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POROSITY AND APPROXIMATE DERIVATIVES

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1. Introduction. In recent years, a considerable amount of research has been devoted to questions involving set porosity, particularly as it relates to differentiation theory. We may express the type of question in which we are interested by using the language of path derivatives and sequential derivatives. A *path derivative* of a function f is defined by writing

$$f'_{\mathbf{E}}(x) = \lim_{y \to x, y \in E_x} \frac{f(y) - f(x)}{y - x}$$

where at each point x a set E_x is given. A special case of the path derivative is the *sequential derivative*, defined by writing

$$f'_{\mathbf{h}}(x) = \lim_{n \to +\infty} \frac{f(x + h_n) - f(x)}{h_n}$$

where h_n is a fixed sequence of nonzero numbers converging to zero. Two natural questions arise in this setting:

(a) what information about the derivatives $f'_{\mathbf{E}}$ or $f'_{\mathbf{h}}$ on a set A implies that f is differentiable or approximately differentiable a.e. in A; and

(b) when such derivatives exist on a set A, on which the approximate derivative f'_{ap} also exists, what conditions will ensure that

$$f'_{\mathrm{ap}}(x) = f'_{\mathbf{E}}(x) \quad \text{or} \quad f'_{\mathrm{ap}}(x) = f'_{\mathbf{h}}(x)$$

a.e. in A?

In regard to these questions, there are a number of classical and recent works that show what information provides such implications. For question (a) the most important geometric conditions on the system of paths **E** that supply information about the differentiability or approximate differentiability properties of the function f, have been density conditions ([**1**], [**5**], [**11**]), intersection conditions ([**4**]), and porosity conditions ([**3**], [**10**], [**12**], [**22**]). In particular it has emerged that in many instances a porosity hypothesis may replace both an intersection condition and a density hypothesis. Several results in Khintchine [**10**] and [**12**] use porosity considerations in questions involving path derivatives, and

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Sindalovskii [**23**] employs similar notions, but in the special setting of sequential derivatives.

These early results of Khintchine [**11**] and related later work of Sindalovskii have been called to our attention by a recent private communication of Woś [**24**]. Dr. Woś points out that the proof in Khintchine's article is not clear and that some applications of this result that appear in Sindalovskii [**23**] may not be correct. There are in fact errors in the proof of Khintchine that invalidate the argument, and the theorem itself is false; also a number of the results asserted in the article of Sindalovskii are not quite correct. One of our secondary concerns in this article is to present a counterexample and an alternative version for the false Khintchine theorem, and to give the arguments to establish the right version of one of Sindalovskii's statements.

We begin by clarifying the Khintchine "theorem". We present, in Section 3, counterexamples to show that the hypotheses of that assertion do not provide any information about the approximate derivative. This shows that, contrary to the assertions in [**11**], porosity hypotheses do not suffice to draw such strong conclusions. In Section 4 we show that there is a correct theorem of this type using porosity, but one which requires an exceptional set of the first category rather than of measure zero. In Section 5 we provide a necessary and sufficient condition under which the result of Sindalovskii will hold; this can be considered a response to question (b) in the setting of sequential derivatives. In Section 6 we show that these conditions are not met in general (contradicting the claim in [**23**]). Finally in Section 7 we consider problem (b) in the larger setting of path derivatives.

2. Notation and terminology. Our results are formulated for real functions defined on the unit interval $[0, 1]$, unless stated otherwise. If A is a set of real numbers then $|A|$ will denote the exterior Lebesgue measure of the set A, and $\mathrm{cl}\, A$ will denote the closure of the set A. We shall also use the customary notations

$$A + h = \{x + h : x \in A\}$$

and

$$A + B = \{x + y : x \in A \text{ and } y \in B\}$$

where A and B are sets of real numbers and h is a real number.

(2.1) Let A be a measurable set and t a positive number. Then we shall write

$$U(A; t) = \{x + \tau : x \in A, |\tau| \leqq t\}.$$

We review next the definition of sequential derivation introduced in [**16**] and in [**13**], and the notion of path derivative introduced in [**4**].

(2.2) A *path leading to* x is a set $E_x \subset [0, 1]$ such that $x \in E_x$ and x is a point of accumulation of E_x.

(2.3) A *system of paths* is a collection

$$\mathbf{E} = \{E_x : x \in [0, 1]\}$$

where each set E_x is a path leading to x.

(2.4) For a function f defined on the interval $[0, 1]$ and a system of paths $\mathbf{E}$ we define the extreme $\mathbf{E}$-derivatives of f at the point x by the expressions

$$\overline{f'}_{\mathbf{E}}(x) = \limsup_{y \to x, y \in E_x} \frac{f(y) - f(x)}{y - x}$$

and

$$\underline{f'}_{\mathbf{E}}(x) = \liminf_{y \to x, y \in E_x} \frac{f(y) - f(x)}{y - x}.$$

Where these are equal we say that f is $\mathbf{E}$-differentiable and use the notation $f'_{\mathbf{E}}$ for this derivative function where it exists.

(2.5) Let $\mathbf{h} = \{h_n\}$ be a sequence of nonzero numbers converging to zero. Then we define the extreme $\mathbf{h}$-derivatives of f at an arbitrary point x by the expressions

$$\overline{f'}_{\mathbf{h}}(x) = \limsup_{n \to \infty} \frac{f(x + h_n) - f(x)}{h_n}$$

and

$$\underline{f'}_{\mathbf{h}}(x) = \liminf_{n \to \infty} \frac{f(x + h_n) - f(x)}{h_n},$$

with an appropriate convention to handle the endpoints. Where these are equal we say that f is $\mathbf{h}$-differentiable and use the notation $f'_{\mathbf{h}}$ for this derivative function where it exists.

Note that these sequential derivatives are precisely the path derivatives relative to the system of paths $\mathbf{E}$ for which each set E_x is given by

$$E_x = \{x + h_n : n = 1, 2, 3, \dots\}.$$

We shall need also the notion of set porosity introduced into analysis by numerous authors. The terminology is due to Dolženko [7] but the basic computations may be found in early writings of Denjoy, Khintchine and others.

(2.6) Let E be a set of real numbers, and a, b any distinct points on the real line. Then by $\lambda(E, a, b)$ we denote the length of the largest subinterval of (a, b) [or of (b, a) if $b < a$] that is disjoint from the set E.

(2.7) Let E be a set of real numbers, and x any point on the real line. Then we write

$$p^+(E; x) = \limsup_{h \to 0+} \frac{\lambda(E, x, x + h)}{h}$$

and

$$p^-(E; x) = \limsup_{h \to 0+} \frac{\lambda(E, x, x - h)}{h}$$

and refer to these numbers as the *right and left porosity* of the set E at the point x.

