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Formal Fibers of Unique Factorization Domains

Adam Boocher, Michael Daub, Ryan K. Johnson, H. Lindo, S. Loepp, and Paul A. Woodard

Abstract. Let (T, M) be a complete local (Noetherian) ring such that $\dim T \geq 2$ and $|T| = |T/M|$ and let $\{p_i\}_{i\in\mathcal{I}}$ be a collection of elements of T indexed by a set $\mathcal{I}$ so that $|\mathcal{I}| < |T|$. For each $i \in \mathcal{I}$, let $C_i := \{Q_{i1}, \dots, Q_{in_i}\}$ be a set of nonmaximal prime ideals containing p_i such that the Q_{ij} are incomparable and $p_i \in Q_{jk}$ if and only if $i = j$. We provide necessary and sufficient conditions so that T is the $\mathbf{m}$-adic completion of a local unique factorization domain $(A, \mathbf{m})$, and for each $i \in \mathcal{I}$, there exists a unit t_i of T so that $p_i t_i \in A$ and C_i is the set of prime ideals Q of T that are maximal with respect to the condition that $Q \cap A = p_i t_i A$.

We then use this result to construct a (nonexcellent) unique factorization domain containing many ideals for which tight closure and completion do not commute. As another application, we construct a unique factorization domain A most of whose formal fibers are geometrically regular.

1 Introduction

Given a complete local ring (T, M), it is natural to ask what kinds of subrings $(A, M \cap A)$ satisfy $\widehat{A} = T$. Throughout the *completion* $\widehat{R}$ of a local ring $(R, \mathbf{m})$ denotes the $\mathbf{m}$-adic completion of R. When T has certain properties we know of specific types of rings A whose completion is T. For example, Heitmann characterizes complete local rings that are the completions of unique factorization domains (UFDs) [4], whereas Loepp provides a characterization of complete local rings that are the completions of excellent domains with characteristic zero [9]. Yet the question of when a complete local ring is the completion of an excellent UFD remains open. There is, however, a partial result. Given a complete local normal domain T and a collection of prime elements $\{p_i\}_{i\in\mathcal{I}} \subset T$ that satisfy certain properties, the authors of [1] constructed a UFD A so that (1) $\widehat{A} = T$, (2) if Q is a prime ideal of A such that $QT = p_i T$ for some $i \in \mathcal{I}$, then the formal fiber over Q is geometrically regular, and (3) all prime ideals of A with height at least two have geometrically regular formal fibers. Thus "excellence" is ensured at these prime ideals of A. Specifically, the authors in [1] showed the following theorem.

Theorem 1.1 (**[1, Theorem 16]**) *Let (T, M) be a complete local normal domain containing the rationals with $|T/M| \geq |\mathbb{R}|$. Suppose P is a nonmaximal prime ideal of T such that T_P is a regular local ring and that K is a set of height-one prime ideals of T such that $|K| < |T/M|$. Suppose also that $\{p_i\}_{i\in\mathcal{I}}$, where $\mathcal{I}$ is an index set satisfying*

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$|\mathfrak{I}| < |T/M|$, is a set of height-one principal prime elements of T. Then there exists a local UFD A contained in T such that

(i) $\widehat{A} = T$.
(ii) *If Q is a nonzero prime ideal of A such that* $\operatorname{ht}(Q) \geq 2$ *or $QT = p_i T$ for some $i \in \mathfrak{I}$, then $T \otimes_A k(Q)$ is a field where $k(Q) := A_Q/QA_Q$.*
(iii) *The generic formal fiber ring of A, $T \otimes_A k((0))$, is regular, and the generic formal fiber is $\{q \in \operatorname{Spec}(T) \mid q \subseteq P\} \cup K$.*
(iv) *If I is an ideal of A such that* $\operatorname{ht}(IT) \geq 2$, *then A/I is complete.*

In [2, 3], the authors considered a different kind of property on A. Given a complete local ring T and a set of prime ideals $S := \{P_1, \dots, P_n\}$ of T, the authors of [2] gave necessary and sufficient conditions for T to be the completion of an integral domain A, where S is the set of maximal elements of the set $\{Q \in \operatorname{Spec} T \mid Q \cap A = (0)\}$. Along the same lines in [3], for an element p of T, the authors gave necessary and sufficient conditions for T to be the completion of an integral domain A such that $p \in A$ is a prime element of A and S is the set of maximal elements of the set $\{Q \in \operatorname{Spec} T \mid Q \cap A = pA\}$. Specifically, in [3], the authors proved the following theorem.

Theorem 1.2 **([3, Theorem 2.13])** *Let (T, M) be a complete local ring, Π the prime subring of T, and $S := \{P_1, \dots, P_n\}$ a finite set of non-maximal incomparable prime ideals of T. Let $p \in \bigcap_{i=1}^{n} P_i$ with $p \neq 0$. Then there exists a local domain A contained in T such that $p \in A$, $\widehat{A} = T$ and pA is a prime ideal whose formal fiber is semilocal with maximal ideals the elements of S if and only if*

(i) $Q \cap \Pi[p] = (0)$ *for all* $Q \in \operatorname{Ass}(T)$,
(ii) *for every $P' \in \operatorname{Ass}(T/pT)$, $P' \subseteq P_i$ for some $i \in \{1, 2, \dots, k\}$,*
(iii) $((P_i \setminus pT)\Pi[p]) \cap \Pi[p] = \{0\}$ *for all $i \in \{1, 2, \dots, k\}$.*

Taking motivation from these results, we seek to study the relationship between a local ring A and its completion $\widehat{A}$ by examining the formal fibers of the ring A. The formal fiber ring of a prime ideal $P \in \operatorname{Spec} A$ is defined to be $\widehat{A} \otimes_A k(P)$, where $k(P) := A_P/PA_P$. We call $\operatorname{Spec}(\widehat{A} \otimes_A k(P))$ the formal fiber of A at P. This approach is promising because there is a natural correspondence between the formal fiber of A at P and the prime ideals $Q \in \operatorname{Spec} \widehat{A}$ such that $Q \cap A = P$. In light of this correspondence, for such a prime ideal Q of $\widehat{A}$, we say, "Q is in the formal fiber of A at P."

In this paper, we prove three main results. First, in Section 3, we generalize Theorem 1.2 by considering a set $\{p_i\}_{i \in \mathfrak{I}}$ of elements in T over an index set $\mathfrak{I}$, where $|\mathfrak{I}| < |T|$ rather than just one element p. We also construct our integral domain A to be a unique factorization domain. In particular, we prove the following result, which appears as Theorem 3.2 of this paper.

Main Theorem *Let (T, M) be a complete local ring of dimension at least two satisfying $|T| = |T/M|$. Let $\{p_i\}_{i \in \mathfrak{I}}$ be a countable (possibly finite) set of regular elements of T. For each $i \in \mathfrak{I}$, let $C_i := \{Q_{i1}, \dots, Q_{in_i}\}$ be a collection of nonmaximal prime ideals of T intersecting the prime subring Π trivially. Furthermore, suppose that the Q_{ij} are* *incomparable, and that $p_i \in Q_{jk}$ if and only if $i = j$. Finally, suppose that for each $i \in \mathcal{I}$ and every $P \in \mathrm{Ass}(T/p_iT)$, $P \subset \bigcup_{Q \in C_i} Q$. Let $C := \bigcup_i C_i$.*

Then the following are equivalent:

(i) *There exists a local UFD $(A, M \cap A) \subset T$ such that $\widehat{A} = T$ with the following properties:*
   (a) *For every $i \in \mathcal{I}$, there is a unit $t_i \in T$ such that $p_it_i \in A$ and the formal fiber of p_it_iA is semilocal with maximal ideals the elements of C_i.*
   (b) *The formal fiber rings of all prime ideals of height at least two are fields.*

(ii) *The following two properties hold:*
   (a) *No element of Π is a zero divisor.*
   (b) $\mathrm{depth}(T) > 1$.

