_{bR} u\left(\sum_{j=1}^{r} a_j \omega_j\right) = 0$$

for all $u \in U$, so $\sum_{j=1}^{r} a_j \omega_j = 0$ on E. Since $\sum a_j \omega_j$ is analytic, $\sum a_j \omega_j \equiv 0$, contradicting the linear independence of the ω's.

Hence $\Phi(U)$ has an interior point $[y_1, \ldots, y_r]$. If $b > 0$ is large, there is a ball with centre $[by_1, \ldots, by_r]$ which is contained in $\Phi(U)$, and which contains a representative of every r-tuple $[c_1, \ldots, c_r]$ modulo 2π.

Fatou's theorem shows that $H^\infty(R)$ can be considered as a subalgebra of $L^\infty(d\mu)$. Lemma 1 can be used, for instance, to deduce that the Šilov boundary of $H^\infty(R)$ is the maximal ideal space of $L^\infty(d\mu)$, by the same proof as that for the disc (cf. **4**, p. 174). To prove the assertion, it suffices to show that for any measurable set F such that $\mu(F) > 0$, there exists an $f \in H^\infty$ such that the essential supremum of $|f|$ on F is strictly greater than the essential supremum of $|f|$ on $bR - F = E$. Let u be the harmonic function with boundary values 1 on F and 0 on E. If $\mu(E) = 0$, there is nothing to prove. If $\mu(E) > 0$, we apply Lemma 1 to find a bounded harmonic function v such that $v \leqq 0$, $v = 0$ on F, and the periods of $^*v$ are congruent to the periods of $-^*u$ modulo 2π. If $f = \exp(u + v + i\,^*u + i\,^*v)$, then $f \in H^\infty$, $|f| = e$ a.e. on F, and $|f| \leqq 1$ a.e. on E. Hence f is the desired function.

Lemma 1 can be used to carry over the proof of Theorem 4 from (**4**) as follows. If $f \in A(R)$, $\|f\| = 1$, and $\int \log[1 - |f|]\, d\mu > -\infty$, we must construct $g \in A(R)$ such that $|g(z)| \leqq 1 - |f(z)|$ for $z \in bR$. Let v be an integrable function on bR such that $v \leqq \log[1 - |f|]$, and v is C^∞ on each open arc of the set where $|f| \neq 1$. Let $E = bR$, and choose a C^∞ function u in accordance with Lemma 1, so that the periods of $^*u$ are congruent to the periods of $-^*v$ modulo 2π. If $g = \exp(u + v + i\,^*u + i\,^*v)$, then $g \in A(R)$ and $|g| \leqq e^v \leqq 1 - |f|$ on bR.

The proof of Theorem 3 is carried over similarly from (**4**). Using Lemma 1 in conjunction with the results of (**8**), the following theorem can also be established for $A(R)$ (cf. **4**, pp. 80, 88).

THEOREM 5. *Every closed ideal in $A(R)$ is the closure of the principal ideal generated by some function in $A(R)$.*

3. Reduction to meromorphic functions. It is useful to consider a class of functions which includes $H^1(d\mu)$: H_m^1 is the class of multiple-valued analytic functions f on R such that $|f|$ is single-valued and $|f|$ has a harmonic majorant on R. If $f \in H_m^1$, then $|f|$ has boundary values a.e. on bR and on bR $|f|$ and $\log|f|$ are in $L^1(d\mu)$ when $f \not\equiv 0$. A function $f \in H_m^1$ is said to be an *outer function* if $\log|f(q)| = \int_{bR} \log|f|\, d\mu$. H_m^∞ is the class of bounded functions in H_m^1. A function $f \in H_m^\infty$ is an *inner function* if $|f| = 1$ a.e. on bR. A *singular function* is an inner function which never vanishes on R. An inner function f with zeros $p_1, p_2, \ldots$ is a *Blaschke product* if $\log|f(p)| = -\sum g(p, p_j)$ for all $p \in R$ where g is the Green's function for R. A Blaschke product is *finite* if it has a finite number of zeros.