(2.8) A set E is said to be *porous* at a point x provided either

$$p^+(E; x) > 0 \quad \text{or} \quad p^-(E; x) > 0.$$

(2.9) A set E is said to be *nonporous* at a point x provided

$$p^+(E; x) = 0 \quad \text{and} \quad p^-(E; x) = 0.$$

(2.10) A set E is said to be *bilaterally strongly porous* at a point x provided

$$p^+(E; x) = 1 \quad \text{and} \quad p^-(E; x) = 1.$$

3. Khintchine's theorems. Let us begin by quoting an important theorem of Khintchine that is well-known and that is closely related to the statement in which we are interested. This is a celebrated result from his fundamental studies, [**11**] and [**12**], into the structure of measurable functions. It appears essentially, reproved, in [**1**], in [**5**], in [**8**], and in [**6**].

(3.1) THEOREM. (Khintchine) *Let f be a measurable function. Then at every point x, with the possible exception of a set of x of measure zero, one of the four conditions below must hold*:

(i) $S_x = \left\{y: \dfrac{f(y) - f(x)}{y - x} > 0\right\}$ *has density* 1 *at* x;

(ii) $T_x = \left\{y: \dfrac{f(y) - f(x)}{y - x} < 0\right\}$ *has density* 1 *at* x;

(iii) $U_x \left\{y: f(y) = f(x)\right\}$ *has density* 1 *at* x;

(iv) *both sets* S_x *and* T_x *have upper density* 1 *at* x *and* U_x *has measure zero.*

Moreover f has a finite approximate derivative at almost every point that is of type (i), (ii) *or* (iii).

In the language of Khintchine a function is "asymptotiquement determinée" (A.D.) at a point x if it is of type (i), (ii) or (iii) at x. Thus at almost every point x a measurable function is A.D. or else it is oscillatory in the sense (iv). A consequence of this is the remarkable fact that for a measurable function f there must be at almost every point x either a finite approximate derivative $f'_{\text{ap}}(x)$, or else $+\infty$ and $-\infty$ are essential derived numbers through sets of upper density 1 on both sides at x. This is easily obtained from the theorem by applying it to each of the functions $f_n(x) = f(x) + nx$ for integers n. We express this as a corollary to the theorem. This was first obtained by Denjoy in 1916, stated for continuous functions, but easily extended to arbitrary measurable functions by means of Lusin's theorem.

(3.2) COROLLARY. *Let f be a measurable function. Then at almost every point x either f has a finite approximate derivative or else for each integer n both sets*

$$\left\{y:\frac{f(y) - f(x)}{y - x} > n\right\} \quad \textit{and} \quad \left\{y:\frac{f(y) - f(x)}{y - x} < -n\right\}$$

have upper density 1 *on both sides at x.*

A further corollary is just a restatement of the theorem in the language of path derivatives. This appears, in nearly the same language, in [**15**, Theorem 10.1, p. 295].

(3.3) COROLLARY. *Let f be a measurable function and let*

$$\mathbf{E} = \{E_x:x \in \mathbf{R}\}$$

be a system of paths such that each set E_x has lower interior density positive on one side at least at x. Then if either

$$\underline{f}'_{\mathbf{E}}(x) > -\infty \quad \textit{or} \quad \overline{f}'_{\mathbf{E}}(x) < +\infty$$

at every point x of a set X then necessarily f is approximately differentiable almost everywhere in X.

A very similar assertion to Theorem (3.1), but for continuous functions, appears in [**11**] and is quoted in [**12**, p. 227]. We present this statement as (*) below. Note that it is only in the statement of fourth condition (iv) that the conclusion differs; here the conclusion reads that a continuous function is, at almost every point x, either A.D. or else oscillatory in a strong porosity sense.

(*) Let f be a continuous function. Then at every point x, with the possible exception of a set of x of measure zero, either

(i) $S_x = \left\{y:\dfrac{f(y) - f(x)}{y - x} > 0\right\}$ has density 1 at x, or

(ii) $T_x = \left\{ y: \dfrac{f(y) - f(x)}{y - x} < 0 \right\}$ has density 1 at x, or

(iii) $U_x = \{ y: f(y) = f(x) \}$ has density 1 at x, or finally

(iv) both sets

$$V_x = \left\{ y: \frac{f(y) - f(x)}{y - x} \leqq 0 \right\} \quad \text{and}$$

$$W_x = \left\{ y: \frac{f(y) - f(x)}{y - x} \geqq 0 \right\}$$

have porosity 1 on both sides at x and U_x has measure zero.

If true, this would be a remarkable result, allowing for example a considerable weakening of the hypotheses in (3.3) above. It is likely that Khintchine was aware that the statement was false. In [**12**, p. 227] he uses this result to obtain a proof of a theorem on Borel derivatives; but then in a postscript, [**12**, pp. 276-279], he claims that the proof is incomplete and provides a completely different proof that does not depend on (*). Indeed, since (*) is little mentioned in the literature (it does not, for example, appear in [**15**], although the main theorem of Khintchine above does), it may have been thought by some analysts of the time to have been false.

We demonstrate that (*) is false by proving the following theorem.

(3.4) THEOREM. *There exists a continuous function f defined on the interval* $[0, 1]$, *and a set $P \subset [0, 1]$ of positive measure such that f does not have an approximate derivative at any point of P and, for each $x \in P$, the associated sets*

$$\{ y: f(y) > f(x) \} \quad \textit{and} \quad \{ y: f(y) < f(x) \}$$

are nonporous at x.

Proof. Let $Q \subset [0, 1]$ be a nowhere dense perfect set of positive measure. Let $\{I_k: k = 1, 2, \dots\}$ be an enumeration of those intervals $((i - 1)/n, i/n)$, for positive integers i and n, which have nonempty intersections with Q. We can select, inductively, a sequence $\{J_k: k = 1, 2, \dots\}$ of distinct intervals contiguous to Q, such that $J_k \subset I_k$ for every k.