Second, in Section 4, we use the main theorem from Section 3 to generalize [10, Theorem 14]. Specifically, in Theorem 4.2, we construct a (nonexcellent) UFD A satisfying the property that there are many principal ideals of A for which tight closure and completion do not commute.

Finally, in Section 5, we use the main theorem from Section 3 to weaken the hypothesis of Theorem 1.1. In particular, we show that with the weaker condition that each p_iT is a radical ideal instead of a prime ideal, there exists an "almost excellent" UFD A, whose completion is T, and A contains an associate p_it_i of each p_i. Further, we obtain that the formal fibers over each prime ideal p_it_iA are geometrically regular, as are the formal fibers over every prime ideal of height at least two. We then prove some facts about the relationship between radical ideals and the construction of excellent UFDs, as well as give necessary conditions for the construction.

Our proof combines the methods used in the papers [1, 3, 4]. In particular, we adapt the construction in [3] to the case when A is a UFD. We adapt the techniques Heitmann introduced in [4], which are essential to our construction. Throughout this paper, by a quasi-local ring, we mean a ring with exactly one maximal ideal. By a local ring, we mean a Noetherian quasi-local ring.

2 Preparation

The framework of our construction is taken from Heitmann [4]. We begin with a subring R_0 of T with certain properties. We maintain these properties, while adjoining to R_0 well-chosen elements one by one until we arrive at the desired subring. Our construction, therefore, hinges on the definition of our starting subring (which we call an NC subring), the types of elements we adjoin, and the lemmas necessary to prove that our properties are maintained. We lay these out in this section. Throughout this paper, we retain the following notation.

Setting 2.1 Let (T, M) be an infinite complete local ring. Let $\{p_i\}_{i \in \mathcal{I}}$ be a set of regular elements in T indexed by the set $\mathcal{I}$, where $\mathcal{I}$ is countable (possibly finite) or $|\mathcal{I}| < |T/M|$. For each $i \in \mathcal{I}$, let $C_i := \{Q_{i1}, \ldots, Q_{in_i}\}$ be a collection of nonmaximal prime ideals of T. Further, suppose that the Q_{ij} are incomparable, and that $p_i \in Q_{jk}$ if and only if $i = j$. Let $C := \bigcup_{i \in \mathcal{I}} C_i$.

Definition 2.2 Let R be a set. Then we define $\Gamma(R) := \sup(\aleph_0, |R|)$.

Definition 2.3 Let $(R, M \cap R)$ be a quasi-local UFD contained in the complete local ring (T, M). Then we call R an *NC subring* if R satisfies:

(i) $|R| \leq \Gamma(T/M)$ with equality only if T/M is countable.
(ii) $Q \cap R = (0)$ for all $Q \in \operatorname{Ass}(T)$.
(iii) If $z \in T$ is regular and $P \in \operatorname{Ass}(T/zT)$, then $\operatorname{ht}(P \cap R) \leq 1$.
(iv) For each $i \in \mathfrak{I}$, and for every $Q \in C_i$, $(Q \setminus p_i T)R \cap R = \{0\}$.

If R satisfies (i)–(iii), but not necessarily (iv), then R is called an *N-subring* in [4].

Property (iv) of Definition 2.3 is similar to the *pT-complement avoiding property* from [3].

Definition 2.4 Let $(R, R \cap M)$ be an *NC* subring contained in the complete local ring (T, M). An *A-extension* of R is a ring S such that

(i) S is an *NC* subring,
(ii) $R \subseteq S \subseteq T$,
(iii) prime elements of R remain prime in S,
(iv) $|S| \leq \Gamma(R)$.

Proposition 2.5 *Let $(R, R \cap M)$ be a UFD contained in the complete local ring (T, M), and suppose R satisfies property* (iii) *of NC subring. Let $a \in R$. Then $aT \cap R = aR$.*

Proof We may suppose $a \neq 0$. Since $aR \subseteq aT \cap R$ is clear, we need show only that $aT \cap R \subseteq aR$. Let $c \in aT \cap R$. First suppose that a is prime in R. Then, by property (iii) of *NC* subrings, $aR = P \cap R$ for every $P \in \operatorname{Ass}(T/aT)$. Thus $c \in aT \cap R \subset P \cap R = aR$.

Now suppose that aR is not necessarily prime in R. If a is not a unit of R, then a is contained in some principal prime pR. Then $c \in aT \cap R \subset pT \cap R = pR$, and thus $c/p \in R$. Now $c/p \in (a/p)T \cap R$. It, therefore, suffices to prove the result with c replaced by c/p and a replaced by a/p. The Noetherian property of T guarantees this factorization process must stop eventually. The element by which we replace c is a unit, and the result follows trivially. ■

Property (iv) of *NC* subring is sometimes cumbersome. In Proposition 2.6, we show that there is an equivalent condition that, on occasion, is more convenient to use.

Proposition 2.6 *Let $(R, R \cap M)$ be a quasi-local UFD contained in the complete local ring (T, M). Suppose that R satisfies properties* (i)–(iii) *of NC subring. Suppose that $i \in \mathfrak{I}$ satisfies $p_i \in R$. Then $(Q \setminus p_i T)R \cap R = \{0\}$ if and only if $Q \cap R = p_i R$ for all $Q \in C_i$.*

Proof We first show the forward direction. Fix $Q \in C_i$. Since $p_i \in R$ and $p_i \in Q \in C_i$, one containment is clear. To see the other, take $q \in Q \cap R$. Suppose $q \notin p_i R$. By Proposition 2.5, $p_i T \cap R = p_i R$. It follows that $q \notin p_i T$, and so $q \cdot 1 \in (Q \setminus p_i T)R \cap R = \{0\}$, a contradiction.

Now we show the backwards direction. Suppose $Q \cap R = p_i R$, for all $Q \in C_i$. Suppose there are $Q \in C_i$ and $f \neq 0$ with $f \in (Q \setminus p_i T)R \cap R$. Then there exist

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$q \in Q \setminus p_i T$ and $g \in R$ such that $f = qg$. Thus, by Proposition 2.5, $qg = f \in gT \cap R = gR$. By property (ii) of *NC* subring, since $g \in R$, g is not a zero divisor of T. It follows that $q \in R$. By assumption, $q \in Q \cap R = p_i R$, a contradiction. ∎

The proof of the following lemma is drawn from a similar lemma for pT-complement avoiding subrings in [3].

Lemma 2.7 *Suppose R is a subring of T, and R is a (not necessarily local) UFD satisfying properties* (i)–(iv) *of NC subring. Then $R_{M\cap R}$ is an NC subring.*

Proof First, $R_{M\cap R}$ is a UFD by [11, Theorem 20.5]. The first two properties of an *NC* subring are also clear. Thus we need show only properties (iii) and (iv).

We first show property (iii). Let $z \in T$ be regular, and let $Q \in \operatorname{Ass}(T/zT)$. Then, since R satisfies the third property of *NC* subrings, $Q \cap R = aR$ for some $a \in R$. We claim that $Q \cap R_{M\cap R} = aR_{M\cap R}$. One inclusion is obvious. To see the other, let $f \in Q \cap R_{M\cap R}$. Then $f = u/v$ for some $u, v \in R$. Then $u = fv \in Q \cap R = aR$, so $f \in aR_{M\cap R}$.

