In order to illustrate the relative closeness of the approximation, the several formulæ numbered (1), (5), and (6), may now be applied to the computation of the value of a joint-life annuity according to the H^{M} 3 per-cent tables. Let it be required for example, to deduce approximate values of $a_{50.48}$, given the values of $a_{50.40}$, $a_{50.45}$, $a_{50.50}$ and $a_{50.55}$.

By formula (1) to first differences,

$$a_{50.48} = \tfrac{1}{5}(2a_{50.45} + 3a_{50.50})$$
$$= 10{\cdot}8331.$$

By formula (5) to second differences,

$$a_{50.48} = \tfrac{1}{25}(7a_{50.45} + 21a_{50.50} - 3a_{50.55})$$
$$= 10{\cdot}8578.$$

By formula (6) employing central second differences,

$$a_{50.48} = \tfrac{1}{25}(16a_{50.45} + 12a_{50.50} - 3a_{50.40})$$
$$= 10{\cdot}8543.$$

The true value, as given in the Institute H^{M} 3 per-cent tables, is 10·8535; and it will be seen that, as stated by Mr. Woolhouse (*J.I.A.*, xi, 73 [note]), the central difference formula gives a more accurate result than that derived from the employment of ordinary second differences.

I am, Sir,
Yours, &c.,
THOMAS G. ACKLAND.

Addiscombe, Croydon,
30 *December* 1895.

GRADUATION—MR. J. A. HIGHAM'S THEOREM.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—In the study of graduation formulas, I have been rather surprised that comparatively little use has been made of the very remarkable theorem given by Mr. J. A. Higham (*J.I.A.*, xxv, 17 and 245–8), by which the result of any number of successive summations of the terms of a series is expressed by means of the first term of the series and its differences. Mr. Higham himself has fully explained the application of his theorem for purposes of graduation, and in his last contribution on the subject (*J.I.A.*, xxxi, 319) he has shown how it may be made to yield Woolhouse's formula. The demonstration of the theorem which Mr. Higham gives seems to me unnecessarily difficult, and I therefore venture to submit the following simpler proof.

Suppose the terms of the series $u_0, u_1, u_2, \ldots u_n$ to be summed continuously in p's; the results to be summed in q's; the next results to be summed in r's, and so on till the series has been reduced to one term only: it is required to find a general expression for the value of this term, which we shall denote by S.

The terms of the original series may be written

$$u_0,\ (1+\Delta)u_0,\ (1+\Delta)^2u_0, \ldots (1+\Delta)^n u_0,$$

and therefore the sum of the first p terms is

$$\{1+(1+\Delta)+(1+\Delta)^2 \ldots +(1+\Delta)^{p-1}\}u_0,$$

or $$\frac{(1+\Delta)^p-1}{\Delta}u_0, \quad \text{which may be denoted by } S_p.$$

S_p, then, is the first term of the new series got by the summation of the original series in p's, and it is clear that the second term of this new series is simply $(1+\Delta)S_p$, and that the others are $(1+\Delta)^2S_p$, $(1+\Delta)^3S_p, \ldots (1+\Delta)^{n-p+1}S_p$.

Similarly, if the terms of this new series are summed in q's, a third series is got, the first term of which is S_{pq}, where

$$S_{pq}=\frac{(1+\Delta)^q-1}{\Delta}S_p=\frac{(1+\Delta)^p-1}{\Delta}\cdot\frac{(1+\Delta)^q-1}{\Delta}\cdot u_0,$$

and the remaining terms are

$$(1+\Delta)S_{pq},\ (1+\Delta)^2S_{pq}, \ldots (1+\Delta)^{n-p-q+2}S_{pq}.$$

Proceeding thus, it is clear that finally we get

$$S=\frac{(1+\Delta)^p-1}{\Delta}\cdot\frac{(1+\Delta)^q-1}{\Delta}\cdot\frac{(1+\Delta)^r-1}{\Delta}\ldots u_0,$$

where, if t summations in all have been made, we must have

$$p+q+r+\ldots=n+t.$$

Let now $p^2+q^2+r^2+\ldots$ be denoted by s_2, and let us denote by σ_m the sum of the mth powers of $(p-1)$, $(q-1)$, so that

$$\begin{aligned}\sigma_1&=(p-1)+(q-1)+\ldots\\&=p+q+r\ldots-t=n\\ \sigma_2&=(p-1)^2+(q-1)^2\ldots\\&=s_2-2(n+t)+t\\&=s_2-2n-t.\end{aligned}$$

Then

$$\begin{aligned}\frac{S}{pqr\ldots}=&\left\{1+\frac{p-1}{2}\Delta+\frac{p-1.p-2}{6}\Delta^2+\frac{p-1.p-2.p-3}{24}\Delta^3+\ldots\right\}\\ &\times\left\{1+\frac{q-1}{2}\Delta+\frac{q-1.q-2}{6}\Delta^2+\frac{q-1.q-2.q-3}{24}\Delta^3+\ldots\right\}\\ &\times\left\{1+\frac{r-1}{2}\Delta+\frac{r-1.r-2}{6}\Delta^2+\frac{r-1.r-2.r-3}{24}\Delta^3+\ldots\right\}\\ &\times\left\{\ \ldots\ldots\ldots\ldots\ldots\ldots\ \right\}u_0.\end{aligned}$$

