SPINOR SPACE AND LINE GEOMETRY

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Synopsis. This is the first of two papers dealing with the projective theory of spinors. It contains the algebraic introduction to the projective spinor analysis which will be dealt with in the second paper.

The leading idea may be roughly described as follows: Let $\mathbf{Q}$ be the ideal quadric of the isotropic cone of a four dimensional centered vector space R_4. The ideal space L_3 of R_4 may be looked upon as a non-euclidean space with the absolute quadric $\mathbf{Q}$. Using the Cartan matrix[1] (0,12) one obtains a "representation" of L_3 by a linear complex Γ in the spinor space S_3 (Theorem (2,2)) with a linear congruence K as an "absolute" (Theorem (1,3)). In particular the biaxial involution (2,1) which leads to Γ is closely connected with Dirac equations (dealt with in the second paper). On the other hand $\mathbf{Q}$ as a two parametric point set is mapped on K, while $\mathbf{Q}$ as a three parameter set of lineal elements (Definition (5,1)) is mapped on S_3: A lineal element is mapped on a couple of spinors (Theorem (5,2)) and *vice versa* a spinor is a map of a lineal element (Theorem (6,3)). Finally the map in S_3 of any "orthogonal" transformation in L_3 is found (Theorem (7,2)) and *vice versa* the map in L_3 of the biaxial involution (2,1) is given (Theorem (8,1)).

The second paper based on these results and on equation (6,5b) will deal with the analysis of the spinor space S_3.

Introduction. Consider a centered four dimensional space R_4 with the isotropic cone[2]

$$\mathbf{Q} \equiv x^{\mathrm{I}}x^{\mathrm{II}} + x^{\mathrm{III}}x^{\mathrm{IV}} = 0. \tag{0,1}$$

The ideal point of the direction defined by a vector $\mathbf{x}(x^{\mathrm{I}}, x^{\mathrm{II}}, x^{\mathrm{III}}, x^{\mathrm{IV}})$ will be denoted also by $\mathbf{x}$. It is obviously determined by its homogeneous coordinates $x^{\mathrm{I}} : x^{\mathrm{II}} : x^{\mathrm{III}} : x^{\mathrm{IV}}$. Hence the ideal space L_3 of R_4 may be looked upon as a non-euclidean three space with the absolute $\mathbf{Q}$. A point[3] $\mathbf{x}$ will be termed *isotropic* (*anisotropic*) if it is (is not) on $\mathbf{Q}$. The group of all projective transformations in L_3 which reproduce the form $x^{\mathrm{I}}x^{\mathrm{II}} + x^{\mathrm{III}}x^{\mathrm{IV}}$ (up to a factor of proportionality) will be denoted by (T).

The group (T) splits in a group (G) of all transformations from (T) which reproduce each of both reguli of $\mathbf{Q}$ and a family (F) of all transformations from (T) which interchange these reguli. The "dot product" $\mathbf{x} \cdot \mathbf{y}$ of two points

$$\mathbf{x} \cdot \mathbf{y} \equiv \tfrac{1}{2}(x^{\mathrm{I}}y^{\mathrm{II}} + x^{\mathrm{II}}y^{\mathrm{I}} + x^{\mathrm{III}}y^{\mathrm{IV}} + x^{\mathrm{IV}}y^{\mathrm{III}}), \tag{0,2}$$

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[1]Cartan [**1**], cf. also Veblen-von Neumann-Givens [**5**].

[2]If we put

$$x^{\mathrm{I}} = x + iy,\ x^{\mathrm{II}} = x - iy,\ x^{\mathrm{III}} = z + ct,\ x^{\mathrm{IV}} = z - ct,$$

we have

$$\mathbf{Q} \equiv x^2 + y^2 + z^2 - c^2t^2 = 0,$$

which is the usual form in the special theory of relativity.

[3]From now on we understand by *point* or *line* a point or line in L_3.

is obviously (T)-invariant (up to a factor of proportionality).

Assuming $\mathbf{Q}$ in parametric form

$$\mathbf{x} = \mathbf{x}(u^{1^\cdot}, u^{2^\cdot}) \tag{0,3}$$

and introducing the symbol[4] $\mathbf{x}_\lambda \equiv \dfrac{\partial \mathbf{x}}{\partial u^\lambda}$ we obtain the conformal metric tensor of $\mathbf{Q}$

$$a_{\lambda\mu} \equiv \mathbf{x}_\lambda . \mathbf{x}_\mu, \tag{0,4a}$$

which transforms under

$$^*\mathbf{x} = \rho(u)\mathbf{x} \tag{0,5}$$

according to

$$^*a_{\lambda\mu} = \rho^2 a_{\lambda\mu}. \tag{0,4b}$$

The rulings $^iR^{AB}$ of the reguli ^iR of $\mathbf{Q}$ may be expressed as follows:[5]

$$\begin{aligned} &{}^1R^{AB}\left(\frac{1}{x^{\mathrm{III}}}, -\frac{1}{x^{\mathrm{III}}}, \frac{1}{x^{\mathrm{I}}}, \frac{x^{\mathrm{I}}}{(x^{\mathrm{III}})^2}, 0, 0\right), \\ &{}^2R^{AB}\left(-\frac{1}{x^{\mathrm{IV}}}, -\frac{1}{x^{\mathrm{IV}}}, 0, 0, \frac{x^{\mathrm{I}}}{(x^{\mathrm{IV}})^2}, \frac{-1}{x^{\mathrm{I}}}\right), \end{aligned} \tag{0,6}$$

provided

$$x^{\mathrm{I}}x^{\mathrm{III}}x^{\mathrm{IV}} \neq 0. \tag{0,7}$$

Throughout this paper we assume that the condition (0,7) is satisfied. If we put

$$P_\lambda{}^{AB} = x^{[A}x_\lambda{}^{B]} \tag{0,8}$$

then the vectors (on $\mathbf{Q}$)

$$4\,{}^ie_\lambda = {}^iR_{AB}P_\lambda{}^{AB}$$

e.g.

$$^1e_\lambda = \frac{x^{\mathrm{II}}x_\lambda{}^{\mathrm{I}} + x^{\mathrm{III}}x_\lambda{}^{\mathrm{IV}}}{x^{\mathrm{III}}} \qquad {}^2e_\lambda = -\frac{x^{\mathrm{I}}x_\lambda{}^{\mathrm{II}} + x^{\mathrm{III}}x_\lambda{}^{\mathrm{IV}}}{x^{\mathrm{IV}}} \tag{0,9}$$

are the null vectors of $a_{\lambda\mu}$, where

$$2a_{\lambda\mu} \equiv {}^1e_\lambda\,{}^2e_\mu + {}^1e_\mu\,{}^2e_\lambda. \tag{0,10}$$

By (0,5) they transform according to

$$^{*i}e_\lambda = \rho\,{}^ie_\lambda. \tag{0,11}$$

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[4] | | | |
|---|---|---|
| A, B, C, D | | I, II, III, IV |
| a, b, c, d | have the range | 1, 2, 3, 4 |
| $\omega, \mu, \lambda, \nu$ | | $1^\cdot, 2^\cdot$ |
| i, j | | 1, 2 |

[5] The homogeneous Plücker point coordinates $R^{AB} = -R^{BA}$ are written in (0,6) in the following order: $R^{\mathrm{I\,II}}$, $R^{\mathrm{III\,IV}}$, $R^{\mathrm{II\,III}}$, $R^{\mathrm{I\,IV}}$, $R^{\mathrm{III\,I}}$, $R^{\mathrm{II\,IV}}$. The same order will be kept for the homogeneous Plücker plane coordinates $R_{AB} = -R_{BA}$. These are related to R^{AB} by

$R_{\mathrm{I\,II}} = R^{\mathrm{III\,IV}}$, $R_{\mathrm{III\,IV}} = R^{\mathrm{I\,II}}$, $R_{\mathrm{II\,III}} = R^{\mathrm{I\,IV}}$, $R_{\mathrm{I\,IV}} = R^{\mathrm{II\,III}}$, $R_{\mathrm{III\,I}} = R^{\mathrm{II\,IV}}$, $R_{\mathrm{II\,IV}} = R^{\mathrm{III\,I}}$.

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Starting with a generic point $\mathbf{x}$, we consider the matrix[6]

$$(0.12)\qquad \begin{pmatrix} \Xi_1{}^1 & \Xi_2{}^1 & \Xi_3{}^1 & \Xi_4{}^1 \\ \Xi_1{}^2 & \Xi_2{}^2 & \Xi_3{}^2 & \Xi_4{}^2 \\ \Xi_1{}^3 & \Xi_2{}^3 & \Xi_3{}^3 & \Xi_4{}^3 \\ \Xi_1{}^4 & \Xi_2{}^4 & \Xi_3{}^4 & \Xi_4{}^4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & x^{\mathrm{I}} & x^{\mathrm{III}} \\ 0 & 0 & x^{\mathrm{IV}} & -x^{\mathrm{II}} \\ x^{\mathrm{II}} & x^{\mathrm{III}} & 0 & 0 \\ x^{\mathrm{IV}} & -x^{\mathrm{I}} & 0 & 0 \end{pmatrix}.$$

It is well known from the spinor algebra that a transformation from (T) induces on the elements $\Xi_b{}^a$ a transformation

$$'\Xi_b{}^a = \Lambda_c{}^a \lambda_b{}^d \Xi_d{}^c, \qquad (\Lambda_c{}^a \lambda_a{}^b = \delta_c{}^b)$$

where the coefficients Λ depend on (T). Hence $\Xi_b{}^a$ may be looked upon as the homogeneous components of a mixed tensor in a three dimensional projective space which we shall denote by S_3 and call a spinor space. Any object in S_3 will be termed a spinor object (spinor point, spinor plane and so on) and denoted by a Greek letter. Sometimes we say briefly "a spinor ξ^a" instead of "a spinor point ξ^a". For $\Xi_b{}^a$ we have from (0,12)

$$(0,13)\qquad \begin{gathered} \Xi_b{}^a \Xi_c{}^b = \mathbf{x}\,.\,\mathbf{x}\delta_c{}^a, \\ \det(\Xi_b{}^a - \lambda\delta_b{}^a) \equiv (\mathbf{x}\,.\,\mathbf{x} - \lambda^2)^2. \end{gathered}$$

The scope of this paper is the investigation of the relationship between L^3 and S_3. The correspondence between an object O_{L_3} of L_3 and an object O_{S^3} of S_3 (and *vice versa*) will be termed *a representation* and denoted by $L_3 \leftrightarrow S_3$. The correspondence between an object $O_{\mathbf{Q}}$ of $\mathbf{Q}$ and an object O_{S_3} of S_3 (and *vice versa*) will be termed a *mapping* and denoted by $\mathbf{Q} \leftrightarrow S_3$. Iu the first part of this paper we shall deal with the representation $L_3 \leftrightarrow S_3$ (starting with some theorems about the mapping of isotropic points). The second part deals with the mapping $\mathbf{Q} \leftrightarrow S_3$.