Finally, we check that property (iv) of an *NC* subring is satisfied. Fix $i \in \mathfrak{I}$. Let $Q \in C_i$, and suppose that $s \in (Q \setminus p_i T)R_{M\cap R} \cap R_{M\cap R}$. Write $s = f/g = qf'/g'$ with $f, g, f', g' \in R$ and g, g' in $R \setminus (M \cap R)$ and $q \in Q \setminus p_i T$. Since R is a domain, $fg' = qf'g$, and so clearly $fg' \in (Q \setminus p_i T)R \cap R = \{0\}$. Since g' is nonzero, $f = 0$, and thus $s = 0$, so that $(Q \setminus p_i T)R_{M\cap R} \cap R_{M\cap R} = \{0\}$. ∎

In order to build up a chain of *NC* subrings ultimately leading to our final ring, we repeatedly adjoin elements and then localize. The previous lemma shows that localization preserves the *NC* subring properties. The next two lemmas allow us to adjoin elements under certain conditions. The first is a lynchpin of our construction. It controls the intersections of ideals in T with our subring. The second allows us to maintain an NC subring while adjoining these new elements.

Lemma 2.8 *Let R be a subring of T. Let Q be a prime ideal in T. Suppose $Q \cap R = pR$. Then, if $x + Q \in T/Q$ is transcendental over $R/(Q \cap R)$, $Q \cap R[x] = pR[x]$.*

Proof One containment is clear. To see the other, take $f \in Q \cap R[x]$. Then $f = \sum_{i=0}^{k} r_i x^i$, where $r_i \in R$. Since $x+Q$ is transcendental over $R/(Q\cap R)$, $r_i \in Q \cap R = pR$ for all i. It follows that $f \in pR[x]$. ∎

Lemma 2.9 *Let $x \in T$ and R be an NC subring containing all the p_i. Then $R[x]_{R[x]\cap M}$ is an A-extension of R if the following condition holds:*

- *$x + Q \in T/Q$ is transcendental over $R/(R \cap Q)$ for all*

$$Q \in C \cup \{P \in \operatorname{Ass}(T/rT) \mid r \in R\} \cup \operatorname{Ass} T.$$

Proof Properties (ii)–(iv) of A-extension and property (i) of *NC* subring are clear. Thus by Lemma 2.7 it is sufficient to show that $R[x]$ is a UFD satisfying the *NC* properties (ii)–(iv). Since $x + Q \in T/Q$ is transcendental over $R/(Q \cap R)$ for all $Q \in \{P \in \operatorname{Ass}(T/rT) \mid r \in R\} \cup \operatorname{Ass} T$, x is transcendental over R, and thus $R[x]$ is a UFD. Property (ii) follows immediately from the hypothesis and Lemma 2.8. Thus we need show only properties (iii) and (iv) of an *NC* subring.

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We first prove property (iii) of NC subrings. Let $z \in T$ be regular and let $P \in \operatorname{Ass}(T/zT)$. First suppose that $P \cap R \neq (0)$. Then $P \cap R = aR$ for some $a \in R$, since R is an NC subring. Now, $PT_P \in \operatorname{Ass}(T_P/zT_P)$. Since $\operatorname{depth}(T_P) = 1$, the maximal ideal of T_P/aT_P consists only of zero divisors. Hence $P \in \operatorname{Ass}(T_P/aT_P)$, and it follows that $P \in \operatorname{Ass}(T/aT)$. In light of the hypothesis and Lemma 2.8, $P \cap R[x] = aR[x]$. Now suppose $P \cap R = (0)$. Then all nonzero $r \in R$ are inverted in the ring $R[x]_{R[x] \cap P}$. Thus, if F is the fraction field of R, $\dim(R[x]_{R[x]\cap P}) \leq \dim(F[x]) \leq 1$. Therefore $\operatorname{ht}(P \cap R[x]) = \dim(R[x]_{R[x]\cap P}) \leq 1$ and so property (iii) of NC subrings holds.

Finally, we show that $R[x]$ has property (iv) of NC subrings. Fix $i \in \mathfrak{I}$, and let $Q \in C_i$. Then since R is an NC subring, $Q \cap R = p_i R$ by Proposition 2.6. Now, by the hypothesis and Lemma 2.8, $Q \cap R[x] = p_i R[x]$. Appealing to Proposition 2.6 again, we have property (iv) of NC subrings. ■

Proposition 2.10 *Fix $i \in \mathfrak{I}$ from Setting 2.1. If $p_i \in R$ and $x + Q$ is transcendental over $R/(Q \cap R)$ for all $Q \in C_i$, then $(Q \setminus p_i T)R[x] \cap R[x] = \{0\}$ for all $Q \in C_i$.*

Proof In light of Proposition 2.6, this is simply the content of Lemma 2.8. ■

For our construction, we use a proposition from [5] to ensure that our final subring has the desired completion.

Proposition 2.11 ([5, Proposition 1]) *If $(A, M \cap A)$ is a quasi-local subring of a complete ring (T, M), $A \to T/M^2$ is onto, and $IT \cap A = I$ for every finitely generated ideal I of A, then A is Noetherian and the natural homomorphism $\widehat{A} \to T$ is an isomorphism.*

In light of Proposition 2.11, one sees that Lemma 2.12 is critical.

Lemma 2.12 *Let R be a subring of T and suppose R is an NC subring such that $p_i \in R$ for every $i \in \mathfrak{I}$. (Here, $\mathfrak{I}$ is from Setting 2.1.) If $I = (y_1, \dots, y_n)R$ is an ideal of R, n a positive integer, and $c \in IT \cap R$, then there exists an A-extension S of R such that $c \in IS$.*

The proof of Lemma 2.12 follows after some preliminary lemmas.

Lemma 2.13 **([4, Lemma 4])** *Let (T, M) be a complete local ring, R an N-subring of T, I a finitely generated ideal of R, and $c \in IT \cap R$. Then there exists a ring S such that $R \subset S \subset T$, S is an N-subring of T, $|S| = |R|$, prime elements in R are prime in S, and $c \in IS$.*

Notes from Heitmann's proof: In the proof of Lemma 2.13, Heitmann inducts on the number of generators of I. He first shows that if I is a principal prime ideal of R, then the lemma holds. He then shows that one may reduce to the case where I is not contained in a principal prime ideal of R. In this case the lemma follows easily, if in addition I is principal. If I is generated by two elements, the proof is much more difficult; Heitmann intersects two carefully chosen Krull domains and shows that the result is the desired N-subring. When I is generated by more than two elements, Heitmann adjoins an element $\tilde{t}$ of T to R so that the image of $\tilde{t}$ in T/P is transcendental over $R/(P \cap R)$ for every $P \in \bigcup_{r \in R} \operatorname{Ass}(T/rT)$. Since $\tilde{t}$ is transcendental over R, prime elements in R will remain prime in S.

For the proof of Lemma 2.12 we adjust the proof above. We again induct on the number of generators of I. By Proposition 2.5, Lemma 2.12 holds when I is principal. We now show, in Lemma 2.14, that Lemma 2.12 holds when I is generated by two elements.

Lemma 2.14 *Lemma* 2.12 *is true when* $n = 2$.