Let g be a continuous function on $[0, 1]$ that is nowhere approximately differentiable in $[0, 1]$. (For the existence of such functions see, for example, [**9**].) If I_k is the interval $((i - 1)/n, i/n)$ then we put

$$m_k = \min \left\{ g(x): x \in \left[\frac{i - 1}{n} - \frac{1}{\sqrt{n}}, \frac{i}{n} + \frac{1}{\sqrt{n}} \right] \right\}$$

and

$$M_k = \max\left\{g(x):x \in \left[\frac{i-1}{n} - \frac{1}{\sqrt{n}}, \frac{i}{n} + \frac{1}{\sqrt{n}}\right]\right\}.$$

For every $k = 1, 2, \ldots$ we define a function f on J_k by requiring that f is continuous on cl J_k, agrees with g at the endpoints of J_k and

$$\min\{f(x):x \in J_k\} = m_k - \frac{1}{k},$$

and

$$\max\{f(x):x \in J_k\} = M_k + \frac{1}{k}.$$

We define

$$f(x) = g(x) \quad \text{for} \quad x \in [0, 1]\setminus \bigcup_{k=1}^{\infty} J_k.$$

Since $M_k - m_k \to 0$ as $k \to \infty$, f is continuous on [0, 1].

Let P denote the set of density points of Q. Obviously f has no approximate derivative at any point of P. It remains to show that the two associated sets are nonporous at each point of P. It is sufficient to prove that, for a fixed $x \in P$, the set

$$Y = \{y:f(y) > f(x)\}$$

is nonporous on the right at x. If this is not the case, then there must exist a positive number ϵ and a sequence of intervals

$$\{L_m\} = \{(a_m, b_m)\} \subset [0, 1]$$

converging to x on the right, with

$$(1) \qquad b_m - a_m > \epsilon(b_m - x),$$

and

$$(2) \qquad f(y) \leqq f(x) \quad \text{for each } y \in L_m.$$

For every $m = 1, 2, \ldots$ we can choose positive integers n_m and i_m such that

$$(3) \qquad \frac{2}{b_m - a_m} < n_m < \frac{3}{b_m - a_m},$$

and

$$(4) \qquad a_m < \frac{i_m - 1}{n_m} < \frac{i_m}{n_m} < b_m.$$

Then, by (1), (3), and (4) we have

(5) $$\frac{1}{n_m} > \frac{\epsilon}{3}\left[\frac{i_m - 1}{n_m} - x\right].$$

This implies that

(6) $$\frac{i_m - 1}{n_m} - \frac{1}{\sqrt{n_m}} < x \quad \text{for } m \text{ sufficiently large.}$$

Since x is a density point of Q, (5) requires that, for sufficiently large m,

$$\left[\frac{i_m - 1}{n_m}, \frac{i_m}{n_m}\right] \cap Q \neq \emptyset.$$

Thus there is an integer m_0 so that for $m \geqq m_0$ there is a k_m such that

$$I_{k_m} = \left(\frac{i_m - 1}{n_m}, \frac{i_m}{n_m}\right).$$

Taking (6) into consideration, we conclude that

$$\max\{f(y):y \in J_{k_m}\} = M_{k_m} + \frac{1}{k_m} > M_{k_m} \geqq f(x).$$

Since $J_{k_m} \subset I_{k_m} \subset L_m$, this contradicts (2) and the proof is complete.

As a consequence of this theorem we may conclude that the existence of a path derivative, even for a nonporous system, need not imply the existence of the approximate derivative. This shows that porosity conditions may not substitute for intersection conditions (**[4]**), or for density conditions (**[1]**, **[5]**, **[12]**) in theorems of this type.

(3.5) COROLLARY. *There exists a continuous function f defined on the interval* $[0, 1]$, *a set* $P \subset [0, 1]$ *of positive measure, and a system of paths* $\mathbf{E} = \{E_x:x \in P\}$, *each set E_x is nonporous at x* ($x \in P$), *so that the derivative $f'_{\mathbf{E}}(x)$ exists and vanishes at every point $x \in P$, and yet f does not have an approximate derivative at any point of P.*

Proof. We let P and f be as in the proof of the theorem. Then we set

$$E_x = \{y:f(y) = f(x)\},$$

and it is easy to verify that the assertion of the lemma is now valid.

We have seen that certain information about the nonporous derivatives or derivatives may give no information about the existence of an approximate derivative. Let us now show that even when the approximate derivative exists, there need be little connection between it and these derivatives. We can arrange for the path derivative $f'_{\mathbf{E}}$ to exist, to be Baire 1, and to differ from the approximate derivative on a set of positive measure, even for a system of paths that is nonporous on both sides. Again we see that a porosity condition may not substitute for either an intersection condition or a density condition.

(3.6) THEOREM. *Let P be a nowhere dense perfect subset of the interval* $[0, 1]$ *and which has positive measure. Then there exists a continuous function* f *on* $[0, 1]$ *and a nonporous system of paths* $\mathbf{E} = \{E_x : x \in [0, 1]\}$ *such that* f *is* $\mathbf{E}$*-differentiable everywhere in* $[0, 1]$, *the derivative* $f'_{\mathbf{E}}$ *is in the first class of Baire,* f *is continuously differentiable on* $[0, 1]\setminus P$, f *is approximately differentiable a.e. in* P *and yet*

$$|\{x \in P : f'_{\mathrm{ap}}(x) \neq f'_{\mathbf{E}}(x)\}| > 0.$$

Proof. Let $I_1, I_2, I_3, \dots$ be the sequence of intervals contiguous to P in $[0, 1]$. Let us write

$$\epsilon_n = \min[\,|I_i| : 1 \leqq i \leqq n],$$

$$P_0 = [0, 1],$$

$$P_n = [0, 1]\setminus \bigcup_{i=1}^{n} I_i,$$

and

$$\|P_n\| = \text{length of largest component interval of } P_n.$$

Since P is nowhere dense it is clear that $\|P_n\| \to 0$, so that we may define an increasing sequence of integers $\{n_k : k = 0, 1, 2, \dots\}$ in such a way that

$$\|P_{n_k}\| < \frac{\epsilon n_{k-1}}{k^2} \quad k = 1, 2, \dots .$$

We now define our function f on the interval $[0, 1]$ so that each of the following is true:

(i) f is continuous on $[0, 1]$,

(ii) f vanishes on P,

(iii) f is continuously differentiable in each I_i,

(iv) f vanishes on all of I_i except a middle portion that is of length $|I_i|\setminus i$, and

(v) if $n_k < i \leqq n_{k+1}$ and $[a_i, b_i]$ denotes that middle portion of I_i then

$$f(a_i) = f(b_i) = 0,$$

$$f\left(a_i + \frac{b_i - a_i}{3}\right) = \|P_{n_{k-1}}\|,$$

$$f\left(a_i + \frac{2(b_i - a_i)}{3}\right) = -\|P_{n_{k-1}}\|.$$

(We wish, in example (7.7), to use this same construction but with a nearly trivial modification. Let us note here that, for the purposes of that example, the function f will be required to be linear on each of the intervals

$$\left[a_i, a_i + \frac{b_i - a_i}{3}\right], \left[a_i + \frac{b_i - a_i}{3}, a_i + \frac{2(b_i - a_i)}{3}\right]$$

and

$$\left[a_i + 2\frac{b_i - a_i}{3}, b_i\right].$$

Of course then we shall not require that f satisfy (iii) above.)