1. Mapping of isotropic points. The following theorems will be proved simultaneously:

THEOREM (1,1). *A necessary and sufficient condition that*

$$(1,1\mathrm{a})\qquad \Xi_b{}^a \xi^b = 0$$

admit at least one spinor[7] ξ^a *is that the point* $\mathbf{x}$ *be isotropic:*

$$(1,1\mathrm{b})\qquad \mathbf{x}\,.\,\mathbf{x} = 0.$$

THEOREM (1,2). *If* (1,1b) *is satisfied then the locus of all spinors* ξ^a *satisfying* (1,1a) *is a spinor line*

$$(1,2)\qquad {}^0\Xi^{ab}(0, 0, x^{\mathrm{II}}, x^{\mathrm{I}}, x^{\mathrm{III}}, x^{\mathrm{IV}}).$$

(The homogeneous Plücker point coordinates ${}^0\Xi^{ab} = -{}^0\Xi^{ba}$ are written in the following order: ${}^0\Xi^{12}, {}^0\Xi^{34}, {}^0\Xi^{23}, {}^0\Xi^{14}, {}^0\Xi^{31}, {}^0\Xi^{24}$. These coordinates are related to the homogeneous Plücker plane coordinates ${}^0\Xi_{ab} = -{}^0\Xi_{ba}$ by

[6]This is substantially the matrix used in Cartan's books [1] where also the theorem (1,1) may be found.

[7]As usual, $\xi^a = 0$ is not regarded as a (spinor) point in S_3.

$$\Xi_{12} = \Xi^{34}, \Xi_{34} = \Xi^{12}, \Xi_{23} = \Xi^{14}, \Xi_{14} = \Xi^{23}, \Xi_{31} = \Xi^{24}, \Xi_{24} = \Xi^{31}.)$$

THEOREM (1,3). *The locus of all spinor lines* (1,2) *as* $\mathbf{x}$ *moves along* $\mathbf{Q}$ *is a linear congruence* K *with the axes*

(1,3)
$$^{1}\Phi^{ab}(0, 1, 0, 0, 0, 0),$$
$$^{2}\Phi^{ab}(1, 0, 0, 0, 0, 0).$$

Hence $\mathbf{Q}$ *as a point set is mapped by* (1,2) *on* K *and this mapping is a one to one correspondence.*

Proof. Theorem (1,1) is an immediate consequence of (0,12) and

$$\Xi_a{}^c\, \Xi_b{}^a\, \xi^b = \mathbf{x}\,.\,\mathbf{x}\xi^c$$

which follows from (0,13). If (1,1b) is satisfied then the four equations (1,1a) reduce to two independent equations (linear in ξ^a) which together with (0,7) lead at once to (1,2). The locus of (1,2) is obviously a two parametric one and the rank of the matrix $({}^0\Xi, \dfrac{\partial}{\partial u^{1\cdot}}\, {}^0\Xi, \dfrac{\partial}{\partial u^{2\cdot}}\, {}^0\Xi)$ is[8] 3. Hence the locus is a congruence of spinor lines (1,2). It is a linear congruence because we have

(1,4)
$$\Gamma_{ab}\, {}^0\Xi^{ab} = {}_*\Gamma_{ab}\, {}^0\Xi^{ab} = 0,$$

where Γ and ${}_*\Gamma$ are two linear complexes

(1,5a)
$$\Gamma^{ab}(1, 1, 0, 0, 0, 0),$$

(1,5b)
$${}_*\Gamma^{ab}(-1, 1, 0, 0, 0, 0).$$

The remaining statement of Theorem (1,3) is obvious.

Note. The equations of ${}^0\Xi^{ab}$ mentioned in the proof are

$$\xi^3 x^{\mathrm{I}} + \xi^4 x^{\mathrm{III}} = \xi^1 x^{\mathrm{II}} + \xi^2 x^{\mathrm{III}} = 0.$$

In the special theory of relativity x^{I} and x^{II} are complex conjugate, so that we may put in this case

(1,6)
$$\bar{\xi}^2 = \xi^4, \quad \bar{\xi}^1 = \xi^3.$$

2. Representation of anisotropic points (Biaxial involution). Let $\mathbf{v}$ be an anisotropic point ($\mathbf{v}\,.\,\mathbf{v} \neq 0$) and $\Omega_b{}^a$ its corresponding matrix built up according to (0, 12). Then the following theorem holds:

THEOREM (2, 1). *The spinor transformation*

(2, 1)
$$'\xi^a = \Omega_b{}^a \xi^b$$

is a biaxial involution with the axes

(2, 2)
$$^{\epsilon}\Omega^{ab}(-\epsilon(\mathbf{v}\,.\,\mathbf{v})^{\frac{1}{2}},\ \epsilon(\mathbf{v}\,.\,\mathbf{v})^{\frac{1}{2}}, v^{\mathrm{II}}, v^{\mathrm{I}}, v^{\mathrm{III}}, v^{\mathrm{IV}})$$
$$(\epsilon = + \text{ or } \epsilon = -).$$

Proof. The double spinor points of the projectivity are obtained from

[8]It has the rank 3 for the parametric equations $x^{\mathrm{I}} = u^{1\cdot}$, $x^{\mathrm{II}} = u^{2\cdot}$, $x^{\mathrm{III}} = -u^{1\cdot}u^{2\cdot}$, $x^{\mathrm{IV}} = 1$ and consequently it has the rank 3 in any allowable parameter system.

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(2, 3a) $$(\Omega_b{}^a - \lambda\delta_b{}^a)\xi^b = 0$$

where, according to (0, 13),

(2, 3b) $$\lambda = \epsilon(\mathbf{v}\,.\,\mathbf{v})^{\frac{1}{2}}.$$

The equations (2, 3a) reduce to two independent equations for the value (2, 3b) and these equations lead at once to (2, 2). Because ${}^+\Omega_{ab}{}^-\Omega^{ab} = 8\mathbf{v}\,.\,\mathbf{v} \neq 0$ the spinor lines (2, 2) are skew. Hence (2, 1) is a biaxial projectivity with the axes (2, 2). A generic spinor ξ^a in the spinor plane $\xi^4 = 0$ on a line meeting the axes may be written

(2, 4a) $$\xi^a = \alpha{}^+\Omega^{a4} + \beta{}^-\Omega^{a4}$$

or

(2, 5a) $$\xi^1 : \xi^2 : \xi^3 : \xi^4 = (\alpha + \beta)v^{\mathrm{I}} : (\alpha + \beta)v^{\mathrm{IV}} : (\alpha - \beta)\,(\mathbf{v}\,.\,\mathbf{v})^{\frac{1}{2}} : 0$$

and its corresponding spinor $'\xi^a$ in (2, 1) is

(2, 5b) $$'\xi^1 : {}'\xi^2 : {}'\xi^3 : {}'\xi^4 = (\alpha - \beta)v^{\mathrm{I}} : (\alpha - \beta)v^{\mathrm{IV}} : (\alpha + \beta)\,(\mathbf{v}\,.\,\mathbf{v})^{\frac{1}{2}} : 0$$

or

(2, 4b) $$'\xi^a = \alpha{}^+\Xi^{a4} - \beta{}^-\Xi^{a4}.$$

Hence if we denote by ${}^+\xi^a\;{}^-\xi^a$ the double spinor points on the line $'\xi\xi$ we obtain for the cross ratio $(\xi\,'\xi,\,{}^+\xi{}^-\xi)$ of these four points according to (2, 4)

$$(\xi\,'\xi,\,{}^+\xi{}^-\xi) = -1.$$

Hence there is at least one couple of corresponding spinor points $\xi\,'\xi$ in involution and consequently the biaxial projectivity (2, 1) is an involution.

Note (i). A biaxial involution is uniquely determined by its axes. In our case the axes are uniquely determined by the point $\mathbf{v}$. Hence *the axes* (2, 2) *of the biaxial involution* (2, 1) *may be looked upon as a representation in* S_3 *of the anisotropic point* $\mathbf{v}$.