Proof In this case, $I = (y_1, y_2)$ with $y_1 \neq 0$ and $y_2 \neq 0$. We first use the proof of Lemma 2.13 to reduce to the case when I is not contained in a principal prime ideal of R. Since $c \in IT$, $c = t_1 y_1 + t_2 y_2$, where $t_1, t_2 \in T$. Then let $x_1 := t_1 + t y_2$ and $x_2 := t_2 - t y_1$ for some $t \in T$ to be determined shortly. Note that $c = x_1 y_1 + x_2 y_2$.

Define the set $\Lambda := C \cup \{P \in \operatorname{Ass}(T/rT) \mid r \in R\} \cup \operatorname{Ass} T$. Then Λ is either countable or $|\Lambda| < |T/M|$.

In order to use the proof of Lemma 2.13, we wish to choose t so that, for each $Q \in \Lambda$, either $x_1 + Q$ or $x_2 + Q$ is transcendental over $R/(R \cap Q)$. In particular, we choose t so that $x_1 + Q$ is transcendental over $R/(Q \cap R)$ if $y_2 \notin Q$ and $x_2 + Q$ is transcendental over $R/(Q \cap R)$ if $y_1 \notin Q$.

Notice that for every $Q \in \Lambda$, y_1, y_2 cannot both be in Q. Otherwise, the definition of NC subring and Proposition 2.6 would imply that $Q \cap R$ would be prinicpal and then $I \subseteq Q \cap R$ would imply that I is contained in a principal ideal, a contradiction to the reduction above. Thus, for each Q, $y_1 \notin Q$ or $y_2 \notin Q$. Also, if $Q \in C_i$ and $y_j \notin Q$, then $y_j \notin Q'$ for all $Q' \in C_i$, since $Q' \cap R = p_i R$ for all $Q' \in C$ by property (iv) of NC subrings and Proposition 2.6.

We choose t to meet these conditions as in the proof of Lemma 2.13. For every $Q \in \Lambda$, either y_1 or y_2 is not an element of Q. First, suppose $y_1 \notin Q$ for some $Q \in \Lambda$. From the argument of the proof of Lemma 2.13, the number of choices for t that yield x_2 with $x_2 + Q \in T/Q$ algebraic over $R/(R \cap Q)$ is either countable or strictly less than $|T/M|$. Similarly, if $y_2 \notin Q$, the number of choices for t that yield an x_1 such that $x_1 + Q$ is algebraic over $R/(R \cap Q)$ is either countable or strictly less than $|T/M|$. From this it follows that there exists a $t \in T$ so that $x_2 + Q \in T/Q$ is transcendental over $R/(R \cap Q)$ for all $Q \in \Lambda$ satisfying $y_1 \notin Q$ and so that $x_1 + Q \in T/Q$ is transcendental over $R/(R \cap Q)$ for all $Q \in \Lambda$ satisfying $y_2 \notin Q$.

With this choice of t, define $S := (R[x_2, y_1^{-1}] \cap R[x_1, y_2^{-1}])_N$ where

$$N := R[x_2, y_1^{-1}] \cap R[x_1, y_2^{-1}] \cap M.$$

To show that S satisfies properties (ii)–(iv) of an A-extension and properties (i)–(iii) of an NC subring, we use the observation from [4, Lemma 4] that, for example, if $x_2 + Q$ is transcendental over $R/(R \cap Q)$ for some $Q \in \Lambda$, then $Q \cap R[x_2] = (Q \cap R)R[x_2] = aR[x_2]$, where $a \in R$ is such that $Q \cap R = aR$. Since $x_2 = (c - x_1 y_1)/y_2$, $x_2 \in R[x_1, y_2^{-1}]$ and so $c = x_1 y_1 + x_2 y_2 \in IS$. We now show that S has property (iv) of NC subring.

Fix $i \in \mathfrak{I}$ and let $Q \in C_i$. Then, by the choice of t, we may suppose that $x_1 + Q$ is transcendental over $R/(Q \cap R)$ with no loss of generality. Now by Proposition 2.6 and Lemma 2.8, $R[x_1]$ satisfies $(Q \setminus p_i T)R[x_1] \cap R[x_1] = \{0\}$.

Claim *For* $R' := R[x_1]$ *and* $Z := (Q \setminus p_i T)R'[y_2^{-1}] \cap R'[y_2^{-1}]$, $Z = 0$.

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If $f \in Z$, then $f = qg$ where $q \in (Q \setminus p_i T)$ and $g \in R'[y_2^{-1}]$. Thus there exists a non-negative integer n such that $fy_2^n \in R'$ and $gy_2^n \in R'$. It follows that $fy_2^n = q(gy_2^n) \in (Q \setminus p_i T)R' \cap R' = \{0\}$. Since f and y_2 are not zero divisors and $y_2 \neq 0$, $f = 0$. This proves the claim.

Thus, for each $i \in \mathcal{I}$ and for each $Q \in C_i$, $(Q \setminus p_i T)B \cap B = \{0\}$ for either $B = R[x_1, y_2^{-1}]$ or $B = R[x_2, y_1^{-1}]$. Hence $R[x_1, y_2^{-1}] \cap R[x_2, y_1^{-1}]$ satisfies property (iv) of *NC* subring. By Lemma 2.7, S satisfies it as well. ■

Proof of Lemma 2.12 We follow the proof of Lemma 2.13, except we use the set Λ described in the previous lemma throughout the construction. Note that Λ is countable or $|\Lambda| < |T/M|$. The construction in Lemma 2.13 yields an element $\tilde{t}$ such that $\tilde{t} + Q \in T/Q$ is transcendental over $R/(R \cap Q)$ for all $Q \in \Lambda$. Heitmann shows properties (i)–(iii) are satisfied [4, Lemma 4]. By Lemma 2.9, property (iv) is satisfied as well. ■

Lemma 2.15 *Suppose* $\dim T \geq 2$ *and R is an NC subring containing p_i for all $i \in \mathcal{I}$. Further suppose that*

- *J is an ideal of T,*
- *$u \in T$, and*
- *$J \nsubseteq Q$ for all $Q \in \Lambda := C \cup \{P \in \operatorname{Ass}(T/rT) \mid r \in R\} \cup \operatorname{Ass} T$.*

Then there exists an A-extension S of R meeting the following conditions:

(i) *$u + J \in \pi(S)$ where π is the natural map $\pi : T \to T/J$.*
(ii) *If $u \in J$, then, for each $Q \in C$, $S \cap J \nsubseteq Q$.*
(iii) *For every finitely generated ideal I of S, $IT \cap S = I$.*

Proof For every $P \in \Lambda$, let $D_{(P)}$ be a full set of coset representatives $t + P$ that make $(u + t) + P$ algebraic over $R/(P \cap R)$. Set $D := \bigcup_{P \in \Lambda} D_{(P)}$. Then there exists $x \in J$ such that $(u + x) + P$ is transcendental over $R/(P \cap R)$, for every $P \in \Lambda$ [4, Lemmas 2 and 3]. Thus, by Lemma 2.9, $S' := R[u + x]_{R[u+x] \cap M}$ is an A-extension of R.

Note that $\pi(u + x) = u + J$. For every $Q \in C$, $u + x + Q$ is transcendental over $R/(Q \cap R)$, and so $u + x \notin Q$. But if $u \in J$, then $u + x \in J \cap S'$. Thus, $J \cap S' \nsubseteq Q$. Our final ring S contains S', so the second property of this lemma holds.