In the expansion of the right-hand side of this equation,

the coefficient of Δ is $\frac{1}{2}\sigma_1 = \frac{n}{2}$;

the coefficient of Δ^2 is $\frac{1}{4}\Sigma(p-1)(q-1) + \frac{1}{6}\Sigma(p-1)(p-2)$

$$= \tfrac{1}{4}\cdot\tfrac{1}{2}(\sigma_1^2 - \sigma_2) + \tfrac{1}{6}\{\Sigma(p-1)^2 - \Sigma(p-1)\}$$

$$= \tfrac{1}{8}(n^2 - s_2 + 2n + t) + \tfrac{1}{6}(s_2 - 2n - t - n)$$

$$= \frac{n(n-2)}{8} + \frac{s_2 - t}{24};$$

the coefficient of Δ^3 is

$$\Sigma\frac{p-1.q-1.r-1}{8} + \Sigma\frac{p-1.p-2.q-1}{12} + \Sigma\frac{p-1.p-2.p-3}{24},$$

of which the first term is

$$\tfrac{1}{8}\cdot\tfrac{1}{3}\{\Sigma(p-1).\Sigma(p-1)(q-1) - \Sigma(p-1)^2.(q-1)\}$$

$$= \tfrac{1}{24}\{\Sigma(p-1).\Sigma(p-1)(q-1) - \Sigma(p-1)^2.\Sigma(p-1) + \Sigma(p-1)^3\}$$

$$= \tfrac{1}{24}\left\{\frac{\sigma_1(\sigma_1^2 - \sigma_2)}{2} - \sigma_2\sigma_1 + \sigma_3\right\};$$

the second term is $\frac{1}{12}\Sigma(p-1)^2.(q-1) - \frac{1}{6}\Sigma(p-1)(q-1)$

$$= \tfrac{1}{12}(\sigma_1\sigma_2 - \sigma_3) - \tfrac{1}{12}(\sigma_1^2 - \sigma_2),$$

and the third term is $\frac{1}{24}\Sigma(p-1)(p-1-1)(p-1-2)$

$$= \tfrac{1}{24}(\sigma_3 - 3\sigma_2 + 2\sigma_1).$$

Hence the coefficient of Δ^3 is $\frac{\sigma_1^3}{48} - \frac{\sigma_1^2}{12} + \frac{\sigma_1}{12} + \frac{\sigma_1\sigma_2}{48} - \frac{\sigma_2}{24}$

$$= \frac{\sigma_1(\sigma_1 - 2)^2}{48} + \frac{\sigma_2(\sigma_1 - 2)}{48}$$

$$= \frac{(\sigma_1 - 2)}{48}\{\sigma_1(\sigma_1 - 4) + \sigma_2 + 2\sigma_1\}$$

$$= \frac{n(n-2)(n-4)}{48} + \frac{(n-2)(s_2 - t)}{48}.$$

If now we assume that the original series proceeds by *third* differences only, so that $\Delta^4 u_0 = \Delta^5 u_0 = \ldots = 0$, we get

$$\frac{S}{pqr\ldots} = \left[1 + \frac{n}{2}\Delta + \left\{\frac{n(n-2)}{8} + \frac{s_2 - t}{24}\right\}\Delta^2 + \left\{\frac{n(n-2)(n-4)}{48} + \frac{(n-2)(s_2 - t)}{48}\right\}\Delta^3\right] u_0,$$

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which is Mr. Higham's expression with a slightly modified notation.

I am, Sir,
Yours faithfully,
ABRAHAM LEVINE.

National Life Assurance Society,
5 *December* 1895.

UNIFORM SENIORITY.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—In Part II of the *Text-Book*, Mr. King gives an investigation into the most general law of human mortality which will give Simpson's rule for joint-life annuities, and deduces as that law the function of Gompertz. It has occurred to me that a similar investigation into the most general law of mortality that will permit the substitution of *two lives of equal ages* for any two given lives, might perhaps interest some of the readers of the *Journal*.

In order to make this substitution we must be able, for any given values of x and y, to determine w, so that ${}_np_{ww}={}_np_{xy}$ for all values of n. This may be otherwise expressed thus:

$$2\log{}_np_w=\log{}_np_x+\log{}_np_y$$

for all values of n. Differentiating with respect to n, and changing the sign of both sides of the equation, we get

$$2\mu_{w+n}=\mu_{x+n}+\mu_{y+n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

for all values of n. Whence differentiating again, we have

$$2\frac{d\mu_{w+n}}{dn}=\frac{d\mu_{x+n}}{dn}+\frac{d\mu_{y+n}}{dn},$$

or, what is the same thing,

$$2\frac{d\mu_{w+n}}{dw}=\frac{d\mu_{x+n}}{dx}+\frac{d\mu_{y+n}}{dy} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

for all values of n. Putting now $n=o$ in equations (1) and (2), we get

$$2\mu_w=\mu_x+\mu_y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$2\frac{d\mu_w}{dw}=\frac{d\mu_x}{dx}+\frac{d\mu_y}{dy} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Supposing now x and y to vary so that w remains constant, we get, by differentiation with respect to x,

$$\frac{d\mu_x}{dx}+\frac{d\mu_y}{dy}\cdot\frac{dy}{dx}=o \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\frac{d^2\mu_x}{dx^2}+\frac{d^2\mu_y}{dy^2}\cdot\frac{dy}{dx}=o \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$