Choose a perfect subset $Q \subset P$ such that P has density 1 at each point of Q and such that Q has positive measure. We construct a system of paths $\mathbf{E} = \{E_x : x \in [0, 1]\}$ in the following manner. We define

$$E_x = \{t : f(t) = t - x\} \quad (x \in Q),$$

$$E_x = \{t : f(t) = 0\} \quad (x \in P \setminus Q),$$

and

$$E_x = [0, 1] \quad (x \in [0, 1] \setminus P).$$

All of the conditions of the theorem are now easy to verify with the exception of the fact that each path E_x is nonporous on both sides at x. Let us compute the porosity $p^+(E_x, x)$ for each x; similar arguments may be applied to obtain the left porosity.

For points x in $[0, 1)\setminus P$ it is trivially true that $p^+(E_x, x) = 0$. For points $x \in P\setminus Q$ it is also immediate that $p^+(E_x, x) = 0$ since the points at which f is nonzero form only a tiny fraction of the complementary intervals $\{I_i\}$. Finally then let us consider a point $x \in Q$.

For such a point x we may compute that

$$(7) \qquad p^+(E_x, x) = \limsup_{y \to x^+, y \in P} \frac{\lambda(E_x, x, y)}{y - x}$$

since any such point x is a point of nonporosity of P. So let us estimate the size of $\lambda(E_x, x, y)$ for x, y in P and x in Q. If (a, b) is the largest subinterval of (x, y) that is disjoint from the set E_x then, because f is continuous, either $f(t) > t - x$ everywhere on (a, b), or $f(t) < t - x$ everywhere on (a, b).

Consider the former situation. For this to be the case (a, b) must be a subinterval of some interval I_i; but then

$$\frac{\lambda(E_x, x, y)}{y - x} \leqq \frac{\lambda(P, x, y)}{y - x},$$

and, since x is a point of nonporosity of P, this is arbitrarily small for y sufficiently close to x. Alternatively let us consider the latter situation. For that we may choose an integer k in such a way that x and y belong to the same component of $P_{n_{k-1}}$, but do not belong to the same component of P_{n_k}; this means that

(8) $\quad \epsilon n_k \leqq y - x \leqq ||P_{n_{k-1}}||$

since there is some interval $I_i \subset (x, y)$ with $n_{k-1} < i \leqq n_k$. Now the interval (a, b) in this case can contain no interval I_i on which the function has values as high as $||P_{n_{k-1}}||$, and so any interval $I_i \subset (a, b)$ must have an index $i > n_{k+1}$. Thus a and b belong to the same component of $P_{n_{k+1}}$ and this gives

(9) $\quad b - a \leqq ||P_{n_{k+1}}||.$

Consequently by (7), (8), and (9),

$$\frac{\lambda(E_x, x, y)}{y - x} \leqq \frac{b - a}{y - x} \leqq \frac{||P_{n_{k+1}}||}{\epsilon n_k} \leqq \frac{1}{(k + 1)^2}.$$

Obviously, if $y \to x$ then $k \to \infty$ and hence $p^+(E_x, x) = 0$ for any $x \in Q$. This completes the proof.

(3.7) Remark. Note that in the construction one may arrange to take the set Q to be nonmeasurable rather than closed, so that the derivative function $f'_{\mathbf{E}}$ is then nonmeasurable. From this we see that a path derivative of a continuous function need not belong to any Borel class, nor be measurable, even under the hypothesis that the system $\mathbf{E}$ is nonporous.

4. A category analogue of Khintchine's theorem. In order to complete the considerations of the preceding section let us state and prove a category analogue of Theorem (3.1). The above results concern the structure of a function with the exception of a set of measure zero; a similar structure theorem is available but with the exception of a set of the first category. Note that this assertion is analogous to the statement of Khintchine's theorem and closely related to the false assertion (*).

(4.1) THEOREM. *Let f be a continuous function. Then at every point x, with the exception only of a set of x of the first category, either*

(i) $S_x = \left\{ y{:}y = x \text{ or } \dfrac{f(y) - f(x)}{y - x} \geqq 0 \right\}$

is a neighborhood of x, or

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(ii) $T_x = \left\{ y : y = x \text{ or } \dfrac{f(y) - f(x)}{y - x} \leqq 0 \right\}$

is a neighborhood of x, *or*

(iii) *both sets* S_x *and* T_x *have porosity* 1 *on both sides at* x.

Moreover f *has a finite derivative at almost every point that is of type* (i), *or* (ii).

Proof. Let X denote the set of points x at which (i), (ii), or (iii) fail. If this set is not first category then there must be numbers $0 < p < 1$, $\delta > 0$, a, b $(0 < b - a < \delta)$ and a set $X_1 \subset X$ with X_1 dense in the interval (a, b) and

$$\lambda(S_x, x, x + t) < pt \quad \text{for } 0 < t < \delta \text{ and } x \in X_1$$

(or the same assertion for T_x or for left rather than right; we argue just for this particular situation).

Since f is not increasing on (a, b), there are points $a < a_1 < b_1 < b$ with $f(b_1) < f(a_1)$ and a point

$$c = \sup\{t \in [a_1, b_1) : f(t) \geqq f(a_1)\}.$$

Since X_1 is dense in (a, b), we may select a sequence of points $x_n \in X_1$ so that $x_n \to c$. Using the above porosity estimate there are, consequently, points $y_n \in S_{x_n}$ such that

$$f(y_n) \geqq f(x_n) \quad \text{and} \quad b_1 - p(b_1 - x_n) < y_n < b_1.$$

Since f is continuous, this will give a point y,

$$b_1 - p(b_1 - c) \leqq y \leqq b_1$$

at which $f(y) \geqq f(c) \geqq f(a_1)$, contradicting the definition of c. From this contradiction we obtain the fact that the set X is first category as required. Finally it is easy to show that a function that satisfies (i) or (ii) everywhere on a set Y is VBG* on Y and from that fact the final assertion of the theorem follows.

In contrast to the situation in the preceding section, the estimates on the path derivates for a system that satisfies a weak porosity condition, can be used to give information on the ordinary derivates of a function outside of the exceptional set of the first category. For completeness let us give first the measure-theoretic version, which may be considered merely an alternative version of the Theorem (3.1) of the preceding section. The category analogue then follows.