If $f \in H^1(d\mu)$ (or H_m^1), then f has the factorization $f = BSG$, where B is a Blaschke product, S is a singular function, and G is an outer function; these factors are unique up to multiplication by constants of modulus one (see **9**).

The factorization of H^1 functions on R is related to the factorization of H^1 functions on Δ in the following way. Let $\tau\colon \Delta \to R$ be an analytic universal covering map of R. For $f \in H_m^1$ let $\hat{f} = f \circ \tau$. If $f \in H_m^1$ and $f = BSG$ is the factorization of f, then $\hat{f} \in H^1(\Delta)$ and $\hat{f} = \hat{B}\hat{S}\hat{G}$ is the factorization of $\hat{f}$. For later reference it is useful to note that τ can be extended to $b\Delta$ a.e. (cf. **8**, p. 500).

The jth *period* of a function $f \in H_m^1$ is the increase in the argument of f when some branch of f is continued around γ_j. The *period vector* of f is the vector $\Pi(f) \in \mathbf{R}^r$ whose jth component is the jth period of f.

In the remainder of the paper, "extreme point" will always refer to the extreme points of the unit ball of $H^1(d\mu)$. The point of departure for the classification of the extreme points is the following lemma of de Leeuw and Rudin (see **2** and **4** for proofs).

LEMMA 2. *Suppose $f \in H^1(d\mu)$ and $||f|| = 1$. f is not an extreme point if and only if there is a non-constant real-valued function $k \in L^\infty(d\mu)$ such that on bR, kf is the boundary function of a function in $H^1(d\mu)$.*

An inner function $F \in H_m^\infty$ is *extremal* if whenever a real-valued function $k \in L^\infty(d\mu)$ is such that kF is the boundary function of a function in H_m^∞ which has the same periods (modulo 2π) as F, then k is a constant.

LEMMA 3. *Suppose $f \in H^1(d\mu)$ and $||f|| = 1$. f is an extreme point if and only if the inner part of f is extremal.*

Proof. Let $f = FG$ be the factorization of f as a product of an inner and an outer function. In view of Lemma 2, f is not an extreme point if F is not extremal. Suppose f is not an extreme point, and let k be the function of Lemma 2 and h the function in $H^1(d\mu)$ such that $h = kf$ on bR. Then on bR, $|h| \leqq ||k||_\infty |G|$, so $|h|/|G|$ is bounded on bR. Since G is an outer function $|h|/|G|$ is bounded on R, thus $h/G \in H_m^\infty$. Since on bR, $h/G = (kFG)/G = kF$, the lemma is proved.

Lemma 3 shows that the extremal properties of functions $f \in H^1(d\mu)$ depend only on their inner parts. Since 1 is an extremal inner function, we have the following corollary.

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COROLLARY. *If f is an outer function in $H^1(d\mu)$ of norm one, then f is an extreme point.*

In (**3**), Forelli showed that if f is extremal, the L^1-closure of $A(R)f$ has co-dimension at most $r/2$ in $H^1(d\mu)$. This shows, in particular, that every non-constant extremal inner function is a Blaschke product with at most $r/2$ zeros, counting multiplicity. We will not use Forelli's result in what follows, but we will restrict ourselves to finite Blaschke products, giving our own derivation of the bound on the number of zeros of extremal inner functions.

LEMMA 4. *Suppose F is a finite Blaschke product. F is not extremal if and only if there is a non-constant meromorphic function h on $\bar{R}$ such that h is real on bR, and every pole of h is a zero of F and the order of the pole is at most the order of the zero.*

Proof. If there exists such a meromorphic function h, then $hF \in H_m^\infty$ has the same periods as F, so F is not extremal. Suppose that F is not extremal. Choose a non-constant real-valued function $k \in L^\infty(d\mu)$ such that kF is the boundary function of $f \in H_m^\infty$, where f has the same periods as F (modulo 2π). Then $h = f/F$ is meromorphic on R, is bounded near bR, and has real boundary values k on bR. Hence h can be continued analytically across bR by reflection. Since hF is analytic, the order of the pole of h at a point p cannot exceed the order of the zero of F at p.