Note (ii). Let $\mathbf{p}$ be an arbitrary point, isotropic or not. Then the corresponding spinor lines which respectively map or represent this point may be written

(2, 6a) $$\Pi^{ab}(-\epsilon(\mathbf{p}\,.\,\mathbf{p})^{\frac{1}{2}},\,\epsilon(\mathbf{p}\,.\,\mathbf{p})^{\frac{1}{2}},\,p^{\mathrm{II}},\,p^{\mathrm{I}},\,p^{\mathrm{III}},\,p^{\mathrm{IV}}).$$

They reduce to the spinor lines of K if $\mathbf{p}\,.\,\mathbf{p} = 0$.

THEOREM (2, 2). *The locus of all spinor lines* (2, 6a) *as* $\mathbf{p}$ *moves along* L_3 *is the linear complex* Γ *defined by* (1, 5a) *and containing* K.

Proof. If $\mathbf{p}$ is not in the plane $x^{\mathrm{IV}} = 0$ we may put $p^{\mathrm{IV}} = 1$ in (2, 6a):

(2, 6b) $$\Pi^{ab}(-\epsilon(\mathbf{p}\,.\,\mathbf{p})^{\frac{1}{2}},\,\epsilon(\mathbf{p}\,.\,\mathbf{p})^{\frac{1}{2}},\,p^{\mathrm{II}},\,p^{\mathrm{I}},\,p^{\mathrm{III}},\,1).$$

The locus of all spinor lines is obviously three parametric and the matrix $\left(\Pi,\,\dfrac{\partial}{\partial p^{\mathrm{I}}}\Pi,\,\dfrac{\partial}{\partial p^{\mathrm{II}}}\Pi,\,\dfrac{\partial}{\partial p^{\mathrm{III}}}\Pi\right)$ is of rank 4. Hence the locus is a complex. Because at least one of the coordinates p^A must be different from zero, our statement holds for all points $\mathbf{p}$ of L_3. Because

$$\Gamma_{ab}\Pi^{ab} = 0$$

the complex is a linear one, namely the complex (1, 5a). The remaining statements are obvious.

Note. According to Theorem (2, 2) we may say that L_3 *as a point set is represented by* Γ (*and in particular* $\mathbf{Q}$ *as a point set is mapped on* K *in* Γ).

THEOREM (2, 3). *The axes* ${}^{\epsilon}\Omega$ *as given by* (2, 2) *are conjugate polars of the linear complex* ${}_*\Gamma$ (*defined by* (1, 5b)) *which is projectively orthogonal to* Γ. *Its rulings consist of spinor lines reproduced* (*not pointwise*) *by the biaxial involutions* (2, 1).

Let ${}_*\Gamma^{ab}$ be the components of any linear complex whatsoever and Λ^{ab} the conjugate polar of ${}^+\Omega$ with respect to ${}_*\Gamma$. Then[9]

$$\Lambda^{ab} = 2{}_*\Gamma^{ab}(\tfrac{1}{2}{}_*\Gamma_{cd}{}^+\Omega^{cd}) - {}^+\Omega^{ab}(\tfrac{1}{2}{}_*\Gamma_{cd}{}^*\Gamma^{cd}). \tag{2, 7a}$$

If we substitute from (1, 5b) we obtain

$$\Lambda^{ab} = 2^-\Omega^{ab}. \tag{2, 7b}$$

Because ${}^-\Omega^{ab}$ is conjugate polar to ${}^+\Omega^{ab}$, the latter must be conjugate polar to ${}^-\Omega$. Hence the congruence with axes ${}^+\Omega$, ${}^-\Omega$ consists of rulings of ${}_*\Gamma$. Because

$$\Gamma_{ab*}\Gamma^{ab} = 0$$

the complexes Γ and ${}_*\Gamma$ are projectively orthogonal.

Note (i). Conjugate polars with respect to a complex Γ^* are skew unless one of them is a ruling of Γ^*. Then both polars coincide. This is exactly the situation with the spinor lines (2, 2) and (1, 2). As long as the point $\mathbf{v}$ is an anisotropic one ${}^+\Omega({}^-\Omega)$ is not a ruling of ${}_*\Gamma$ (for ${}_*\Gamma_{ab}{}^{\epsilon}\Omega^{ab} = -4\epsilon(\mathbf{v}\,.\,\mathbf{v})^{\frac{1}{2}} \neq 0$) and consequently the conjugate polars ${}^+\Omega$, ${}^-\Omega$ are skew. If on the contrary $\mathbf{x}$ is an isotropic point, then ${}^0\Xi^{ab}$ as given by (1, 2) is a ruling of K which is the intersection of the complexes Γ and ${}_*\Gamma$, and consequently it is a ruling of ${}_*\Gamma$. Hence we have ${}^+\Xi = {}^-\Xi = {}^0\Xi$: both conjugate polars coincide.

Note (ii). If ξ^a is a generic spinor its spinor polar plane ξ_a with respect to Γ is

$$\xi_a = \Gamma_{ab}\xi^b \tag{2, 8a}$$

or

$$\xi_1 = \xi^2,\ \xi_2 = -\xi^1,\ \xi_3 = \xi^4,\ \xi_4 = -\xi^3. \tag{2, 8b}$$

Hence we have in the special theory of relativity by virtue of (1, 6)

$$\xi_1 = \xi^2,\ \xi_2 = -\xi^1,\ \xi_3 = \bar{\xi}^2 = \bar{\xi}_1,\ \xi_4 = -\bar{\xi}^1 = \bar{\xi}_2. \tag{2, 8c}$$

[9]Cf. Hlavatý [**2**], where also other notions of line geometry, which are used here, are discussed.

Hence ξ_1, ξ_2, $\bar{\xi}_1$, $\bar{\xi}_2$ are "the covariant components" of ξ^1, ξ^2, $\bar{\xi}^1$, $\bar{\xi}^2$ in the theory of quanta.[10]

3. Representation of a line. Let $\mathbf{x}$ be an isotropic point, v^ν a vector on $\mathbf{Q}$ defined at $\mathbf{x}$. Put

$$\mathbf{l} \equiv v^\lambda \mathbf{x}_\lambda, \tag{3, 1a}$$

$$\mathbf{v} \equiv v^0 \mathbf{x} + w\mathbf{l}, \tag{3, 1b}$$

and denote by ${}^0\Xi^{ab}$ and ${}^\epsilon\Lambda^{ab}$ the spinor lines which map $\mathbf{x}$ and which represent $\mathbf{l}$, respectively. As v^0, w change the point $\mathbf{v}$ describes a line L tangent to $\mathbf{Q}$ at $\mathbf{x}$.

THEOREM (3, 1). *The spinor lines* ${}^\epsilon\Omega^{ab}$ *representing the points* $\mathbf{v}$ *of* L *belong to the pencil*

$$ {}^\epsilon\Omega^{ab} = v^0\, {}^0\Xi^{ab} + w\, {}^\epsilon\Lambda^{ab} \tag{3, 2}$$

(*of rulings of* Γ) *in the polar plane of its vertex* ${}^\epsilon\xi$ *with respect to* Γ. *Both pencils* (3, 2) *coincide if and only if* v^ν *is the null vector of* $a_{\lambda\mu}$ (*e.g. if* L *is a ruling of* $\mathbf{Q}$). *Then the spinor plane of this pencil is the focal plane of* K *of the focal spinor point* ${}^+\xi = {}^-\xi$ *of* K.

Proof. Because $\mathbf{x}$ and $\mathbf{x}_\lambda$ are conjugate points (with respect to $\mathbf{Q}$) we have

$$\begin{aligned} \mathbf{v} \,.\, \mathbf{v} &= (v^0\mathbf{x} + w\mathbf{l}) \,.\, (v^0\mathbf{x} + w\mathbf{l}) = w^2 v^\lambda v^\mu \mathbf{x}_\lambda \,.\, \mathbf{x}_\mu \\ &= w^2 v^\lambda v^\mu a_{\lambda\mu} = w^2 \mathbf{l} \,.\, \mathbf{l}. \end{aligned} \tag{3, 3}$$

Substituting from (3, 3) and (3, 1) in (2, 2) and remembering (1, 2) we obtain the pencils (3, 2) of rulings of Γ. Hence each of these pencils must be in the polar spinor plane (with respect to Γ) of its vertex ${}^\epsilon\xi$. According to the previous results we have ${}^+\Omega = {}^-\Omega$ if and only if $\mathbf{v} \,.\, \mathbf{v} = 0$, which yields by virtue of (3, 3) either $w = 0$ (e.g. $\mathbf{v} = \mathbf{x}$) or (for $\mathbf{v} \neq \mathbf{x}$)

$$v^\lambda v^\mu a_{\lambda\mu} = 0$$

and L is a ruling of $\mathbf{Q}$. Each of its points $\mathbf{v}$ is mapped on a spinor line of the pencil (3, 2) all of whose rulings belong to K. The last statement of the theorem follows at once from these facts.

Note. As in the previous case *we may look upon the pencils* (3, 2) *as representing the points of a tangential line* L *of* $\mathbf{Q}$. In the next section we shall be concerned with a representation of points of a non-tangential line.

4. Continuation: Line-"sphere" transformation. Let ${}_1\mathbf{x}$, ${}_2\mathbf{x}$ be two isotropic points not situated on the same ruling of $\mathbf{Q}$, $({}_1\mathbf{x} \,.\, {}_2\mathbf{x} \neq 0)$. Denote by M the line ${}_1\mathbf{x}{}_2\mathbf{x}$, by

$$\mathbf{v} \equiv {}^i w\, {}_i\mathbf{x}. \tag{4, 1}$$

[10] Cf. van der Waerden [4].