(4.2) THEOREM. *Let* f *be a measurable function and let*

$$\mathbf{E} = \{E_x : x \in \mathbf{R}\}$$

be a system of paths such that each set E_x has lower interior density positive on one side at least at x. Then if

$$\underline{f}'_{\mathbf{E}}(x) > s$$

at every point x of a set X, then there is a denumerable partition

$$X = N \cup \bigcup_{i=1}^{\infty} X_i$$

of the set X such that N has measure zero and the function $f_s(x) = f(x) - sx$ is increasing on each set X_n.

Proof. The proof follows routine arguments and we shall omit it. Note that essentially this theorem is just an alternative version of the Denjoy-Khintchine theorem referred to earlier.

(4.3) THEOREM. *Let f be a continuous function and let*

$$\mathbf{E} = \{E_x : x \in \mathbf{R}\}$$

be a system of paths such that each set E_x has porosity less than 1 *on one side at least at x. Then if*

$$\underline{f}'_{\mathbf{E}}(x) > s$$

at every point x of a set X, then there is a denumerable partition

$$X = N \cup \bigcup_{i=1}^{\infty} X_i$$

of the set X such that N is first category in **R** *and the function $f_s(x) = f(x) - sx$ is increasing on each set X_n.*

Proof. The assertion is an easy consequence of Theorem (4.1). Note that the conclusion of the theorem really is meant to assert that, since the function f is continuous, f_s is increasing on any interval within which some set X_n is dense.

This theorem allows us to give a statement analogous to (3.3) of the preceding section. A continuous function that has a finite path derivate on an interval relative to a system of paths satisfying a weak porosity condition, is differentiable on a substantial set. This feature of derivatives has been proved earlier in [**3**].

(4.4) THEOREM. *Let f be a continuous function and let*

$$\mathbf{E} = \{E_x : x \in \mathbf{R}\}$$

be a system of paths such that each set E_x has porosity less than 1 *on one side at least at x. Then if either*

$$\underline{f}'_{\mathbf{E}}(x) > -\infty \quad or \quad \overline{f}'_{\mathbf{E}}(x) < +\infty$$

at every point x of an interval $[a, b]$ then necessarily f is differentiable almost everywhere on an open dense set in $[a, b]$.

Proof. This is a consequence of Theorem (4.3).

5. Sindalovskii's theorems. In the article [**23**] Sindalovskii states the following two results, that appear as [**23**, Theorem 1, p. 945] and [**23**, Lemma 3, pp. 953-958] respectively. Note that the property (D) of the first assertion is, for decreasing sequences, equivalent to the assertion that the range of the sequence is not strongly porous on the right at 0.

(**) Let $\mathbf{h} = \{h_n\}$ be a sequence of positive numbers converging to zero and for which the following property (labelled as property D) holds

$$\text{(D)} \qquad \liminf_{n\to+\infty} \frac{h_n+1}{h_n} > 0.$$

Then if f is a continuous function on the interval [0, 1] and

$$-\infty < \underline{f}'_{\mathbf{h}}(x) \leqq \overline{f}'_{\mathbf{h}}(x) < +\infty$$

at every point x of a set A then necessarily

$$\underline{f}'_{\mathbf{h}}(x) = \overline{f}'_{\mathbf{h}}(x)$$

at almost every point of A.

(***) Let $\mathbf{h} = \{h_n\}$ be an arbitrary sequence of positive numbers converging to zero. If f is a measurable function on the interval [0,1], which has a finite approximate derivative everywhere on a set A, and which has

$$\overline{f}'_{\mathbf{h}}(x) < +\infty$$

at every point x of the set A, then necessarily

$$f'_{\text{ap}}(x) = \overline{f}'_{\mathbf{h}}(x)$$

at almost every point of A.

The validity of both of these results is in question. An error appears in the proof of (***); (**) would then follow as an application of (***) and (*). Since neither of these statements is true, (**) remains unproved; we do not know whether the statement itself is true. In this section we present an analysis of the statement (***) that will show exactly what condition is needed in order that it becomes valid. This analysis arises directly from a study of the arguments used by Sindalovskii. The property that we require is a rather technical property of a sequence that appears in the arguments given for [**23**, Lemma 3].

(5.1) *Definition.* A decreasing sequence $\mathbf{h} = \{h_n\}$ of positive real numbers converging to zero will be said to satisfy the property (S) if the following is true. Whenever $P, P_1, P_2, P_3, P_4, \ldots$ are closed sets in the interval [0, 1] such that

(a) every point $x \in P$ belongs to infinitely many of the sets P_i;

(b) $|P| > 0$;

(c) $P \cap [P_i + h_i] = \emptyset$ for each index i,

then necessarily for every positive number C there are indices i and j with $i < j$, $h_i > Ch_j$ and

$$[P_i + h_i] \cap [P + h_j] \neq \emptyset.$$

The arguments that Sindalovskii uses in attempting to establish (***) may be divided into two parts. He first shows that for any sequence $\mathbf{h}$ that has the property (S) the assertion (***) is valid. This is true and we shall reproduce this in the proof of Theorem (5.4) below. Indeed we shall show that this property (S) is both necessary and sufficient in order that this conclusion be valid. He then argues that every sequence must have this property (S). This is not correct; in fact in the next section, Theorem (6.1), we shall show that there exist sequences, even "arbitrarily slow" sequences, that fail to have the property (S).

Let us begin by stating and proving a simple lemma that we shall require in the proof of (5.4) and several later results.

(5.2) LEMMA. *Let A be a measurable set and let $\{h_n\}$ be a sequence of numbers converging to zero. Then for almost every $x \in A$ there are infinitely many indices n for which*

$$x + h_n \in A.$$

Proof. We can suppose that A is bounded. Let $\epsilon > 0$. Then there is a finite sequence of closed intervals $\{ [a_i, b_i] \}$ so that

$$\left| A \,\Delta \bigcup_{i=1}^{m} [a_i, b_i] \right| < \epsilon.$$

Let us write

$$A_n = \{x \in A : x + h_n \in A\}.$$

We show that $|A_n| \to |A|$ and the result evidently follows. If $x \in A$ and $0 < h_n < t$ but $x + h_n$ is not in A then either $x + h_n$ is in an interval $[b_i, b_i + t]$ or else x or $x + h_n$ is in the above small set difference. For sufficiently small t the set of such points has small measure and this supplies the proof.

We need the following property of approximate derivatives. It may be obtained as an application of a well known theorem of Whitney [**25**], or a theorem of O'Malley [**14**], or from the proof of Theorem (10.8), Chapter VII in [**15**].

(5.3) Let f be a measurable function that has an approximate derivative everywhere on a measurable set A of positive measure. Then there is a measurable subset A_1 of A, with $|A_1| > 0$, and such that

$$\lim_{y\to x, y\in A_1} \frac{f(y) - f(x)}{y - x} = f'_{\mathrm{ap}}(x)$$

for every point x in A_1.