4. Application of the Riemann-Roch theorem.

For δ a divisor on R, $M[\delta]$ is the space of meromorphic functions on $\bar{R}$ which are real on bR and are multiples of δ; $D[\delta]$ is the space of meromorphic differentials on $\bar{R}$ which are real along bR and are multiples of δ; $A[\delta]$ and $B[\delta]$ are the real dimensions of $M[\delta]$ and $D[\delta]$, respectively; and $d[\delta]$ is the degree of δ. In (**5**), Royden showed that

$$A[1/\delta] = B[\delta] + 2d[\delta] - r + 1. \tag{1}$$

The divisor associated with a meromorphic function or differential will be denoted by $(\cdot)$. Lemma 4 then states that a finite Blaschke product F is not extremal if and only if $A[(1/F)] \geqq 2$.

LEMMA 5. *If F is an inner function which has more than $r/2$ zeros, counting multiplicity, then F is not extremal.*

Proof. Let δ be a divisor such that $d[\delta] > r/2$ and δ divides (F). By (1), $A[1/\delta] \geqq 2$. Hence there is a non-constant meromorphic function $h \in M[1/\delta]$. Now $hF \in H_m^\infty$, and hF has the same periods as F. Since h is real on bR, F is not extremal.

There is one case in which we can compute $A[1/\delta]$ easily.

THEOREM 6. *Suppose that R is a parallel slit domain with slits along the real axis. An integral divisor δ is the divisor of an extremal Blaschke product if and*

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only if $d[\delta] \leqq r/2$ and whenever p divides δ, then $\bar{p}$ (the complex conjugate of p) does not divide δ. In particular, for $r \geqq 2$, a point $p \in R$ is the zero of an extremal Blaschke product if and only if p is not real.

Proof. If $p \in R$ and p is real, then $1/(z - p) \in M[1/p]$, so $A[1/p] \geqq 2$. If $p \in R$ is complex, then $1/(z - p)(z - \bar{p}) \in M[1/p\bar{p}]$, so $A[1/p\bar{p}] \geqq 2$. Hence no extremal Blaschke product can have real zeros, or two zeros which are complex conjugates. This proves the forward implication.

To prove the reverse implication, we suppose that $d[\delta] \leqq r/2$ and that δ contains no real or complex conjugate points. A basis for the space of analytic differentials real along bR is $\{dz/f, z\,dz/f, \ldots, z^{r-1}\,dz/f\}$, where $f(z)^2 = (z - e_1) \ldots (z - e_{2r+2})$, and $e_1, \ldots, e_{2r+2}$ are the end points of the slits (cf. **7**, p. 293). Hence $D[\delta]$ consists of all differentials of the form $g\,dz/f$, where g is a polynomial of degree at most $r - 1$ with real coefficients and δ is a divisor of (g). Then $\bar{\delta}$ must also be a divisor of (g), so $\delta\bar{\delta}$ is a divisor of (g). Hence we see that there are $r - d[\delta\bar{\delta}]$ linearly independent polynomials g such that $g\,dz/f \in D[\delta]$. That is, $B[\delta] = r - d[\delta\bar{\delta}] = r - 2d[\delta]$. By (1), $A[1/\delta] = 1$, and the finite Blaschke product F such that $(F) = \delta$ is extremal.

Proof of Theorem 2. If R is conformally equivalent to a parallel slit domain with slits along the real axis, and Z is an analytic function which effects the equivalence, then no point $p \in R$ such that $Z(p)$ is real can be the zero of an extreme point.