Finally, to obtain the third property, we define the set

$$\Omega := \{(I, c) \mid I \text{ is a finitely generated ideal of } S' \text{ and } c \in IT \cap S'\}.$$

Next we adjust the proof of [4, Lemma 7] using Lemma 2.12 in place of [4, Lemma 4] to obtain the desired ring S. In particular, $S = \bigcup_{i=1}^{\infty} R_i$ where, for every positive integer i, $R_{i-1} \subseteq R_i$, R_i is an A-extension of R_{i-1}, and $IT \cap R_{i-1} \subseteq IR_i$ for all finitely generated ideals I of R_{i-1}. ■

The following lemma is a slight modification of [4, Lemma 6].

Lemma 2.16 *Suppose we have the following situation:*

- *R_0 is an NC subring of T.*
- *Ω is a well-ordered set with least element 0 such that either Ω is countable or, for all $\alpha \in \Omega$, $|\{\beta \in \Omega \mid \beta < \alpha\}| < |T/M|$.*

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- $\gamma(\alpha) = \sup\{\beta \in \Omega \mid \beta < \alpha\}$.
- $\{R_\alpha \mid \alpha \in \Omega\}$ *is an ascending collection of rings such that*

$$\begin{cases} R_\alpha \text{ is an } A\text{-extension of } R_{\gamma(\alpha)} & \text{if } \gamma(\alpha) < \alpha, \\ R_\alpha = \bigcup_{\beta<\alpha} R_\beta & \text{if } \gamma(\alpha) = \alpha. \end{cases}$$

- $S := \bigcup_{\alpha\in\Omega} R_\alpha$.

Then S satisfies all conditions of Definition 2.3 *for an NC subring of T except for* (i)*, the cardinality condition. Instead, $|S| \le \sup(\aleph_0, |R_0|, |\Omega|)$. Furthermore, elements that are prime in some R_α remain prime in S.*

Proof By [4, Lemma 6], S satisfies properties (i)–(iii) of *NC* subring, except that here $|S| \le \sup(\aleph_0, |R_0|, |\Omega|)$ and S satisfies the condition that prime elements in some R_α remain prime in S. To see that S satisfies property (iv) of *NC* subring, fix $i \in \mathfrak{I}$ and let $Q \in C_i$. Now, if $f \in (Q \setminus p_iT)S \cap S$, then $f = qf'$ for some $q \in Q$ and $f' \in S$. Thus, there exists some α such that both f and f' are in R_α and, since R_α is an *NC* subring, we have $f = 0$. ■

3 Main Result

Recall that we wish to construct a UFD A containing an associate p_it_i of each p_i such that the formal fiber of the prime ideal p_it_iA is semilocal with the set of maximal ideals equal to C_i. The preparation section of this paper provides us with the tools for the construction. The hardest part of the construction is finding an *NC* subring that contains an associate of each p_i. We begin this section by doing that.

Lemma 3.1 *Let (T, M) be a complete local ring of dimension at least two such that $|T/M| = |T|$. Let $\{p_i\}_{i\in\mathfrak{I}}$ be a countable (possibly finite) set of regular elements of T. For each $i \in \mathfrak{I}$, let $C_i := \{Q_{i1}, \dots, Q_{in_i}\}$ be a collection of nonmaximal prime ideals in T. Furthermore, suppose that the Q_{ij} are incomparable, and that $p_i \in Q_{jk}$ if and only if $i = j$. Set $C := \bigcup_i C_i$.*

Suppose that the following conditions hold:

- *$Q \cap \Pi = (0)$ for all $Q \in C$, where Π is the prime subring of T.*
- *For each $i \in \mathfrak{I}$ and every $P \in \operatorname{Ass}(T/p_iT)$, $P \subset \bigcup_{Q\in C_i} Q$.*
- *No element of the prime subring of T is a zero divisor.*

Then there exists an NC subring of T that contains an associate of each p_i.

Proof The idea of the proof is to start with the prime subring Π of T and then use [8, Lemma 4] to find a set of units $\{t_i\} \subset T$, so that we can safely adjoin the p_it_i to Π while maintaining the *NC* subring properties. This we now do.

Let $R_0 := \Pi_{M\cap\Pi}$. The hypotheses imply that R_0 is an *NC* subring. By assumption, $Q \cap R_0 = (0)$ for all $Q \in C$. Assume inductively that R_{i-1} has been constructed to be an *NC* subring and $Q \cap R_{i-1} = (0)$ for all $Q \in C_j$ with $j > i - 1$. By [8, Lemma 4], there exists a unit $t_i \in T$ such that $p_it_i + Q$ is transcendental over $R_{i-1}/(Q \cap R_{i-1})$ for each prime ideal Q in the set

$$\Lambda_i := \bigcup_{k\neq i} C_k \cup \operatorname{Ass} T \cup \bigcup_{r\in R_{i-1}} \operatorname{Ass}(T/rT).$$

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Let $S_i := R_{i-1}[p_it_i]$, and define R_i to be S_i localized at $S_i \cap M$. Note that $Q \cap S_i = (0)$ for all $Q \in C_j$ with $j > i$. Indeed, suppose that $f \in Q \cap S_i = Q \cap R_{i-1}[p_it_i]$ for such a Q. Then $f = r_n(p_it_i)^n + \cdots + r_0$ with $r_\ell \in R_{i-1}$ for all $\ell \in \{0, 1, 2, \ldots, n\}$. Since $p_it_i + Q$ is transcendental over $R_{i-1}/(Q \cap R_{i-1})$, each $r_\ell \in Q \cap R_{i-1} = (0)$. Hence $f = 0$ and so $S_i \cap Q = (0)$. It follows that $R_i \cap Q = (0)$ as well. Also, by Lemma 2.9 and Proposition 2.10, S_i satisfies properties (i)–(iii) of *NC* subring and, for each $k \neq i$ and each $Q \in C_k$, $(Q \setminus p_kT)S_i \cap S_i = \{0\}$. Now to complete the inductive step, we show that $(Q \setminus p_iT)S_i \cap S_i = \{0\}$ for each $Q \in C_i$.

By Proposition 2.6 it suffices to show that $Q \cap S_i = p_it_iS_i$ for all $Q \in C_i$. Clearly, $p_it_iS_i \subseteq Q \cap S_i$. For $Q \in C_i$, $Q \cap R_{i-1} = (0)$, as shown above. If $g \in Q \cap S_i$, $g = s_n(p_it_i)^n + \cdots + s_0$, $s_j \in R_{i-1}$. Since $g - s_0 \in (p_it_i)T \subset Q$, $s_0 \in Q$. Therefore, $s_0 \in Q \cap R_{i-1} = (0)$, and so $g \in p_it_iS_i$ as required.

We construct a sequence of *NC* subrings R_i that are easily seen to be successive A-extensions. Letting $B := \bigcup_{i=0}^{\infty} R_i$, we see by Lemma 2.16 that B is an *NC* subring containing each p_it_i. ■

We now prove our main result.

Theorem 3.2 *Let (T, M) be a complete local ring of dimension at least two satisfying $|T| = |T/M|$. Let $\{p_i\}_{i\in\mathfrak{I}}$ be a countable (possibly finite) set of regular elements of T. For each $i \in \mathfrak{I}$, let $C_i := \{Q_{i1}, \ldots, Q_{in_i}\}$ be a collection of nonmaximal prime ideals of T intersecting the prime subring Π trivially. Further, suppose that the Q_{ij} are incomparable, and that $p_i \in Q_{jk}$ if and only if $i = j$. Finally, suppose that for each $i \in \mathfrak{I}$ and every $P \in \operatorname{Ass}(T/p_iT)$, $P \subset \bigcup_{Q\in C_i} Q$. Let $C := \bigcup_i C_i$.*

Then the following are equivalent:

(i) *There exists a local UFD $(A, M \cap A) \subset T$ such that $\widehat{A} = T$ with the following properties:*
   - (a) *For every $i \in \mathfrak{I}$, there is a unit $t_i \in T$ such that $p_it_i \in A$ and the formal fiber of p_it_iA is semilocal with maximal ideals the elements of C_i.*
   - (b) *The formal fiber rings of all prime ideals of height at least two are fields.*

(ii) *The following two properties hold:*
   - (a) *No element of Π is a zero divisor.*
   - (b) $\operatorname{depth}(T) > 1$.