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its generic point and by

(4, 2) $$^{\epsilon}\Omega^{ab}(-\epsilon(\mathbf{v}\,.\,\mathbf{v})^{\frac{1}{2}},\ \epsilon(\mathbf{v}\,.\,\mathbf{v})^{\frac{1}{2}},\ v^{\mathrm{II}},\ v^{\mathrm{I}},\ v^{\mathrm{III}},\ v^{\mathrm{IV}})$$

the spinor lines representing $\mathbf{v}$.

THEOREM (4, 1). *The locus of all spinor lines* (4, 2) *as* $\mathbf{v}$ *moves along* M *is a regulus* $R(M)$.

The proof will be accomplished in three steps.

(a) Let $\mathbf{v}, \mathbf{v}'$ be two distinct points on M. Because M is not a tangent line to $\mathbf{Q}$ we must have

(4, 3) $$(\mathbf{v}\,.\,\mathbf{v})^{\frac{1}{2}}\,(\mathbf{v}'\,.\,\mathbf{v}')^{\frac{1}{2}} \neq \pm\,\mathbf{v}\,.\,\mathbf{v}'.$$

(b) If $^{\epsilon}\Omega^{ab}$ and $^{\epsilon'}\Omega'^{ab}$ are the lines representing $\mathbf{v}$ and $\mathbf{v}'$ we have, according to (4, 2) and (4, 3),

$$\tfrac{1}{2}\,{}^{\epsilon}\Omega_{ab}\,{}^{\epsilon'}\Omega'^{ab} = 2\epsilon\epsilon'(-(\mathbf{v}\,.\,\mathbf{v})^{\frac{1}{2}}(\mathbf{v}'\,.\,\mathbf{v}')^{\frac{1}{2}} + \epsilon\epsilon'\mathbf{v}\,.\,\mathbf{v}') \neq 0.$$

Hence the spinor lines of the locus do not meet each other.

(c) Let $\sum_{ab}$ be an arbitrary spinor line not belonging to the locus. It intersects each spinor line $^{\epsilon}\Omega^{ab}$ for which

$$\textstyle\sum_{ab}{}^{\epsilon}\Omega^{ab} = 0$$

or

(4, 4) $$-\,\epsilon(\mathbf{v}\,.\,\mathbf{v})^{\frac{1}{2}}(\textstyle\sum_{34} - \sum_{12}) = \sum_{23}v^{\mathrm{II}} + \sum_{14}v^{\mathrm{I}} + \sum_{31}v^{\mathrm{III}} + \sum_{24}v^{\mathrm{IV}}.$$

Substituting in (4, 4) from (4, 1) we obtain an equation for $^{1}w : {}^{2}w$. If it is not identically satisfied, it has only two roots $^{1}w : {}^{2}w$ (which may coincide). Hence $\sum_{ab}$ meets either all spinor lines of the locus or only two of them (which may coincide). The statement of the theorem follows from the results in (b) and (c).

THEOREM (4, 2). *The regulus* $R(M)$ *has the following properties*:

(1) *With any spinor line* $^{+}\Omega(^{-}\Omega)$ *it contains also the spinor line* $^{-}\Omega(^{+}\Omega)$.

(2) *It meets the congruence* K *in two distinct spinor lines* ${}_{i}{}^{0}\Xi$ *mapping the points* ${}_{i}\mathbf{x}$.

(3) *If* $^{\epsilon}\Omega$, $^{\epsilon}\Omega'$ *represent two points* $\mathbf{v}$, $\mathbf{v}'$ *of* M *then*[11]

(4, 5) (a) $$({}^{+}\Omega,\ {}^{+}\Omega',\ {}_{1}{}^{0}\Xi,\ {}_{2}{}^{0}\Xi) = ({}^{-}\Omega,\ {}^{-}\Omega',\ {}_{1}{}^{0}\Xi,\ {}_{2}{}^{0}\Xi),$$

(b) $$(\mathbf{v},\ \mathbf{v}',\ {}_{1}\mathbf{x},\ {}_{2}\mathbf{x}) = ({}^{+}\Omega,\ {}^{+}\Omega',\ {}_{1}{}^{0}\Xi,\ {}_{2}{}^{0}\Xi)^{2}.$$

(4) *The conjugate regulus*[12] $*R(M)$ *to* $R(M)$ *consists of rulings of the complex* $*\Gamma$.

Proof. The statement (2) is obvious, the corresponding lines are

(4, 6) $${}_{i}{}^{0}\Xi^{ab}(0,\ 0,\ {}_{i}x^{\mathrm{II}},\ {}_{i}x^{\mathrm{I}},\ {}_{i}x^{\mathrm{III}},\ {}_{i}x^{\mathrm{IV}}).$$

[11]The symbols in (4,5a) denote cross ratios of spinor lines *taken on* $R(M)$, the symbol at left in (4,5b) denotes a cross ratio of points on M.

[12]e.g. the regulus belonging with $R(M)$ to the same quadric.

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On the other hand a generic spinor line Ψ^{ab} of $R(M)$ may be expressed by

$$\Psi^{ab} \equiv x_1\, {}_1{}^0\Xi^{ab} + x_2\, {}_2{}^0\Xi^{ab} + x_3{}^+\Omega^{ab} \tag{4, 7a}$$

where

$$x_1x_2({}_1{}^0\Xi_{ab}\, {}_2{}^0\Xi^{ab}) + x_1x_3({}_1{}^0\Xi_{ab}{}^+\Omega^{ab}) + x_2x_3({}_2{}^0\Xi_{ab}{}^+\Omega^{ab}) = 0$$

or[13]

$$x_1x_2 + x_1x_3{}^2w + x_2x_3{}^1w = 0, \tag{4, 7b}$$

and this equation is in particular satisfied by

$$x_1 : x_2 : x_3 = -\,2{}^1w : -\,2{}^2w : 1. \tag{4, 7c}$$

Substituting from (4, 7c) in (4, 7a) we obtain

$$\Psi^{ab} = -\,{}^-\Omega^{ab}. \tag{4, 8}$$

Hence if $R(M)$ contains ${}^+\Omega$ it contains also ${}^-\Omega$. The remaining part of the statement (1) may be proved in a similar manner. The spinor line $\sum_{ab}$ involved in (4, 4) is a ruling of the conjugate regulus ${}_*R(M)$ if and only if the equation (4, 4) is satisfied identically (for any point $\mathbf{v}$ on M), and this yields $\sum_{12} = \sum_{34}$ so that

$$\textstyle\sum_{ab*}\Gamma^{ab} = 0. \tag{4, 9}$$

The equation (4, 9) proves the statement (4). One of these rulings (belonging also to K) is

$$\textstyle\sum^{ab}(0, 0, {}_2x^{\mathrm{II}}{}_1x^{\mathrm{IV}}, {}_1x^{\mathrm{I}}{}_2x^{\mathrm{IV}}, -\,{}_1x^{\mathrm{I}}{}_2x^{\mathrm{II}}, {}_1x^{\mathrm{IV}}{}_2x^{\mathrm{IV}}). \tag{4, 10}$$

If ${}_i\xi^a$ and ${}^\epsilon\omega^a$ are the intersection points of $\sum^{ab}$ with ${}_i{}^0\Xi^{ab}$ and ${}^\epsilon\Omega^{ab}$, respectively, then

$${}^\epsilon\omega^a = {}^2w({}_2x^{\mathrm{III}}{}_1x^{\mathrm{IV}} + {}_2x^{\mathrm{II}}{}_1x^{\mathrm{I}})\,{}_2\xi^a + \epsilon\, {}_2x^{\mathrm{II}}(\mathbf{v}\,.\,\mathbf{v})^{\frac{1}{2}}{}_1\xi^a. \tag{4, 11}$$

Consequently we obtain for the cross ratio of the points ${}^\epsilon\omega$, ${}^\epsilon\omega'$, ${}_1\xi$, ${}_2\xi$ the expression

$$({}^\epsilon\omega, {}^\epsilon\omega', {}_1\xi, {}_2\xi) = \frac{({}^1w'{}^2w')^{\frac{1}{2}}}{{}^2w'}\,\frac{{}^2w}{({}^1w{}^2w)^{\frac{1}{2}}}, \tag{4, 12}$$

and this cross ratio is obviously equal to the cross ratio of the four rulings ${}^\epsilon\Omega$, ${}^\epsilon\Omega'$, ${}_1{}^0\Xi$, ${}_2{}^0\Xi$ of $R(M)$ taken on $R(M)$:

$$({}^\epsilon\Omega, {}^\epsilon\Omega', {}_1{}^0\Xi, {}_2{}^0\Xi) = ({}^\epsilon\omega, {}^\epsilon\omega', {}_1\xi, {}_2\xi). \tag{4, 13}$$

On the other hand we have for the cross ratio of the four points $\mathbf{v}$, $\mathbf{v}'$, ${}_1\mathbf{x}$, ${}_2\mathbf{x}$ by virtue of (4, 1)

$$(\mathbf{v}, \mathbf{v}', {}_1\mathbf{x}, {}_2\mathbf{x}) = \frac{{}^2w{}^1w'}{{}^1w{}^2w'}. \tag{4, 14}$$

[13] ${}^iw$ are the numbers defining $\mathbf{V}$ by (4,1). ${}^+\Omega$ is one of the spinor lines ${}^\epsilon\Omega$ representing $\mathbf{V}$. The equation (4,7b) results from the previous one by virtue of ${}_1\mathbf{X}\,.\,{}_2\mathbf{X} \neq 0$ and (4,1).

The equations (4, 5ab) follow from (4, 12) – (4, 14).