Conversely, suppose that $p \in R$ is the zero of no extreme point. Let h be a non-constant meromorphic function on $\bar{R}$ such that h is real on bR and has only one pole, a simple pole at p. Let $\tilde{h}$ be the extension of h to the doubled surface $\tilde{R}$ of R; that is, if ϕ is the anticonformal reflection of $\tilde{R}$ across bR, then $\tilde{h}$ is defined so that $\tilde{h}(\phi(w)) = \overline{\tilde{h}(w)}$. $\tilde{h}$ has two poles on $\tilde{R}$, at p and at $\phi(p)$. Hence $\tilde{h}$ is a two-to-one map of $\tilde{R}$ onto the extended complex plane, mapping each boundary curve onto a segment of the real axis.

Let z be complex, $z \notin h(bR)$, and let Γ be a curve joining ∞ and z which does not cross $h(bR)$. $\tilde{h}^{-1}(\Gamma)$ consists of two curves on $\tilde{R}$ which do not meet bR, one curve passing through $p \in R$ and the other through $\phi(p) \in \tilde{R} - \bar{R}$. Hence one of the curves is contained in R, and the other in $\tilde{R} - \bar{R}$. In particular, h assumes the value z once and only once on R. So h is a conformal equivalence of R and the extended complex plane with the slits $h(bR)$ deleted.

As an example, we consider surfaces R for which $r = 2$. In this case, every extreme point has at most one zero. If the surface is not planar, it is topologically a torus with a disc excised and every point $p \in R$ is the zero of an extreme point. If the surface is planar, it is necessarily equivalent to a parallel slit domain with three slits along the real axis. If, for instance, R is a disc with two smaller discs excised, there is a unique circle (or line) which is orthogonal to the boundaries of the three discs. The points $p \in R$ which lie on this circle are the points which cannot be the zeros of extreme points.

5. The closure of the extreme points. This section is devoted to proving Theorem 1. Lemma 6 allows us to approximate finite Blaschke products with no more than $r/2$ zeros by extremal Blaschke products. Lemma 9 allows us to approximate singular inner functions by outer functions.

LEMMA 6. *Suppose* $1 \leqq k \leqq r/2$, $p_1 \ldots p_k$ *is a divisor on* R, *and* U_j *is a neighbourhood of* p_j, $j = 1, 2, \ldots, k$. *Then there are points* $q_j \in U_j$, $1 \leqq j \leqq k$, *such that* $q_i \neq q_j$ *for* $i \neq j$, *and* $A[1/(q_1 \ldots q_k)] = 1$.

Proof. We will prove the lemma by induction on k. Suppose $k = 1$. Let e be the unit divisor. Then if $f \in M[e]$, f is analytic on R and real on bR and so by reflection f has an analytic extension to the doubled surface $\tilde{R}$. So f is constant, $A[e] = 1$ and by (1), $B[e] = r$. Let $\phi_1, \ldots, \phi_r$ be a basis for $D[e]$. Choose q_1 in U_1 such that $\phi_1(q_1) \neq 0 \neq \phi_2(q_1)$ and $(\phi_1/\phi_2)(q_1)$ is not real. Let z be a local parameter near q_1 so that $z(q_1) = 0$ and let $\phi_j = (u_j + iv_j)dz$, where u_j and v_j are real. Now $D[q_1]$ consists of those differentials of the form $\sum_1^r t_j \phi_j$, $[t_1, \ldots, t_r] \in R^r$, such that $(\sum t_j \phi_j)(q_1) = 0$. In terms of local coordinates this equation becomes the system

$$\sum_1^r t_j u_j(0) = 0$$

$$\sum_1^r t_j v_j(0) = 0.$$

Since $(u_1(0) + iv_1(0))/(u_2(0) + iv_2(0))$ is not real,

$$\begin{vmatrix} u_1(0) & u_2(0) \\ v_1(0) & v_2(0) \end{vmatrix} \neq 0$$

and the solution system has dimension $r - 2$. Hence $B[q_1] = r - 2$ and by (1), $A[1/q_1] = 1$. Hence the lemma is true for $k = 1$.