(5.4) THEOREM. *For every decreasing sequence* $\mathbf{h} = \{h_n\}$ *of positive numbers converging to zero the following three assertions are equivalent.*

(i) *If f is a continuous function on the interval* $[0, 1]$ *such that everywhere on a measurable set A the approximate derivative* $f'_{\mathrm{ap}}(x)$ *exists and* $\overline{f}'_{\mathbf{h}}(x) < +\infty$ *then*

$$f'_{\mathrm{ap}}(x) = \overline{f}'_{\mathbf{h}}(x) \text{ a.e. } on\ A.$$

(ii) *If f is a measurable function on the interval* $[0, 1]$ *such that everywhere on a measurable set A the approximate derivative* $f'_{\mathrm{ap}}(x)$ *exists and* $\overline{f}'_{\mathbf{h}}(x) < +\infty$ *then*

$$f'_{\mathrm{ap}}(x) = \overline{f}'_{\mathbf{h}}(x) \text{ a.e. } on\ A.$$

(iii) *The sequence* $\mathbf{h}$ *has property* (S).

Proof. Let us begin by proving that (iii) implies (ii). Suppose that the sequence $\mathbf{h}$ has the property (S) and let f be a measurable function on the interval $[0, 1]$ such that

$$(10) \qquad f'_{\mathrm{ap}}(x) \neq \overline{f}'_{\mathbf{h}}(x) < +\infty$$

everywhere on a measurable set A_1 of positive measure. From this we will obtain a contradiction, thus proving the implication. By (5.3) there must be a measurable subset A_2 of A_1 that also has positive measure and for which

$$\lim_{y\to x, y\in A_2} \frac{f(y) - f(x)}{y - x} = f'_{\mathrm{ap}}(x)$$

at every point x in A_2. For almost every $x \in A_2$ one has, by (5.2),

$$x + h_n \in A_2$$

for infinitely many indices n and thus

$$f'_{\mathrm{ap}}(x) \leqq \overline{f}'_{\mathbf{h}}(x) \text{ a.e. on } A_2.$$

Hence there are real numbers p, q and r, with $p < q < r$ and a further measurable subset A_3 of A_2, that again has positive measure, and for which

$$(11) \qquad f'_{\mathrm{ap}}(x) < p < q < \overline{f}'_{\mathbf{h}}(x) < r$$

at every point x of A_3.

From this in turn, using (11), there must be a further measurable subset A_4 of A_3, again with positive measure, and an integer N so that

$$(12) \qquad \frac{f(x + h_n) - f(x)}{h_n} < r \quad \text{for } x \in A, \text{ and } n \geqq N.$$

Yet again, from the inequality (11) above, there must be a measurable subset A_5 of A_4, again with positive measure so that

$$(13) \qquad \frac{f(y) - f(x)}{y - x} < p \quad \text{for every } x, y \in A_5.$$

By Lusin's theorem, we can find a closed set of positive measure $P \subset A_5$ such that $f(x + h_i)$ is continuous on P for every $i = 1, 2, \ldots$. We obtain our contradiction by applying the property (S) to this set P and the sequence of sets $\{P_i : i = N, N + 1, \ldots\}$ defined by writing

$$P_i = \left\{x \in P : \frac{f(x + h_i) - f(x)}{h_i} \geqq q\right\} \quad (i \geqq N).$$

Note that each set P_i is a closed subset of the interval [0, 1]. Now we verify conditions (a), (b), and (c) of definition (5.1). Condition (b) holds by the choice of P and (a) readily follows from the fact that

$$\overline{f}'_{\mathbf{h}}(x) > q \quad (x \in P).$$

Finally condition (c) holds since if x is any point in both of the sets P and $P_i + h_i$ for some $i \geqq N$ then $x = y + h_i$ for some $y \in P_i$, which will require, because of (13), that

$$q < \frac{f(y + h_i) - f(y)}{h_i} = \frac{f(y) - f(x)}{y - x} < p$$

which is certainly impossible.

Accordingly, since the sequence $\mathbf{h}$ has the property (S), then, by definition, for every positive number C there are indices i and j with $N \leqq i < j$, $h_i > Ch_j$ and

$$(14) \qquad [P_i + h_i] \cap [P + h_j] \neq \emptyset.$$

We will show that this cannot happen for $C \geqq (r - p)/(q - p)$.

Choose a point y in the intersection (14) above, and set $x = y - h_i$. This gives $x \in P_i$, and $y - h_j \in P$ and therefore from the estimates (12) and (13) above, and from the definition of the sets P_i, we must have

$$\frac{f(y) - f(x)}{h_i} \geqq q,$$

$$\frac{f(y) - f(y - h_j)}{h_j} < r,$$

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and

$$\frac{f(y - h_j) - f(x)}{h_i - h_j} < p.$$

Putting these inequalities together in the obvious way, we obtain

$$\begin{aligned} qh_i &\leqq f(y) - f(x) \\ &= [f(y) - f(y - h_j)] \\ &+ [f(y - h_j) - f(x)] \\ &< h_j r + p(h_i - h_j) < \frac{r - p}{C} h_i + ph_i = h_i \left[\frac{r - p}{C} + p \right] \end{aligned}$$

so that

$$C < \frac{r - p}{q - p}.$$

As this is a contradiction we obtain the proof of the required implication.

The implication (ii) $\rightarrow$ (i) is trivial. Thus it remains to establish the implication (i) $\rightarrow$ (iii), and the theorem is proved. For this let us suppose that $\mathbf{h}$ is a decreasing sequence for which the property (S) fails; we shall construct a continuous function f for which (i) fails, and this gives our result.

There must be closed subsets $P, P_1, P_2, \ldots$ of the interval $[0, 1]$ for which the assertions (a), (b), and (c) of definition (5.1) each hold for this sequence $\mathbf{h}$ and yet there is a number $C > 1$ such that whenever

$$i < j \quad \text{and} \quad h_i > Ch_j$$

then

$$(15) \qquad [P_i + h_i] \cap [P + h_j] = \emptyset.$$

We may suppose that each set $P_i \subset P$, and, to simplify the arithmetic, that $h_n < C^{-1}$ for all n.

Define the sets

$$Q_k = \cup[P_i + h_i : C^{-k-1} \leqq h_i < C^{-k}] \quad (k = 1, 2, \ldots),$$
$$A_k = \cup[P + h_i : C^{-k-1} \leqq h_i < C^{-k}] \quad (k = 1, 2, \ldots).$$

Note that each set Q_k is closed, that (because of (c))

$$Q_k \cap P = \emptyset \text{ for all indices } k,$$

and that (because of (15)),

$$Q_{k_1} \cap Q_{k_2} = \emptyset \text{ for all indices } k_1, k_2 \text{ with } |k_1 - k_2| > 1.$$

Also, every x not in P belongs to finitely many A_k.