Proof The implication (i) implies (ii) follows from [4, Theorem 1].

To show that (ii) implies (i), suppose the conditions in (ii) hold. Let R_0 be the *NC* subring constructed in Lemma 3.1. Let p_it_i be the associate of p_i contained in R_0. With an abuse of notation we denote p_it_i by p_i. We justify this by noticing that R_0 is still an *NC* subring with respect to this new set $\{p_i\}$. Thus, we are in the situation of the preparation section of this paper, and may use its notation freely. For a ring R, define

$$\Lambda(R) := C \cup \operatorname{Ass} T \cup \bigcup_{r\in R} \operatorname{Ass}(T/rT).$$

To construct the ring A, we follow a construction similar to that in [4, 9]. Define

$$\Omega := \{u + J \in T/J \mid J \text{ is an ideal of } T\}.$$

Well order Ω so that each element has fewer than $|\Omega| = |T|$ predecessors. Let λ_0 denote the first element of Ω and let the subring R_0 from above correspond to λ_0. We define NC subrings R_λ for each $\lambda \in \Omega$ with $\lambda > \lambda_0$ in the following way. Suppose inductively that NC subrings R_β have been defined for each $\beta < \lambda$. Suppose that $\gamma(\lambda) < \lambda$. Then if $\lambda = u + J$ and J is not contained in an element of $\Lambda(R_{\gamma(\lambda)})$, using Lemma 2.15, we construct a ring S_λ that is an A-extension of $R_{\gamma(\lambda)}$. Furthermore, $u + J \in \pi(S_\lambda)$, where π is the natural map $\pi\colon S_\lambda \to T/J$, and $IT \cap S_\lambda = IS_\lambda$ for each finitely generated S_λ-ideal I.

We are now ready to succinctly define R_λ in general. If $\gamma(\lambda) < \lambda$, and J is not contained in an element of $\Lambda(R_{\gamma(\lambda)})$, then define $R_\lambda := S_\lambda$. If J is contained in such a prime ideal, then let $R_\lambda := R_{\gamma(\lambda)}$. It is clear that R_λ is an A-extension of $R_{\gamma(\lambda)}$ by construction.

If $\gamma(\lambda) = \lambda$, define $B_\lambda := \bigcup_{\beta<\lambda} R_\beta$, which is an NC subring. Suppose $\lambda = u + J$. If J is not contained in an element of $\Lambda(B_\lambda)$, define R_λ to be the NC subring obtained from B_λ using Lemma 2.15. Otherwise, define $R_\lambda := B_\lambda$.

Finally, define $A := \bigcup_{\lambda\in\Omega} R_\lambda$. By Lemma 2.16, A is a UFD satisfying all properties of NC subring except the cardinality condition. Let J be an ideal of T so that J is contained in no element of $\Lambda(A)$. Then by construction, the map $A \to T/J$ is surjective. In particular, the map $A \to T/M^2$ is surjective by the depth condition. Furthermore, since A is a union, it is straightforward to see that $IT \cap A = I$ for every finitely generated A-ideal I. Thus, $\widehat{A} = T$ by Proposition 2.11. Note that $p_i \in A$ for all $i \in \mathfrak{I}$.

We now prove property (i)(a) of the theorem. Fix $i \in \mathfrak{I}$. Note that A has property (iv) of NC subring and contains all the p_i. By Proposition 2.6, $Q \cap A = p_iA$ for all $Q \in C_i$. It follows that each p_i is prime in A and every $Q \in C_i$ is in its formal fiber. To see that the only elements in the formal fiber of p_iA are contained in $\bigcup_{Q\in C_i} Q$, let $P \in \operatorname{Spec} T$ with $P \nsubseteq Q$ for all $Q \in C_i$. We claim that $P \cap A \neq p_iA$. There are two cases to consider.

Case 1 $P \subseteq Q'$ for some $Q' \in C_j$ $(j \neq i)$. Then $P \cap A \subseteq Q' \cap A = p_jA$. If $P \cap A = p_iA$, we would have a contradiction, for then $p_i \in Q' \in C_j$.

Case 2 For all $Q \in C$, $P \nsubseteq Q$. Suppose first that for all $a \in A$, $P \notin \operatorname{Ass}(T/aT) \cup \operatorname{Ass} T$. Let $Q \in C_i$. Then $P \nsubseteq Q$ and hence, by construction (see the second conclusion of Lemma 2.15), $P \cap A \nsubseteq Q \cap A = p_iA$. Thus P is not in the formal fiber of p_iA. Suppose that $P \in \operatorname{Ass}(T/aT) \cup \operatorname{Ass} T$. Then by property (iii) of NC subring, $P \cap A = cA$ is principal. If $P \cap A = p_iA$, then $cA = p_iA$. Thus $cT = p_iT$. It follows that $P \in \operatorname{Ass}(T/p_iT)$, but, by hypothesis, P is contained in some $Q \in C$, a contradiction.

Thus, in either case, $P \cap A \neq p_iA$, and so P is not in the formal fiber of p_iA, proving (a).

Finally, we prove property (i)(b). Let P be a prime ideal in A with $\operatorname{ht} P \geq 2$. Then if $Q \in \operatorname{Spec} T$ is in the formal fiber of P, we claim that $Q = PT$. Indeed, let $Q \cap A = P$. We assume Q is not contained in an element of C, since then $Q \cap A$ would be p_iA for some $i \in \mathfrak{I}$. Similarly, for all $a \in A$, Q is not in $\operatorname{Ass}(T/aT)$. Thus the map $A \to T/Q$

is onto, and so $A/(A \cap Q) \cong T/Q$. Since T/Q is complete, $A/(Q \cap A)$ is as well. Thus

$$\frac{T}{(Q \cap A)T} \cong \frac{A}{Q \cap A} \cong \frac{T}{Q}.$$

Since $(Q \cap A)T \subseteq Q$, $(Q \cap A)T = Q$, and so $PT = Q$. This shows that the formal fiber ring of P is a field. ∎

Remark 3.3 To simplify the notation in the preceding proofs, the cardinality of the set $\{p_i\}_{i \in \mathfrak{I}}$ was chosen to be at most countable; however, the result remains true so long as $|\mathfrak{I}| < |T|$. The only modification is that the inductions in Lemma 3.1 must be taken to be transfinite in a way similar to that of Theorem 3.2.

4 An Application to Tight Closure

Our first application of Theorem 3.2 is to construct a UFD A for which there are many principal ideals of A, where tight closure and completion do not commute. For an introduction to the theory of tight closure, see [6].

In [10], the authors constructed a (nonexcellent) local UFD A that contains a height-one prime ideal P such that if y is a nonzero element of P, then there is a natural number r such that, for the ideal $y^r A$, tight closure and completion do not commute (see [10, Theorem 14]). In this section, we generalize this result, so that instead of one such height-one prime ideal P of A, there are many. If I is an ideal of a ring of positive prime characteristic, we use the notation I^* to denote the tight closure of I.