Suppose $1 < k \leqq r/2$ and the lemma is true for the points $p_1, \ldots, p_{k-1}$. Then there are distinct points $q_j \in U_j$, $1 \leqq j \leqq k - 1$, such that

$$A[1/(q_1 \ldots q_{k-1})] = 1.$$

By (1), $B[q_1 \ldots q_{k-1}] = r - 2k + 2$ is the dimension of $D[q_1 \ldots q_{k-1}]$. Let $\alpha_1, \ldots, \alpha_n$ $(n = r - 2k + 2)$ be a basis for $D[q_1 \ldots q_{k-1}]$ and choose $q_k \in U_k$, $q_k \neq q_j$, $j = 1, 2, \ldots, k - 1$, so that $\alpha_1(q_k) \neq 0 \neq \alpha_2(q_k)$ and $(\alpha_1/\alpha_2)(q_k)$ is not real. Now $D[q_1 \ldots q_k]$ consists of the differentials of the form

$$\sum_1^n t_j \alpha_j \quad \text{such that} \quad \left(\sum_1^n t_j \alpha_j\right)(q_k) = 0.$$

By the same argument used in the first induction step the solution space has dimension $B[q_1 \ldots q_k] = n - 2 = r - 2k$. By (1), $A[1/(q_1 \ldots q_k)] = 1$.

Let z_j be a uniformizer for the jth boundary curve Γ_j so that $\Gamma_j = \{|z_j| = 1\}$, $j = 1, \ldots, t$. A singular function S determines a unique positive singular finite measure $d\sigma$ on bR (which depends on the choice of boundary uniformizers) such that $S = \exp[-(u + i\,{}^*u)]$, where

$$u(z) = \frac{1}{2\pi}\sum_{j=1}^{t} \int_{\Gamma_j} \frac{\partial g}{\partial n}\,(z, e^{i\theta_j})\, d\sigma(\theta_j),$$

g is the Green's function for R, and $z_j = r_j \exp i\theta_j$ near Γ_j. (Cf. **6**.) The Schottky differential ω_k is given along Γ_j by $q_k(\theta_j)d\theta_j$, where q_k is real. The kth period of ${}^*u$ is

$$\sum_{j=1}^{t} \int_{\Gamma_j} q_k(\theta_j)\, d\sigma(\theta_j).$$

For convenience of notation we will assume that there is only one boundary curve with uniformizer $z = re^{i\theta}$. The adaptation of the proof to the general case involves only a change of notation. In the simplified notation, the kth period of ${}^*u$ is given by $\int_{bR} q_k(\theta)\, d\sigma(\theta)$.

LEMMA 7. *Suppose $A_1, A_2, \ldots$ is a sequence of points in $\mathbf{R}^r$ which converges to 0. Then there is a sequence of singular functions $S_1, S_2, \ldots$ such that A_n is the period vector of S_n, S_n is determined by a finite linear combination of point masses on bR, and $\hat{S}_n \to 1$ a.e. on $b\Delta$.*

Proof. According to (**1**, § 4.2), there exist $r + 1$ points $w_1, \ldots, w_{r+1}$ on bR such that if B_j is the period vector of the singular function T_j corresponding to a unit point mass at w_j, then $B_1, \ldots, B_{r+1}$ are the vertices of a simplex in $\mathbf{R}^r$ which contains 0 as an interior point. We can write

$$A_n = \sum_{j=1}^{r+1} b(n, j)B_j,$$

where $b(n, j) \geqq 0$ and $b(n, j) \to 0$ as $n \to \infty$ for $j = 1, \ldots, r + 1$. Then the singular function

$$S_n = T_1^{b(n,1)} \ldots T_{r+1}^{b(n,\, r+1)}$$

has A_n as its period vector, and $\hat{S}_n \to 1$ everywhere on $\tau^{-1}(bR)$, except at the points $\tau^{-1}(w_j)$, $j = 1, \ldots, r + 1$, so $\hat{S}_n \to 1$ a.e. on $b\Delta$.