Let us define functions ψ_1 and ψ_2 at every point x of the closed set

$$Q = P \cup \bigcup_{i=1}^{\infty} (P + h_i)$$

as follows:

$$\psi_1(x) = \begin{cases} C^{-k}; & x \in Q_k \setminus Q_{k-1} \\ 0 \quad ; & x \in Q \setminus \bigcup_1^{\infty} Q_k \end{cases}$$

and

$$\psi_2(x) = \begin{cases} 0 \quad ; & x \in P \\ C^{-k+1}; & x \notin P, x \in A_k \setminus \bigcup_{j>k} A_j. \end{cases}$$

We may verify that $0 \leqq \psi_1 \leqq \psi_2$ everywhere on Q, that ψ_1 is upper semi-continuous on Q, and that ψ_2 is lower semi-continuous there. The first part of this presents no difficulties, using (15).

To see that ψ_2 is lower semi-continuous on Q, let $x \in Q$ be arbitrary. If $\psi_2(x) = 0$, there is nothing to prove. If $\psi_2(x) > 0$ then $x \notin P$, $x \in A_{k(x)}$ and

$$\psi_2(x) = C^{-k(x)+1}.$$

It is obvious that the set

$$P \cup \bigcup_{i=k(x)+1}^{\infty} A_i$$

is closed and does not contain x. Hence x is a local minimum of ψ_2 and, a fortiori, ψ_2 is lower semi-continuous at x.

Because of these properties of ψ_1 and ψ_2 there must exist a continuous function f defined on the entire interval $[0, 1]$ and such that $\psi_1 \leqq f \leqq \psi_2$ on the set Q. We check that the function f has the following properties:

(16) $\quad f(x) = 0 \quad \text{if } x \in P,$

since $\psi_2(x) = \psi_1(x) = 0$ for such points;

(17) $\quad f(x) \geqq C^{-k-1} \quad \text{if } x \in Q_k,$

since such a point x may belong possibly to one of the sets Q_{k-1} or Q_{k+1} but to no other set Q_i and consequently $\psi_1(x) \geqq C^{-k-1}$; and finally

(18) $\quad 0 \leqq f(x) \leqq C^{-k+1} \quad \text{if } x \in A_k$

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since for any point x we have $k(x) \geqq k$ and hence

$$\psi_2(x) \leqq C^{-k+1}.$$

Let x be an arbitrary point of the set P. If i and k are indices for which

$$C^{-k-1} \leqq h_i < C^{-k}$$

then from the observations (16) and (18) we must have invariably

$$(19) \qquad 0 \leqq \frac{f(x + h_i) - f(x)}{h_i} = \frac{f(x + h_i)}{h_i} \leqq \frac{C^{-k+1}}{h_i} \leqq C^2.$$

On the other hand if such a point x also belongs to the set P_i so that $x + h_i$ is in Q_k then, by (16) and (17), we have

$$(20) \qquad \frac{f(x + h_i) - f(x)}{h_i} = \frac{f(x + h_i)}{h_i} \geqq \frac{C^{-k-1}}{h_i} \geqq C^{-1}.$$

But for each point x in P, $x \in P_i$ for infinitely many indices i, so that the inequalities (19) and (20) together show that

$$C^{-1} \leqq \limsup_{i\to\infty} \frac{f(x + h_i) - f(x)}{h_i} \leqq C^2.$$

Expressing this in the language of the **h**-derivatives, we have shown that

$$C^{-1} \leqq \bar{f}'_{\mathbf{h}}(x) \leqq C^2 < +\infty \quad \text{for } x \in P.$$

But we have supposed that P has positive measure, and f vanishes on P so that a.e. on P the approximate derivative $f'_{\mathrm{ap}}(x) = 0$ and this gives

$$0 = f'_{\mathrm{ap}}(x) < \bar{f}'_{\mathbf{h}}(x) < +\infty \quad \text{a.e. on } P.$$

Thus (i) of the theorem does indeed fail, and the proof is complete.

This theorem gives precisely the condition on a sequence **h** (albeit a technical condition) for which the assertion (***) of Sindalovskii is valid. We point out now some situations in which this property (S) may be verified. Later, in Section 6, we shall prove that not all sequences have this property. It should be remarked that Sindalovskii, in the proof of [**23**, Lemma 3], uses the hypothesis of this next lemma but incorrectly assumes that every sequence has this property.

(5.5) LEMMA. *Let* $\mathbf{h} = \{h_n\}$ *be a decreasing sequence of positive numbers converging to zero. If* $h_n/h_k \to +\infty$ *as* $k - n \to +\infty$ *then* **h** *has the property* (S).

Proof. Suppose that $P, P_1, P_2, \ldots$ satisfy (a), (b), and (c) of (5.1) and let $C > 0$ be given. Then by our assumption, there is a $K > 0$ such that

$$j - i \geqq K \quad \text{implies} \quad h_i > Ch_j.$$

Condition (a) implies

$$\sum_{i=1}^{\infty} |P_i \cap P| = \infty.$$

Since the set

$$\bigcup_{i=1}^{\infty} [(P_i \cap P) + h_i]$$

is bounded, this implies that there is a point x which belongs to more than K of the sets $(P_i \cap P) + h_i$. Thus there are indices i and j with $i < j, j \geqq i + K$ and

$$[(P_i \cap P) + h_i] \cap [(P_j \cap P) + h_j] \neq \emptyset.$$

This proves the assertion.

(5.6) COROLLARY. *Let* $\mathbf{h} = \{h_n\}$ *be a sequence of positive numbers such that*

$$\limsup \frac{h_{n+1}}{h_n} < 1$$

then $\mathbf{h}$ *has the property* (S).

Proof. It is an elementary exercise to show that a sequence that has the stated property satisfies the hypotheses of the lemma. Note that, for decreasing sequences, this condition requires the range of the sequence to be porous on the right at 0.

As an application of the preceding results, we may state a corrected, but weak, version of Sindalovskii's assertion (**).