Note (i). According to (4, 5b) a non-euclidean line metric on $R(M)$ may be introduced (with ${}_1{}^0\Xi$, ${}_2{}^0\Xi$ as absolute lines) which is related to the angular metric in R_4 in the following way. The angle of the vectors $\mathbf{v}$, $\mathbf{v}'$ in R_4 is equal to twice the "distance" of the rulings ${}^{\epsilon}\Omega$, ${}^{\epsilon}\Omega'$ on $R(M)$.

Note (ii). The correspondence $M \rightarrow R(M)$ may be looked upon as a line-"sphere" transformation. It is not a usual contact transformation, because it carries the intersection point $\mathbf{l}$ of two lines M and M' *into a couple* of common rulings ${}^{\epsilon}\Lambda$ of two reguli[14] $R(M)$ and $R'(M')$

5. Mapping of a lineal element of S_3. Definition (5, 1). *By a lineal element* $(\mathbf{x}, v^{\nu})$ *we understand an isotropic point* $\mathbf{x}$ (*the support of the element*) *and a set of vectors*[15] ρv^{ν} *of* $\mathbf{Q}$ *defined at* $\mathbf{x}$ (*the direction of the element*). *If* v^{ν} *is* (*is not*) *the null vector of* $a_{\lambda\mu}$ [*e.g. if the direction of the element is* (*is not*) *on a ruling of* $\mathbf{Q}$] *the lineal element will be called an isotropic* (*an anisotropic*) *lineal element.*

In this section we shall be concerned with mapping of the lineal elements $(\mathbf{x}, v^{\nu})$ and introduce for this purpose a standard notation

$$^{i}v \equiv {}^{i}e_{\lambda}v^{\lambda}$$

where ${}^{i}e_{\lambda}$ are defined by (0, 9). A null vector v^{ν} leads to one of the rulings (0, 6) which may be written

$$R^{AB} = v^{\lambda}x^{\,[A}x_{\lambda}{}^{B]}\,.$$

Comparing this equation with (0, 6) one sees easily that ${}^{i}v = 0$ defines the ruling ${}^{j}R^{AB} (i \neq j)$. We use this fact in the next theorem which deals with an isotropic lineal element $(\mathbf{x}, v^{\nu})$ (where as usual the coordinates of its support are supposed to satisfy (0, 7)).

Theorem (5, 1). *Let* $(\mathbf{x}, v^{\nu})$ *be an isotropic lineal element. This element is mapped on the spinor*

$$(5, 1) \qquad (a)\quad {}_1\psi^a(0, 0, -x^{\mathrm{III}}, x^{\mathrm{I}}) \quad or \quad (b)\quad {}_2\psi^a(x^{\mathrm{I}}, x^{\mathrm{IV}}, 0, 0)$$

if v^{ν} *defines the direction of the ruling* ${}^1R^{AB}$ (*e.g.* ${}^2v = 0$) *or* ${}^2R^{AB}$ (*e.g.* ${}^1v = 0$). *The spinors* (5, 1) *are the focal spinors of the spinor line* ${}^0\Xi^{ab}$ *which maps the support* $\mathbf{x}$ *of the lineal elements* $(\mathbf{x}, v^{\nu})$.

Proof. Let ${}^0\Xi^{ab}$ be given by (1, 2). If we denote by ${}_j\psi^a$ the intersection of ${}^0\Xi$ with the axis ${}^{j}\Phi$ (given by (1, 3)) then these spinors, given by (5, 1), are obviously the focal spinors of ${}^0\Xi$ in K. On the other hand, if v^{ν} is a null vector of $a_{\lambda\mu}$ then the spinor lines[16] of the pencil (3, 2) are rulings of K and conse-

[14]If $\mathbf{l}$ is an isotropic point the couple of common rulings reduces to one ruling (belonging to K).
[15]$\rho \neq 0$ is an arbitrary factor.
[16]Each of these spinor lines maps one point of the point set (3,1b).

quently (cf. Theorem (3, 1)) meet at the common focal spinor which maps the direction of the corresponding ruling of **Q**. The focal spinors of ${}^{\epsilon}\Lambda$ from (3, 1) are

$$ {}_1\lambda^a(0, 0, -v^\lambda x_\lambda{}^{\mathrm{III}}, v^\lambda x_\lambda{}^{\mathrm{I}}), \quad {}_2\lambda^a(v^\lambda x_\lambda{}^{\mathrm{I}}, v^\lambda x_\lambda{}^{\mathrm{IV}}, 0, 0). $$

It is easily seen that

$$ {}_1\lambda^a = {}_1\psi^a \text{ or } {}_2\lambda^a = {}_2\psi^a $$

requires ${}^2v = 0$ or ${}^1v = 0$. The remaining statements of our theorem are obvious.

In the next theorem we shall be concerned with a mapping of an anisotropic lineal element $(\mathbf{x}, v^\nu)$. Because in this case ${}^1v^2v \neq 0$ we may assume without loss of generality

$$ {}^1v > 0. \tag{5, 2} $$

THEOREM (5, 2). *Let* (5, 1) *be the focal spinors of* ${}^0\Xi^{ab}$ *which maps the support* $\mathbf{x}$ *of an anisotropic element* $(\mathbf{x}, v^\nu)$. *This element is mapped on a couple of spinors*[17]

$$ {}^{\epsilon}\xi^a = ({}^1v)^{\frac{1}{2}}{}_1\psi^a + \epsilon({}^2v)^{\frac{1}{2}}{}_2\psi^a. \tag{5, 3} $$

Proof. Each spinor line ${}^{\epsilon}\Omega^{ab}$ of the pencil (3, 2) represents a point on the line L whose direction is defined by v^ν. Hence the vertex of the pencil (3, 2) is the map of our lineal element. We obtain it as the intersection point of the spinor lines ${}^0\Xi$ and ${}^{\epsilon}\Lambda$. A generic spinor point on ${}^0\Xi$ may be written

$$ \xi^a \equiv \lambda_1\psi^a + \mu_2\psi^a. \tag{5, 4} $$

It is also on ${}^{\epsilon}\Lambda$ if and only if

$$ \lambda : \mu = -(x^{\mathrm{I}}l^{\mathrm{II}} + x^{\mathrm{IV}}l^{\mathrm{III}}) : \epsilon x^{\mathrm{III}}(1\,.\,1)^{\frac{1}{2}}, \tag{5, 5} $$

where $\mathbf{v}$ is defined by (3, 1). Because, by virtue of (0, 9) and (0, 10),

$$ \begin{aligned} -(x^{\mathrm{I}}l^{\mathrm{II}} + x^{\mathrm{IV}}l^{\mathrm{III}}) &= -v^\lambda[x^{\mathrm{I}}x_\lambda{}^{\mathrm{II}} + x_\lambda{}^{\mathrm{III}}x^{\mathrm{IV}}] = v^\lambda[x^{\mathrm{II}}x_\lambda{}^{\mathrm{I}} + x^{\mathrm{III}}x_\lambda{}^{\mathrm{IV}}] \\ &= v^{\lambda 1}e_\lambda x^{\mathrm{III}} = {}^1vx^{\mathrm{III}}, \\ 1\,.\,1 = v^\lambda v^\mu \mathbf{x}_\lambda\,.\,\mathbf{x}_\mu &= v^\lambda v^\mu a_{\lambda\mu} = \tfrac{1}{2}v^\lambda v^\mu({}^1e_\lambda{}^2e_\mu + {}^2e_\lambda{}^1e_\mu) = {}^1v^2v, \end{aligned} \tag{5, 6} $$

we obtain by virtue of (5, 2) the equation (5, 3) from (5, 4) – (5, 6).

THEOREM (5, 3). *Let* $(\mathbf{x}, v^\nu)$ *be a set of lineal elements with a fixed support* $\mathbf{x}$, *which is mapped on the spinor line* ${}^0\Xi$. *If* (5, 1) *are focal spinor of* ${}^0\Xi$, *then the set* $(\mathbf{x}, v^\nu)$ *is mapped on an involution on* ${}^0\Xi$ *whose double points are* ${}_1\psi$, ${}_2\psi$.

The proof follows at once from the cross ratio

$$ ({}^+\xi, {}^-\xi, {}_1\psi, {}_2\psi) = -1. \tag{5, 7} $$

THEOREM (5, 4). *Let* $\mathbf{v}$ *be an anisotropic point. The set of all lineal elements* $(\mathbf{x}, v^\nu)$ *common to* **Q** *and the circumscribed cone to* **Q**, *which has* $\mathbf{v}$ *for its vertex, is mapped on the axis* (2, 2) *of the biaxial involution* (2, 1).

[17]By $(\ )^{\frac{1}{2}}$ we understand always the positive square root.

Proof. The spinor lines (3, 2) have the coordinates (2, 2). If we keep $\mathbf{v}$ fixed, ${}^{\epsilon}\Omega$ do not change and intersect the spinor lines ${}^{0}\Xi$ (which represent the supports of the corresponding lineal elements $(\mathbf{x}, v^{\nu})$) in (5, 3).

The locus of the supports of lineal elements $(\mathbf{x}, v^{\nu})$ dealt with in the previous theorem is a conic section. We denote by l^{ν} its tangential vector at $\mathbf{x}$ and shall consider next the set of lineal elements $(\mathbf{x}, l^{\nu})$. The following three theorems will be proved simultaneously.

THEOREM (5, 5a). *The lineal element* $(\mathbf{x}, l^{\nu})$ *is mapped on the couple of spinors*

$$(5, 8) \qquad {}^{\epsilon}\lambda^{a} = ({}^{1}v)^{\frac{1}{2}}{}_{1}\psi^{a} + \epsilon(-{}^{2}v)^{\frac{1}{2}}{}_{2}\psi^{a}$$

and the locus of these spinors is a couple of spinor lines ${}^{\epsilon}\Lambda^{ab}$.