LEMMA 8. *For $\epsilon > 0$, there is a real function $v \in C^\infty(bR)$ such that for $I = (-\epsilon, \epsilon)$,*

(a) $|v| < \epsilon$ *on* $bR - I$,

(b) $-\epsilon < v$ *on* I,

(c) $\displaystyle\int_{-\epsilon}^{\epsilon} v(e^{i\theta})\, d\theta = 1,$

(d) $\displaystyle\int_{-\pi}^{\pi} v(e^{i\theta})q_k(\theta)\, d\theta = q_k(0), \qquad 1 \leqq k \leqq r.$

Proof. Let V be all those real functions $v \in C^\infty(bR)$ with properties a, b, and c. For $v \in V$, let $\Phi(v) = [\int_{bR} v\omega_1, \ldots, \int_{bR} v\omega_r]$. Then $\Phi(V)$ is a convex

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set in $\mathbf{R}^r$. If the lemma is not true, then there are real constants $\alpha_1, \ldots, \alpha_r$, not all zero, such that for all $v \in V$

$$\sum \alpha_k q_k(0) \leqq \int_{bR} v \sum \alpha_k \omega_k. \tag{2}$$

Let J be an arc on bR disjoint from I. Since $\omega_1, \ldots, \omega_r$ are linearly independent, $\sum \alpha_j \omega_j$ is not zero on R and hence not zero along J. Hence there is a real function $v_0 \in C^\infty(bR)$ which is zero on $bR - J$ such that $|v_0| < \epsilon$ and $-\int_{bR} v_0 \sum \alpha_j \omega_j = \delta > 0$. Let $u \in V$ such that $u = 0$ on $bR - I$. Setting $v = v_0 + u$ in (2) we get

$$\sum \alpha_k q_k(0) \leqq -\delta + \int_{-\pi}^{\pi} u(e^{i\theta}) \left(\sum \alpha_k q_k(\theta) \right) d\theta$$

for all such u. But this is impossible since there is a sequence $u_n \in V$ with $u_n = 0$ on $bR - I$, $n = 1, 2, \ldots$, such that

$$\int_{-\pi}^{\pi} u_n(e^{i\theta}) \left(\sum \alpha_k q_k(\theta) \right) d\theta \to \sum \alpha_k q_k(0) \qquad \text{as} \quad n \to \infty.$$

LEMMA 9. *If $S \in H_m^\infty$ is singular, there is a sequence $\{S_n\}_{n=1}^\infty$ of outer functions in H_m^∞ such that S_n has the same periods as S, $||S_n||_\infty \leqq 1$, and $\hat{S}_n \to \hat{S}$ a.e. on $b\Delta$.*

Proof. If $S = \exp[-(u + i\,{}^*u)]$ is the singular function obtained by planting a unit point mass at $z = 1$, the jth period of ${}^*u$ is $q_j(0)$. If v_n is the function of Lemma 8 for $\epsilon = 1/n$, the periods of ${}^*v_n$ are the same as those of ${}^*u$. Let $S_n = \exp[-(1/n) - (v_n + i\,{}^*v_n)]$, then $S_n \in H_m^\infty$ and $||S_n||_\infty \leqq 1$. Since $v_n\, d\theta$ converges weak * to the unit point mass at $\theta = 0$, v_n converges to u uniformly on compact subsets of R.

Passing to a subsequence, we can assume that $\hat{S}_n$ converges weakly in $L^2(b\Delta)$ to some function $g \in H^2(\Delta)$. Since $|\hat{S}_n|$ converges to $|\hat{S}|$ on compact subsets of Δ, $|\hat{S}| = |g|$ on Δ. Consequently, $g = e^{i\theta_0}S$ for some real θ_0. Absorbing this constant into ${}^*v_n$, we can assume that $\hat{S}_n$ converges weakly in $L^2(b\Delta)$ to $\hat{S}$. Since $||\hat{S}_n||_2 \leqq ||\hat{S}||_2$, we must have $||\hat{S}_n||_2 \to ||\hat{S}||_2$, and S_n converges to S strongly in $L^2(b\Delta)$. Passing to a further subsequence, we can assume that $\hat{S}_n$ converges to $\hat{S}$ a.e. on $b\Delta$.