Proposition 4.1 **([10, Proposition 2])** *Let $(A, \mathbf{m})$ be a local normal domain of prime characteristic. Assume that $\widehat{A}$ is an integral domain and not normal. Suppose there is a height-one prime ideal $\widehat{P}$ of $\widehat{A}$ such that $\widehat{A}_{\widehat{P}}$ is not normal and such that $\widehat{P} \cap A = P \neq (0)$. Then, for every $y \in P - (0)$, there is a natural number r such that $y^r \widehat{A} \neq (y^r \widehat{A})^*$. In particular, for this ring A, tight closure and completion do not commute.*

By [6, Example 1.6.1], every principal ideal of A in the above proposition is tightly closed. Armed with Proposition 4.1 and Theorem 3.2, we are now ready to construct the desired UFD A. Since the UFD A we construct is normal, but its completion is not, A is not excellent.

Theorem 4.2 *Let (T, M) be a complete local domain of prime characteristic. Suppose T has dimension at least 2 and satisfies $|T/M| = |T|$. Let $\{p_i\}_{i \in \mathfrak{I}}$ be a countable (possibly finite) set of regular elements in T satisfying the property that every prime ideal in $\operatorname{Ass}(T/p_i T)$ is minimal over $p_i T$. For each $i \in \mathfrak{I}$, let $C_i := \operatorname{Ass}(T/p_i T)$ and let $C := \bigcup_i C_i$. Suppose that $p_i \in \bigcup_{Q \in C_j} Q$ if and only if $i = j$, and that for every $Q \in C$, T_Q is not normal. Then there exists a local unique factorization domain A such that $\widehat{A} = T$ and if $Q \in C_i$ then $Q \cap A = (p_i t_i)A$ for some unit $t_i \in T$. Moreover, for every nonzero $y \in Q \cap A = (p_i t_i)A$, there is a natural number r with $y^r \widehat{A} \neq (y^r \widehat{A})^*$. In particular, for the ring A, tight closure and completion do not commute for these ideals $y^r A$ of A.*

Proof Use Theorem 3.2 to construct A. The result now follows from Proposition 4.1. ■

Example 4.3 Define $S := K[[y_1, y_2, \dots, y_n, z_1, z_2, \dots, z_n]]$, where K is the quotient field of the ring $\mathbb{Z}_7[[x]]$, and $n \in \mathbb{N}$ with $n \geq 2$. Let I be the ideal of S given by $I := (y_1^3 - z_1^2, y_2^3 - z_2^2, \dots, y_n^3 - z_n^2)$. Then $T := S/I$ and $p_i := y_i$ for $i = 1, 2, \dots, n$ satisfy the conditions for Theorem 4.2. For this example, $C_i = \{(y_i, z_i)\}$ for every $i = 1, 2, \dots, n$. By Theorem 4.2, there exists a local unique factorization domain A such that $\widehat{A} = T$ and $(y_i, z_i) \cap A = (y_i t_i)A$ for some unit $t_i \in T$. Moreover, for every nonzero $w \in (y_i, z_i) \cap A = (y_i t_i)A$, there is a natural number r with $w^r\widehat{A} \neq (w^r\widehat{A})^*$. In particular, for the ring A, tight closure and completion do not commute for these ideals w^rA of A.

5 Almost Excellent UFD's

The second application of the main result is related to the construction of excellent UFDs. Let T be a complete local ring. We seek conditions on T such that there is a subring A of T with A an excellent UFD and $\widehat{A} = T$. In light of [4, 9], we see that a necessary condition for this to occur is that T be normal. Due to results of Lipman [7], requiring that T be a UFD is too strong a condition. A natural question to study is "Which elements of T can be chosen to be elements of A?". In this section we discuss some of the necessary conditions, and, using our main result, construct an "almost excellent" UFD A whose completion is T.

Definition 5.1 Let $p \in T$ be a regular element. Then we call p *radical* if $\sqrt{pT} = pT$.

Radical elements are closely related to those elements of T that can be chosen to be elements of our excellent UFD A. The following two lemmas help establish this relationship.

Lemma 5.2 *Suppose $p \in T$ is regular and pT has no embedded associated primes. Then p is radical if and only if $pT_P = PT_P$ for all $P \in \operatorname{Ass}(T/pT)$.*

Proof Let $pT = Q_1 \cap \cdots \cap Q_n$, where n is a positive integer, be a minimal primary decomposition of pT, with $\sqrt{Q_i} = P_i$. Suppose p is radical. Then

$$pT_{P_i} = (\sqrt{pT})_{P_i} = (P_1 \cap \cdots \cap P_n)T_{P_i} = P_i T_{P_i}$$

for all $i = 1, \dots, n$.

Suppose $pT_P = PT_P$ for all $P \in \operatorname{Ass}(T/pT)$. Clearly, $pT \subset \sqrt{pT}$. Now we show that $\sqrt{pT} \subset Q_i$ for all $i = 1, \dots, n$. Let $r \in \sqrt{pT} = P_1 \cap \cdots \cap P_n$. Then $r \in P_i T_{P_i}$ for all i. Since pT has no embedded associated primes, $Q_i \not\subset P_j$ for all $i \neq j$. It follows that

$$P_i T_{P_i} = pT_{P_i} = (Q_1 \cap \cdots \cap Q_n)T_{P_i} = Q_i T_{P_i}.$$

Therefore $r \in Q_i T_{P_i}$ for all i. By a basic fact of localization (see [12, Lemma 5.29]), we conclude that $r \in Q_i$ for all i. It follows that $r \in pT$, and $\sqrt{pT} = pT$. ■

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Lemma 5.3 *Suppose T is a normal complete local ring with* $\dim T \geq 2$, *and A is an excellent UFD such that $\widehat{A} = T$. Let Q be a height-one prime ideal of A. Then $Q = pA$, where p is a radical element of T.*

Proof Since Q is a height-one prime ideal of a UFD, $Q = pA$ for some nonzero $p \in A$. Since A is excellent, $B := T \otimes_A k(pA)$ is a regular ring. Therefore, when we localize at a prime ideal of B, we get a regular local ring. Since T is normal, $\operatorname{ht} P = 1$ for $P \in \operatorname{Ass}(T/pT)$. Therefore $\operatorname{ht}(P \cap A) \leq 1$. Since $p \in P \cap A$, $P \cap A \neq (0)$, and it follows that $P \cap A = pA$. Now P corresponds to a prime ideal of B, and so B_P is a regular local ring. Then

$$B_P \cong \left(\overline{A \setminus pA}\right)^{-1}\left(\frac{T}{pT}\right)_P \cong \left(\frac{T}{pT}\right)_P \cong \frac{T_P}{pT_P},$$

where $\overline{A - pA} := \{\pi(a) \mid a \in A - pA\}$ and $\pi\colon T \to T/pT$ is the natural map. Since T is normal, and $\operatorname{ht} P = 1$, T_P is a DVR, and so it has exactly two prime ideals, (0) and PT_P. But, if $\frac{T_P}{pT_P}$ is a regular local ring, then it must be a domain, and so, since $pT_P \neq (0)$, $pT_P = PT_P$. By Lemma 5.2, p must be radical in T. ■

As shown above, if A is an excellent UFD whose completion is T, and $p \in A$ is a prime element, then p must be a radical element of T. The following application of the main result of Section 3 shows a partial converse.