THEOREM (5, 5b). *The spinor lines* ${}^{\epsilon}\Omega^{ab}$ ${}^{\epsilon}\Lambda^{ab}$ *and the axes* ${}^{i}\Phi^{ab}$ *(given by* (1, 3)) *are rulings of the same regulus* R. *The spinor lines* ${}^{\epsilon}\Lambda^{ab}$ *are the only common couple of the involutions on* R, *whose double rulings are* ${}^{i}\Phi$ *and* ${}^{\epsilon}\Omega$. *The conjugate regulus* ${}_{*}R$ *to* R *consists of ruling of* ${}_{*}\Gamma$.

THEOREM (5, 5c). *The regulus* R *intersects the complex* Γ *only in* ${}^{\epsilon}\Omega$.

Proof. The vector v^{ν} from Theorem (5,4) is conjugate (with respect to $a_{\lambda\mu}$) to l^{ν}. Hence

$$(5, 9a) \qquad {}^{1}v^{2}l + {}^{2}v^{1}l = 0.$$

Substituting from (5, 9a) in (5, 3) we obtain (5, 8). Furthermore the couples ${}^{\epsilon}\Omega$ and ${}^{i}\Phi$ are couples of conjugate polars with respect to the same linear complex ${}_{*}\Gamma$ and consequently are rulings of the same regulus, which we denote by R. Hence the conjugate regulus ${}^{*}R$ consists of rulings of[18] ${}_{*}\Gamma$.

Moreover the locus of spinors (5, 8) is obviously on R. Because

$$(5, 9b) \qquad ({}^{+}\xi, {}^{-}\xi, {}^{+}\lambda, {}^{-}\lambda) = ({}^{+}\lambda, {}^{-}\lambda, {}_{1}\psi, {}_{2}\psi) = -1$$

the locus in question is constituted by two spinor lines which have the properties mentioned in Theorem (5, 5b).

In order to prove Theorem (5, 5c), let us write for the generic ruling $\sum^{ab}$ of R

$$(5, 10) \qquad \textstyle\sum^{ab} = x_1{}^{+}\Omega^{ab} + x_2{}^{-}\Omega^{ab} + x_3{}^{1}\Phi^{ab},$$

where the x_1, x_2, x_3 are subject to

$$(5, 11) \qquad 4x_1x_2(\mathbf{v}\,.\,\mathbf{v})^{\frac{1}{2}} - x_1x_3 + x_2x_3 = 0.$$

From (5, 10) we obtain

$$(5, 12) \qquad \tfrac{1}{2}\textstyle\sum_{ab}\Gamma^{ab} = x_3$$

and the theorem follows from (5, 10) — (5, 12).

[18]Because K belongs to Γ the rulings of ${}_{*}R$ are also (rulings of K and consequently) rulings of Γ.

6. Mapping of spinors on lineal elements. Throughout this section we shall consider spinors whose coordinates satisfy the conditions

$$(6,1) \qquad \xi^1\xi^2\xi^3\xi^4 \neq 0$$

(and consequently are not incident with the axes ${}^i\Phi$ of the congruence ${}^2\Gamma$).

THEOREM (6, 1). *Through a given spinor ξ^a there is only one ruling ${}^0\Xi$ of K namely*

$$(6,2) \qquad {}^0\Xi^{ab}(0, 0, \xi^2\xi^3, \xi^1\xi^4, -\xi^3\xi^1, \xi^2\xi^4),$$

and this spinor line maps the point

$$(6,3) \qquad x^{\mathrm{I}} : x^{\mathrm{II}} : x^{\mathrm{III}} : x^{\mathrm{IV}} = \xi^1\xi^4 : \xi^2\xi^3 : -\xi^3\xi^1 : \xi^2\xi^4.$$

The focal spinors of ${}^0\Xi$ are

$$(6,4) \qquad {}_1\psi^a(0, 0, \xi^3, \xi^4), \quad {}_2\psi^a(\xi^1, \xi^2, 0, 0).$$

Proof. If η^a are the coordinates of a generic spinor in S_3 then the equations of a spinor line of K going through ξ^a are

$$\eta^1\xi^2 - \eta^2\xi^1 = 0, \quad \eta^3\xi^4 - \eta^4\xi^3 = 0,$$

and these equations lead at once to (6, 2). Comparing (6, 2) and (1, 2) we obtain (6, 3) and comparing (6, 3) and (5, 1) we obtain (6, 4).

Note (i). If the locus of the spinors ξ^a is a curve or a surface in S_3 then the equations (6, 3) define the locus of the supports of the corresponding lineal elements.

Note (ii). From (5, 3) and (5, 1) we obtain

$$(6,5a) \qquad {}^{\epsilon}\xi^a(\epsilon({}^2v)^{\frac{1}{2}}x^{\mathrm{I}}, \epsilon({}^2v)^{\frac{1}{2}}x^{\mathrm{IV}}, -({}^1v)^{\frac{1}{2}}x^{\mathrm{III}}, ({}^1v)^{\frac{1}{2}}x^{\mathrm{I}}),$$

where the x^A are defined by (0, 3). Hence if we denote by p the parameter which defines $v^{\bullet}$ at x^A we may write

$$(6,5b) \qquad \xi^a = \xi^a(u^{1^{\bullet}}, u^{2^{\bullet}}, p).$$

In the following considerations we shall use the symbol $\xi_\lambda{}^a \equiv \dfrac{\partial \xi^a}{\partial u^\lambda}$ and the spinor lines

$$(6,6) \qquad \Pi_\lambda{}^{ab} \equiv \xi^{[a}\xi_\lambda{}^{b]}.$$

Moreover, we shall deal with the complex Γ and the axis ${}^i\Phi$ which we normalize in the following way: As the coordinates of the complex Γ we shall use the expressions $\dfrac{2\Gamma^{ab}}{(\Gamma_{cd}\Gamma^{cd})^{\frac{1}{2}}}$ and denote them again by Γ^{ab}. As the coordinates for ${}^i\Phi$ we shall use the expressions $\dfrac{2\,{}^i\Phi^{ab}}{{}^i\Phi_{cd}\Gamma^{cd}}$ and denote them again by ${}^i\Phi^{ab}$. These normalized coordinates do not depend on factor of proportionality and are given by the corresponding numbers in (1, 5a) and (1, 3).

THEOREM (6, 2). *The vectors* (0, 9) *may be written*

(6, 7a) $$ {}^1e_\lambda = \frac{\xi^4}{\xi^1}\,{}^1\Phi_{ab}\Pi_\lambda{}^{ab}, \quad {}^2e_\lambda = \frac{\xi^1}{\xi^4}\,{}^2\Phi_{ab}\Pi_\lambda{}^{ab}, $$

and moreover

(6, 7b) $$ a_{\lambda\mu} = {}^1\Phi_{ab}{}^2\Phi_{cd}\Pi_{(\lambda}{}^{ab}\Pi_{\mu)}{}^{cd}. $$

Proof. The coordinates in (6, 3) being homogeneous we may put without loss of generality

$$ x^{\mathrm{I}} = \xi^1\xi^4, \quad x^{\mathrm{II}} = \xi^2\xi^3, \quad x^{\mathrm{III}} = -\,\xi^3\xi^1, \quad x^{\mathrm{IV}} = \xi^2\xi^4. $$

Substituting these values in (0, 9), we obtain

(6,7c) $$ {}^1e_\lambda = 2\frac{\xi^4}{\xi^1}\,\Pi_\lambda{}^{12}, \qquad {}^2e_\lambda = 2\,\frac{\xi^1}{\xi^4}\,\Pi_\lambda{}^{34}, $$

and (6,7a) follows from (6,7c) and (1,3). From (6,7a) and (0,10) we obtain (6,7b).

THEOREM (6,3). *A given spinor ξ^a is mapped on a lineal element* $(\mathbf{x}, v^\nu)$ *whose support is given by* (6,3) *and whose direction vector v^ν has the covariant components*

(6,8) $$ v_\lambda = \Gamma_{ab}\Pi_\lambda{}^{ab}. $$

Proof. Starting with (6,5a) and (6,3) we see that there must be two factors α, β such that

$$ \alpha^\epsilon\xi^1 = \epsilon\beta({}^2v)^{\frac{1}{2}\epsilon}\xi^{1\epsilon}\xi^4,\ \alpha^\epsilon\xi^2 = \epsilon\beta({}^2v)^{\frac{1}{2}\epsilon}\xi^2\,{}^\epsilon\xi^4,\ \alpha^\epsilon\xi^3 = \beta({}^1v)^{\frac{1}{2}}\,{}^\epsilon\xi^1\,{}^\epsilon\xi^3,\ \alpha^\epsilon\xi^4 = \beta({}^1v)^{\frac{1}{2}}\,{}^\epsilon\xi^1\,{}^\epsilon\xi^4, $$

and consequently by virtue of (6,1)

$$ \alpha = \epsilon\beta({}^2v)^{\frac{1}{2}}\,{}^\epsilon\xi^4 = \beta({}^1v)^{\frac{1}{2}}\,{}^\epsilon\xi^1. $$

Hence

(6.9) $$ {}^1v = ({}^\epsilon\xi^4)^2\sigma, \quad {}^2v = ({}^\epsilon\xi^1)^2\sigma, \qquad \sigma \neq 0. $$

On the other hand,

(6,10) $$ 2v_\lambda = 2v^\nu a_{\nu\lambda} = {}^1v^2e_\lambda + {}^2v^1e_\lambda. $$

Substituting in (6,10) from (6,7a) and (6,9) we obtain

(6,11) $$ 2v_\lambda = \sigma^\epsilon\xi^{1\epsilon}\xi^4({}^2\Phi_{ab}+{}^1\Phi_{ab})\Pi_\lambda{}^{ab} = \sigma^\epsilon\xi^1\,{}^\epsilon\xi^4\,\Gamma_{ab}\Pi_\lambda{}^{ab}. $$

Because the direction of the lineal element $(\mathbf{x}, v^\nu)$ is defined by the ratio $v^{1\cdot} : v^{2\cdot}$—or $v_{1\cdot} : v_{2\cdot}$—we omit in (6,11) the factors 2 and $\sigma^\epsilon\xi^1\,{}^\epsilon\xi^4$ and obtain (6,8).