This proves Lemma 9 in the case that S is determined by a point mass at one point. By taking products of singular functions corresponding to single point masses, one sees that Lemma 9 is valid for singular functions determined by finite linear combinations of point masses.

Now suppose S is a singular function determined by an arbitrary positive singular measure ν, and let $\{\nu_k\}_{k=1}^\infty$ be a sequence of finite positive linear combinations of point masses which converges weak * to ν. If S_k is the singular function determined by ν_k, we can assume, as earlier, that $\hat{S}_k \to \hat{S}$ in $L^2(b\Delta)$. Let $\{T_{k,n}\}_{n=1}^\infty$ be the sequence of outer functions constructed above for S_k. $\{\hat{T}_{k,n}\}_{n=1}^\infty$ converges in $L^2(b\Delta)$ to $\hat{S}_k$, and so the diagonal process yields a

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sequence $\{T_n\}_{n=1}^{\infty}$ chosen from the $\{T_{k,n}\}$ such that $\hat{T}_n \to \hat{S}$ in $L^2(b\Delta)$. Passing again to a subsequence, we can assume that $\hat{T}_n \to \hat{S}$ a.e. on $b\Delta$.

The period vectors $\Pi(T_n)$ must converge to the period vector $\Pi(S)$. According to Lemma 7, we can choose singular functions U_n determined by finite linear combinations of point masses so that $\hat{U}_n \to 1$ a.e. on $b\Delta$, and $\Pi(U_n) = \Pi(S) - \Pi(T_n)$. Since Lemma 9 is valid for singular functions determined by finite linear combinations of point masses, we can find V_n outer such that $\Pi(V_n) = \Pi(U_n)$ and $\hat{V}_n \to 1$ a.e. on $b\Delta$. Then $\Pi(V_n T_n) = \Pi(S)$, and $\hat{V}_n \hat{T}_n$ converges a.e. on $b\Delta$ to S. Also $||V_n T_n||_\infty \leqq 1$.

Proof of Theorem 1. Suppose $f \in H^1(d\mu)$ is such that $||f|| = 1$ and f has k zeros on R, $k \leqq r/2$. Let $f = BSG$ be the canonical factorization of f. By Lemma 6, we can find a sequence of extremal Blaschke products B_n whose zeros $\{p_{n1}, p_{n2}, \ldots, p_{nk}\}$ converge to the zeros of f. Since each B_n can be written explicitly as

$$B_n(p) = \exp\left(-\sum_{j=1}^{k} [g(p, p_{nj}) + i^*g(p, p_{nj})]\right)$$

we see that $\Pi(B_n) \to \Pi(B)$ and $\hat{B}_n \to \hat{B}$ a.e. on $b\Delta$.

By Lemma 7, there are singular functions S_n such that $\hat{S}_n \to 1$ a.e. on $b\Delta$ and $\Pi(S_n) = \Pi(B) - \Pi(B_n)$. Now choose T_n outer so that $|T_n| \leqq 1$, $\Pi(T_n) = \Pi(S_n S)$, and $\hat{T}_n \to \hat{S}_n \hat{S}$ in $L^2(b\Delta)$. Now

$$\Pi(B_n T_n G) = \Pi(B_n) + \Pi(T_n) + \Pi(G) = \Pi(BSG),$$

so $g_n = B_n T_n G \in H^1(d\mu)$. Also $|B_n T_n| \leqq 1$ a.e. Since $\hat{B}_n \hat{T}_n \to \hat{B}\hat{S}$ in $L^2(b\Delta)$, we can, passing to a subsequence, assume that $\hat{B}_n \hat{T}_n \to \hat{B}\hat{S}$ a.e. Hence $|g_n| \leqq |G|$ and $g_n \to f$ a.e. If $f_n = g_n/||g_n||$, then f_n is an extreme point, and $f_n \to f$ in $L^1(d\mu)$.

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