Corollary 5.4 *Let (T, M) be a normal complete local ring containing the rationals. Suppose* $\dim T \geq 2$ *and $|T| = |T/M|$. Let $\{p_i\}_{i \in \mathcal{I}}$ be a countable (possibly finite) set of radical elements that share no associated prime ideals. Then there exists a local UFD $(A, M \cap A) \subset T$ containing an associate $p_i t_i$ of each p_i, where $t_i \in T$ is a unit, with the following properties:*

(i) $\widehat{A} = T$.
(ii) *For all $i \in \mathcal{I}$, $p_i t_i$ is prime in A.*
(iii) *The fiber over* (0)*, fibers over the prime ideals $(p_i t_i)A$, and fibers over all prime ideals of height at least two, are geometrically regular.*

Proof We first use Theorem 3.2 with $C_i := \operatorname{Ass}(T/p_i T)$ to construct a ring A satisfying items (i) and (ii). We now show the third property. We first study the fiber over (0). Since T contains the rationals, $k(0)$ is a field of characteristic zero. Hence, to show that the fiber over (0) is geometrically regular, it suffices to show that $T \otimes_A k(0)$ is a regular ring. Since T is normal and local, $\operatorname{Ass} T = \{(0)\}$. Also, T satisfies Serre's (S_2) condition, and therefore, for each $i \in \mathcal{I}$, all elements of $\operatorname{Ass}(T/p_i T)$ have height one. It follows by construction (see Lemma 2.15) that $S \cap J \neq (0)$ for all primes J that are height at least two. Thus the only primes that intersect to (0) are prime ideals of height at most one. Therefore a maximal ideal of the ring $B := T \otimes_A k(0)$ corresponds to a height-one prime ideal Q of T. It follows that B localized at a maximal ideal is isomorphic to T_Q for some height-one prime ideal Q of T. But T_Q is a regular local ring since T is normal and so satisfies Serre's condition (R_1). Hence, the fiber over (0) is geometrically regular.

Next we show that the formal fibers over the prime ideals $(p_i t_i)A$ are geometrically regular. Since T contains the rationals, $k((p_i t_i)A)$ is a field of characteristic zero. It

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follows that we need show only that, for each $i \in \mathcal{I}$, the ring $B_i := T \otimes_A k((p_i t_i)A)$ is a regular ring. By Theorem 3.2, the maximal ideals of B_i are in one-to-one correspondence with the elements of $C_i = \operatorname{Ass}(T/(p_i t_i)T)$. By Lemma 5.2, $p_i T_Q = QT_Q$ for all $Q \in C_i$. Since $Q \cap A = (p_i t_i)A$,

$$(B_i)_Q \cong \left(\overline{A \setminus (p_i t_i)A}\right)^{-1} \left(\frac{T}{(p_i t_i)T}\right)_Q \cong \left(\frac{T}{p_i T}\right)_Q \cong \frac{T_Q}{p_i T_Q} \cong \frac{T_Q}{QT_Q}.$$

Since QT_Q is the maximal ideal of T_Q, $(B_i)_Q$ is a regular local ring. Hence, the formal fiber of $(p_i t_i)A$ is geometrically regular.

Finally, since the formal fiber rings over all prime ideals of height at least two in A are fields by Theorem 3.2, these formal fibers are also geometrically regular. ■

Since T is an integral domain, it is also equidimensional, and so we get that A in Corollary 5.4 is universally catenary. Therefore, Corollary 5.4 gives an "almost excellent" UFD A. The obstacle in proving that A is excellent, of course, is that the fibers over the remaining height-one prime ideals may not be geometrically regular.

Suppose A is an excellent local UFD with completion T. We know several necessary conditions that elements in A must satisfy. For example, recall from Lemma 5.3 that prime elements in A must be radical in T. In fact, as we show in Proposition 5.6, all elements $q \in A$ satisfy $\sqrt{qT}$ is principal. Furthermore, in Proposition 5.6, we describe how elements in A factor in terms of radical irreducible elements of T.

Lemma 5.5 *Let $p, q \in T$ be regular elements such that pT and qT have no embedded associated prime ideals. Then pq is radical if and only if both p and q are radical and* $\operatorname{Ass}(T/pT) \cap \operatorname{Ass}(T/qT) = \varnothing$.

Proof Throughout the proof, we use the equivalent characterization of radical supplied by Lemma 5.2. Suppose pq is radical but p is not radical. Then $pT_P \neq PT_P$ for some $P \in \operatorname{Ass}(T/pT)$. If $q \in P$, then $pq \in P^2$, and so $pqT_P \neq PT_P$, a contradiction since pq is radical. If $q \notin P$, then $q/1$ is a unit in T_P, and so $pT_P = pqT_P = PT_P$. This is also a contradiction, since we chose P such that $pT_P \neq PT_P$. Therefore, p must be radical. By a similar argument, q must also be radical. Now suppose $\operatorname{Ass}(T/pT) \cap \operatorname{Ass}(T/qT) \neq \varnothing$. Then there exists P such that $p, q \in P$. It follows that $pq \in P^2$, implying that $pqT_P \neq PT_P$. This contradicts the fact that pq is radical.

Now suppose p and q are radical and $\operatorname{Ass}(T/pT) \cap \operatorname{Ass}(T/qT) = \varnothing$. Let $P \in \operatorname{Ass}(T/pqT)$. Then $P \in \operatorname{Ass}(T/pT) \cup \operatorname{Ass}(T/qT)$. The associated prime ideals of p and q are disjoint and so, without loss of generality, assume $p \in P$ and $q \notin P$. Then $q/1$ is a unit in T_P and, since p is radical, $pqT_P = pT_P = PT_P$. It follows from Lemma 5.2 that pq is radical. ■

Proposition 5.6 *Let A be an excellent UFD such that $\widehat{A} = T$. If $a \in A$, then a factors in T as $a = uc_1^{m_1} \cdots c_n^{m_n}$, where u is a unit in A, and the c_i are irreducible radical elements in T with* $\operatorname{Ass}(T/c_i T) \cap \operatorname{Ass}(T/c_j T) = \varnothing$ *for all $i \neq j$. In particular, elements of A have principal radicals in T.*

Proof Since A is a UFD, factor a in A as $a = ub_1^{m_1} \cdots b_n^{m_n}$, where u is a unit in A, and the b_i are distinct primes in A (and radical elements by Lemma 5.3). Since T is

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Noetherian, we can express each $b_i = b_{i1} \cdots b_{ip_i}$ as the product of irreducibles in T. Since the b_i are radical, the b_{ij} are all radical. Also, since the b_i have distinct associated primes, the b_{ij} have distinct associated primes. Thus,

$$a = u(b_{11} \cdots b_{1p_1})^{m_1} \cdots (b_{n1} \cdots b_{np_n})^{m_n},$$

and relabeling, we get the desired result. ■

We end with an example tying together our applications.

Example Let $T := \mathbb{C}[[x, y, z, w]]/(xy - zw)$. The elements $a_n = x^n - y$ are prime in T. Define $p_n := a_{2n}a_{2n+1}$. Then, using Lemmas 5.3 and 5.5, we apply Corollary 5.4 to yield an "almost excellent" UFD A containing an associate of each p_n.

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University of California, Berkeley, CA 94709, USA
e-mail: aboocher@math.berkeley.edu
mwdaub@math.berkeley.edu

University of Chicago, Chicago, IL 60637, USA
e-mail: rkj@uchicago.edu

Williams College, Williamstown, Massachusetts 01267, USA
e-mail: 08hml@williams.edu
sloepp@williams.edu
08paw@williams.edu