7. Mapping of (G) **and** (F). If the coordinates x^A of a point undergo a transformation from (T)

(7,1) $$ 'x^A = L_B{}^A x^B, \qquad \det L_B{}^A \neq 0, $$

then the spinor coordinates undergo a transformation

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(7,2) $$'\xi^a = \Lambda_b{}^a\, \xi^b, \qquad \det \Lambda_b{}^a \neq 0,$$

where the $\Lambda_b{}^a$ are function of the $L_B{}^A$. In this section we shall be concerned with the relationship of (7,1) and (7,2) referring to (7,2) as *a map*[19] *of* (7,1). We shall use also the symbols

(7,3a) $$\delta_{ab}{}^A \equiv \delta_{[ab]}{}^A,$$

defined in the following way:

(7,3b) $$\delta_{14}{}^{\mathrm{I}} = \delta_{23}{}^{\mathrm{II}} = \delta_{31}{}^{\mathrm{III}} = \delta_{24}{}^{\mathrm{IV}} = 1, \text{ the remaining } \delta_{ab}{}^A = 0.$$

THEOREM (7,1). *The map* (7,2) *of* (7,1) *has the following properties:*[20]

(7,4G) $\Lambda_b{}^a = 0$ *for* $a = 1$ *or* 2, $b = 3$ *or* 4 *and for* $a = 3$ *or* 4, $b = 1$ *or* 2,

(7,4F) $\Lambda_b{}^a = 0$ *for* $a, b = 1$ *or* 2 *and* $a, b = 3$ *or* 4,

(7,5G) $\frac{1}{2}\Delta \equiv \Lambda_1{}^{[1}\Lambda_2{}^{2]} = \Lambda_3{}^{[3}\Lambda_4{}^{4]}$,

(7,5F) $\frac{1}{2}\Delta \equiv \Lambda_1{}^{[3}\Lambda_2{}^{4]} = \Lambda_3{}^{[1}\Lambda_4{}^{2]}$,

(7,6) $$\delta_{cd}{}^B L_B{}^A = \delta_{ab}{}^A \Lambda_{[c}{}^a \Lambda_{d]}{}^b.$$

Proof. Any transformation (7,1) belonging to (G) $[(F)]$ reproduces each of the reguli of **Q** [interchanges these reguli]. Consequently its map reproduces [interchanges] the axis ${}^i\Phi$ of K. The equations (7,4) express analytically this fact. Any transformation (7,1) from (T) carries a lineal element in a lineal element. Consequently by virtue of Theorems (5,4) and (2,2) the map (7,2) of (7,1) reproduces the complex Γ. Because of (1,5a) (7,4) and

$$'\Gamma^{ab} = \Lambda_c{}^{[a}\Lambda_d{}^{b]}\, \Gamma^{cd}$$

the complex Γ is reproduced if and only if (7,5) holds. The equations (2,2) yield

(7,7a) $$2v^A = \delta_{ab}{}^A\, {}^\epsilon\Omega^{ab}.$$

The transformations (7,1) and (7,2) carry v^A and ${}^\epsilon\Omega^{ab}$ respectively into

(7,8) $$'v^A = L_B{}^A v^B, \; {}'{}^{\epsilon'}\Omega^{ab} = \Lambda_c{}^{[a}\Lambda_d{}^{b]}\, {}^\epsilon\Omega_c{}^d,$$

where again

(7,7b) $$2\, 'v^A = \delta_{ab}{}^A\, {}'{}^{\epsilon'}\Omega^{ab}.$$

The equations (7,6) follow from (7,7ab) and (7,8).

Note (i): Because

$$'{}^{\epsilon'}\Omega^{12} = -\, '{}^{\epsilon'}\Omega^{34} = -\, '\epsilon('\mathbf{v}.'\mathbf{v})^{\frac{1}{2}},$$
$${}^\epsilon\Omega^{12} = -\, {}^\epsilon\Omega^{34} = -\, \epsilon(\mathbf{v}.\mathbf{v})^{\frac{1}{2}},$$

we have by virtue of (7,5)

[19](7,1) induces a transformation of lineal elements $(\mathbf{x}, v^\nu)$ and (7,2) may be thought of as a map of this transformation.

[20]The equations (G) [(F)] refer to the case of (7,1) belonging to (G) $[(F)]$.

(7,9aG) $$'{}^{\epsilon'}\Omega^{12} = \Delta\,{}^{\epsilon}\Omega^{12} \qquad '{}^{\epsilon'}\Omega^{34} = \Delta\,{}^{\epsilon}\Omega^{34},$$

(7,9aF) $$'{}^{\epsilon'}\Omega^{12} = -\Delta\,{}^{\epsilon}\Omega^{12} \qquad '{}^{\epsilon'}\Omega^{34} = -\Delta\,{}^{\epsilon}\Omega^{34}.$$

Consequently by virtue of (7,2)

(7,9b) $$'\mathbf{v}.'\mathbf{v} = \Delta^2\mathbf{v}.\mathbf{v}.$$

On the other hand we have by virtue of (7,1)

(7,9c) $$'\mathbf{v}.'\mathbf{v} = L^2\mathbf{v}.\mathbf{v},$$

where L^2 is a well defined function of the $L_B{}^A$. Hence

(7,10) $$L^2 = \Delta^2,$$

and this equation prescribes a condition for the $\Lambda_b{}^a$.

Note (ii). Because any transformation from (G) [(F)] reproduces (interchanges) the reguli of $\mathbf{Q}$ it may be easily proved that

(7,11G) $$L_{\mathrm{I}}{}^{[\mathrm{I}}L_{\mathrm{IV}}{}^{\mathrm{IV}]}L_{\mathrm{III}}{}^{[\mathrm{III}}L_{\mathrm{I}}{}^{\mathrm{I}]} \neq 0,$$

(7,11F) $$L_{\mathrm{III}}{}^{[\mathrm{I}}L_{\mathrm{I}}{}^{\mathrm{IV}]}L_{\mathrm{I}}{}^{[\mathrm{III}}L_{\mathrm{IV}}{}^{\mathrm{I}]} \neq 0.$$

Hence the number

(7,12G) $$q^4 \equiv \frac{L_{\mathrm{I}}{}^{[\mathrm{I}}L_{\mathrm{IV}}{}^{\mathrm{IV}]}}{L_{\mathrm{III}}{}^{[\mathrm{III}}L_{\mathrm{I}}{}^{\mathrm{I}]}}$$

(7,12F) $$q^4 \equiv \frac{L_{\mathrm{III}}{}^{[\mathrm{I}}L_{\mathrm{I}}{}^{\mathrm{IV}]}}{L_{\mathrm{I}}{}^{[\mathrm{III}}L_{\mathrm{IV}}{}^{\mathrm{I}]}}$$

is different from zero. We use it in the following theorem, where without loss of generality we assume that $L_{\mathrm{I}}{}^{\mathrm{I}}$ is a real number and moreover

(7,13) $$L_{\mathrm{I}}{}^{\mathrm{I}} > 0.$$

THEOREM (7,2). *Any transformations from* (G) *and* (F) *respectively are mapped on two always distinct*[21] *transformations*:

(7,14G) $$\begin{aligned} '\xi^1 &= \frac{1}{q}(L_{\mathrm{I}}{}^{\mathrm{I}}\xi^1 + L_{\mathrm{IV}}{}^{\mathrm{I}}\xi^2)\\ '\xi^2 &= \frac{1}{q}(L_{\mathrm{I}}{}^{\mathrm{IV}}\xi^1 + L_{\mathrm{IV}}{}^{\mathrm{IV}}\xi^2)\\ '\xi^3 &= \epsilon q(L_{\mathrm{III}}{}^{\mathrm{III}}\xi^3 - L_{\mathrm{I}}{}^{\mathrm{III}}\xi^4)\\ '\xi^4 &= \epsilon q(-L_{\mathrm{III}}{}^{\mathrm{I}}\xi^3 + L_{\mathrm{I}}{}^{\mathrm{I}}\xi^4)\end{aligned}$$

(7,14F) $$\begin{aligned} '\xi^1 &= \frac{1}{q}(L_{\mathrm{III}}{}^{\mathrm{I}}\xi^3 - L_{\mathrm{I}}{}^{\mathrm{I}}\xi^4)\\ '\xi^2 &= \frac{1}{q}(L_{\mathrm{III}}{}^{\mathrm{IV}}\xi^3 - L_{\mathrm{I}}{}^{\mathrm{IV}}\xi^4)\\ '\xi^3 &= \epsilon q(L_{\mathrm{I}}{}^{\mathrm{III}}\xi^1 + L_{\mathrm{IV}}{}^{\mathrm{III}}\xi^2)\\ '\xi^4 &= \epsilon q(-L_{\mathrm{I}}{}^{\mathrm{I}}\xi^1 - L_{\mathrm{IV}}{}^{\mathrm{I}}\xi^2)\end{aligned}$$

where $\epsilon = +1$ *or* $\epsilon = -1$ *and* q *is any one of the four fourth roots of* q^4.

Proof. Let us first prove the equations (7,14G). From (7,4G) and (7,6) for $c = 1$, $d = 4$, $A = I$ we obtain $L_{\mathrm{I}}{}^{\mathrm{I}} = \Lambda_1{}^1\Lambda_4{}^4$.

If we put $\Lambda_1{}^1 = 1/p$ then we obtain[22] from (7,13) and (7,6)

[21]E.g. two distinct transformations (7,14G) [(7,14F)].

[22]The equations (7,15) constitute the conditions for the $L_B{}^A$ to be the coefficients of a transformation from (G).

$$
\begin{aligned}
&\Lambda_2{}^1 = \frac{L_{\mathrm{IV}}{}^{\mathrm{I}}}{L_{\mathrm{I}}{}^{\mathrm{I}}p} = \frac{L_{\mathrm{IV}}{}^{\mathrm{III}}}{L_{\mathrm{I}}{}^{\mathrm{III}}p} = -\frac{L_{\mathrm{II}}{}^{\mathrm{III}}}{L_{\mathrm{III}}{}^{\mathrm{III}}p} = -\frac{L_{\mathrm{II}}{}^{\mathrm{I}}}{L_{\mathrm{III}}{}^{\mathrm{I}}p},\\
(7,15)\qquad &\Lambda_1{}^2 = \frac{L_{\mathrm{I}}{}^{\mathrm{IV}}}{L_{\mathrm{I}}{}^{\mathrm{I}}p} = \frac{L_{\mathrm{III}}{}^{\mathrm{IV}}}{L_{\mathrm{III}}{}^{\mathrm{I}}p} = -\frac{L_{\mathrm{I}}{}^{\mathrm{II}}}{L_{\mathrm{I}}{}^{\mathrm{III}}p} = -\frac{L_{\mathrm{III}}{}^{\mathrm{II}}}{L_{\mathrm{III}}{}^{\mathrm{III}}p},\\
&\Lambda_2{}^2 = \frac{L_{\mathrm{IV}}{}^{\mathrm{IV}}}{L_{\mathrm{I}}{}^{\mathrm{I}}p} = \frac{L_{\mathrm{II}}{}^{\mathrm{II}}}{L_{\mathrm{III}}{}^{\mathrm{III}}p} = -\frac{L_{\mathrm{II}}{}^{\mathrm{IV}}}{L_{\mathrm{III}}{}^{\mathrm{I}}p} = -\frac{L_{\mathrm{IV}}{}^{\mathrm{II}}}{L_{\mathrm{I}}{}^{\mathrm{III}}p},
\end{aligned}
$$

$$
(7,16)\qquad
\begin{aligned}
&\Lambda_3{}^3 = L_{\mathrm{III}}{}^{\mathrm{III}}p, &&\Lambda_4{}^3 = -L_{\mathrm{I}}{}^{\mathrm{III}}p,\\
&\Lambda_3{}^4 = -L_{\mathrm{III}}{}^{\mathrm{I}}p, &&\Lambda_4{}^4 = L_{\mathrm{I}}{}^{\mathrm{I}}p.
\end{aligned}
$$

Multiplying the first right hand member of these equations by $(L_{\mathrm{I}}{}^{\mathrm{I}})^{\frac{1}{2}}$ we obtain

$$
(7,17)\quad \begin{pmatrix}\Lambda_1{}^1\Lambda_2{}^1\\ \Lambda_1{}^2\Lambda_2{}^2\end{pmatrix} = \frac{1}{p(L_{\mathrm{I}}{}^{\mathrm{I}})^{\frac{1}{2}}}\begin{pmatrix}L_{\mathrm{I}}{}^{\mathrm{I}}L_{\mathrm{IV}}{}^{\mathrm{I}}\\ L_{\mathrm{I}}{}^{\mathrm{IV}}L_{\mathrm{IV}}{}^{\mathrm{IV}}\end{pmatrix},\quad \begin{pmatrix}\Lambda_3{}^3\ \Lambda_4{}^3\\ \Lambda_3{}^4\ \Lambda_4{}^4\end{pmatrix} = p(L_{\mathrm{I}}{}^{\mathrm{I}})^{\frac{1}{2}}\begin{pmatrix}L_{\mathrm{III}}{}^{\mathrm{III}}-L_{\mathrm{I}}{}^{\mathrm{III}}\\ -L_{\mathrm{III}}{}^{\mathrm{I}}-L_{\mathrm{I}}{}^{\mathrm{I}}\end{pmatrix}.
$$

Hence we must have by virtue of (7,5G)

$$
(7,18)\qquad \Lambda_1{}^{[1}\Lambda_2{}^{2]} = \frac{1}{p^2L_{\mathrm{I}}{}^{\mathrm{I}}}\ L_{\mathrm{I}}{}^{[\mathrm{I}}L_{\mathrm{IV}}{}^{\mathrm{IV}]} = \Lambda_3{}^{[3}\Lambda_4{}^{4]} = p^2L_{\mathrm{I}}{}^{\mathrm{I}}L_{\mathrm{III}}{}^{[\mathrm{III}}L_{\mathrm{I}}{}^{\mathrm{I}]},
$$

or

$$
(7,19)\qquad q^4 \equiv p^4\,L_{\mathrm{I}}{}^{\mathrm{I}^2} = \frac{L_{\mathrm{I}}{}^{[\mathrm{I}}L_{\mathrm{IV}}{}^{\mathrm{IV}]}}{L_{\mathrm{III}}{}^{[\mathrm{III}}L_{\mathrm{I}}{}^{\mathrm{I}]}}\,.
$$

Denoting by $q_k(k = 1, 2, 3, 4)$ the fourth roots of q^4 and substituting in (7,17), we obtain four transformations. Choosing conveniently the names of the roots, we have $q_1 = -q_3$, $q_2 = -q_4 = iq_1$ which means that the four transformations (7,17) reduce to two, one for q_1 and one for iq_1. The coefficients $\Lambda_b{}^a$ in (7,17) being homogeneous we obtain (7,14G) for $\epsilon = +1$ if we put $q = q_1$ and (7,14G) for $\epsilon = -1$ if we put $q \equiv iq_1$ and multiply the so obtained coefficients $\Lambda_b{}^a$ by i. The equations (7,14) F may be proved in a similar manner.

Note (i). The transformations (7,14) reproduce the complex Γ because by virtue of (7,18) and (7,19) the equations (7,5) are satisfied. On the other hand (7,10) is obviously a necessary condition that Γ be reproduced. Hence (7,10) is satisfied by the coefficients of the transformations (7,14).

Note (ii). For the identity $L_B{}^A = \delta_B{}^A$ we have $q^4 = 1$ and the equations (7,14G) reduce to identity and to

$$
'\xi^1 = \xi^1,\ '\xi^2 = \xi^2,\ '\xi^3 = -\xi^3,\ '\xi^4 = -\xi^4.
$$

This transformation reproduces each of the couples (5,3) interchanging $^+\xi$ and $^-\xi$, as was to be expected.

8. Mapping of the biaxial involution (2,1). In this section we shall be concerned with the transformation on **Q** which is mapped on (2,1) and we shall call it the map of (2,1). A projective transformation in L_3 will be termed a **v**-*reflection* if it reproduces **Q**, **v** and each point of the polar plane of **v** with respect to **Q**, where **v** is assumed to be an anisotropic point.

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THEOREM (8,1). *The biaxial involution* (2,1) *is a map of the transformation of lineal elements on* **Q** *induced by a* **v**-*reflection*

$$'x^A = x^A \mathbf{v}.\mathbf{v} - 2v^A \mathbf{x}.\mathbf{v}. \tag{8,1}$$

Proof. The equations (7,4F) and (7,5F) are satisfied for (2,1) (the matrix $\Omega_b{}^a$ being built up according to (0,13)). If there is a map of (2,1) then its coefficients must satisfy the equations (7,6) (where we put $\Omega_b{}^a$ instead of $\Lambda_b{}^a$). These equations define the matrix $L_B{}^A$ of the transformation (8,1). This transformation obviously reproduces **v** and each conjugate point **x** of **v** (with respect to **Q**). Moreover we have from (8,1)

$$'\mathbf{x}.'\mathbf{x} = (\mathbf{x}.\mathbf{x})(\mathbf{v}.\mathbf{v})^2$$

and consequently (8.1) reproduces **Q**. Hence (8,1) is a **v**-reflection.

Note. The transformation of lineal elements induced by (8,1) may be easily obtained. If **x** is an isotropic point for which $\mathbf{x}.\mathbf{v} \neq 0$, then its corresponding isotropic point is the second intersection points of **Q** and the line **xv**. An arbitrary plane through **xv** intersects the tangential planes at **x** and **'x** in the lines which define the corresponding directions of the line elements at **x** and **'x**.

In the next paper we shall deal with the analysis of S_3 starting with (6,5b).

Note added in April 1951:

At the last International Congress of Mathematicians (Cambridge, September 1950) Professor O. Veblen was kind enough to mention to me his paper "Geometry of four-component spinors" (Proc. Nat. Ac. Sci., vol. 19 (1933), 503-517) which was unknown to me. Although his paper and this present one follow quite different lines, it is nevertheless interesting to observe that they have two points in common: (1) the equations (2, 10) together with (3,2) in Professor Veblen's paper are substantially the same as my equations (6, 3), and (2) the elements of the first two of the matrices (4, 8) of Professor Veblen are substantially the Plücker coordinates of my complexes (1, 5).

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