(5.7) COROLLARY. *Let* $\mathbf{h} = \{h_n\}$ *be a sequence of positive numbers such that*

$$\limsup \frac{h_{n+1}}{h_n} < 1.$$

Then if f is a measurable function on the interval [0, 1] *such that*

$$-\infty < \underline{f}'_{\mathbf{h}}(x) \leqq \overline{f}'_{\mathbf{h}}(x) < +\infty$$

everywhere on a set A, and f is approximately differentiable a.e. *on A, then* a.e. *on the set A,*

$$\underline{f}'_{\mathbf{h}}(x) = \overline{f}'_{\mathbf{h}}(x) = f'_{\mathrm{ap}}(x).$$

The hypothesis that f is approximately differentiable is unfortunate, but we do not know at present whether it may be replaced by giving further restrictions on the sequence $\mathbf{h}$. There remain a number of problems we leave unanswered in connection with these investigations. These are posed here as problems. See also **[13]** for several related problems, that appear still to be unanswered.

(5.8) *Problem.* Let $\mathbf{h}$ be a sequence of positive numbers such that

$$0 < \liminf \frac{h_{n+1}}{h_n} \leqq \limsup \frac{h_{n+1}}{h_n} < 1.$$

Then if f is a continuous function on the interval [0, 1] such that $f'_{\mathbf{h}}$ exists on a set P, must it be the case that f is approximately differentiable a.e. on P?

(5.9) *Problem.* Let $\mathbf{h}$ be a sequence of positive numbers such that

$$0 < \liminf \frac{h_{n+1}}{h_n} \leqq \limsup \frac{h_{n+1}}{h_n} < 1.$$

Then if f is a continuous function on the interval [0, 1] such that

$$-\infty < \underline{f}'_{\mathbf{h}}(x) \leqq \overline{f}'_{\mathbf{h}}(x) < +\infty$$

everywhere on a set P, must it be the case that $\underline{f}'_{\mathbf{h}} = \overline{f}'_{\mathbf{h}}$ a.e. on P?

6. Existence of sequences without property (S). To complete the concerns of the preceding section we now show that the property (S) of Definition (5.1), that is critical to the veracity of statement (***), is not enjoyed by all sequences $\mathbf{h}$, even those for which some condition is imposed regarding the distribution of the values.

(6.1) THEOREM. *For every decreasing sequence of positive numbers $\{\alpha_n\}$ converging to zero there is a sequence $\mathbf{h} = \{h_n\}$ of positive numbers converging to zero that does not have the property* (S) *and such that for every index k there is at least one index i for which*

$$h_i \in (\alpha_k, \alpha_{k-1}).$$

Proof. We shall construct a sequence of numbers $\mathbf{h} = \{h_n\}$, a closed set P of positive measure in the interval [0, 1], and a sequence of closed subsets $P_1, P_2, P_3, P_4, \ldots$ of P in such a way that (a), (b), and (c) of (5.1) are satisfied, so that for every index k there is an index i for which

$$h_i \in (\alpha_k, \alpha_{k-1}),$$

and with the property that whenever there are indices i and j with $i < j$, and

$$[P_i + h_i] \cap [P + h_j] \neq \emptyset,$$

then for some k,

$$h_i, h_j \in [\alpha_k, \alpha_{k-1}).$$

If α_{k-1}/α_k is bounded (which can be achieved by adding new elements to the sequence $\{\alpha_k\}$) then such a construction supplies the sequence whose existence is asserted in the statement of the theorem. The proof will be given after we have established four lemmas. Throughout we assume that the sequence $\{\alpha_n\}$ is as given in the statement of the theorem with, for convenience, $\alpha_0 = 1$.

(6.2) LEMMA. *Let C be a compact set of real numbers with $|C| < \alpha$. Then there is a positive number δ so that*

$$|\text{cl } U(C; t)| < \alpha$$

for every $0 < t < \delta$.

Proof. Since C is compact and has measure smaller than α there is a finite covering of C by open interals $I_1, I_2, \ldots, I_N$ with total length less than α. There is then a positive number δ so that if $|t| < \delta$ then the set $U(C; t)$ also remains covered by these intervals. From this the statement evidently follows.

(6.3) LEMMA. *For any numbers α, β, and ϵ with $\alpha < \beta$ and $\epsilon > 0$, there are closed intervals $I_1, I_2, \ldots, I_N$ in* $[0, 1]$ *and real numbers $h_1, h_2, h_3, \ldots, h_N$ from (α, β) such that*

$$\bigcup_{i=1}^{N} I_i = [0, 1],$$

and

$$\left| \bigcup_{i=1}^{N} \left[(I_i + h_i) \cup \bigcup_{k=0}^{\infty} (I_i + h_i - \alpha_k) \right] \right| < \epsilon.$$

Proof. Let us choose an integer M larger than $1/(\beta - \alpha)$. The closure of the set

$$B = \left[\frac{j}{M} - \alpha_k : j = 0, 1, 2, \ldots, M - 1 \text{ and } k = 0, 1, 2, \ldots \right]$$

is of measure zero and so there must be, by (6.2), a positive number δ so that

$$|\text{cl } U(B; \delta)| < \epsilon.$$

Let K be an integer larger than $1/\delta$, let $N = KM$, write

$$I_i = \left[\frac{i-1}{N}, \frac{i}{N} \right] \quad (i = 1, 2, 3, \ldots, N)$$

and, for $i = Kj + r$, $(j = 0, 1, 2, \ldots, M - 1;\ r = 1, 2, \ldots K)$, write

$$h_i = \beta - \frac{r}{N}.$$

Then we have for each index $i = 1, 2, 3, \ldots N$,

$$\beta > h_i \geqq \beta - \frac{K}{N} = \beta - \frac{1}{M} > \alpha.$$

We must verify that the sequence of intervals $\{I_i\}$ has the properties asserted in the lemma. The first of the properties is immediate and the second will follow directly from the following set inclusion:

$$\bigcup_{i=1}^{N} \left[(I_i + h_i) \cup \bigcup_{k=0}^{\infty} (I_i + h_i - \alpha_k) \right]$$

$$\subset \mathrm{cl}\, U(B; \delta) + \beta.$$

To see this we need show only that

(21) $\quad I_i + h_i - \alpha_k \subset U(B; \delta) + \beta$

for every pair of indices i and k. Let $i = Kj + r$ for $0 \leqq j \leqq M - 1$, and $1 \leqq r \leqq K$. Then

$$I_i + h_i = \left[\frac{Kj - 1}{N} + \beta, \frac{Kj}{N} + \beta \right] = \left[\frac{j}{M} - \frac{1}{N}, \frac{j}{M} \right] + \beta.$$

This gives

$$I_i + h_i - \alpha_k = \left[\frac{j}{M} - \alpha_k - \frac{1}{N}, \frac{j}{M} - \alpha_k \right] + \beta$$

and since $1/N \leqq 1/K < \delta$ this proves assertion (21) above and the lemma.

(6.4) LEMMA. *Let Q be a bounded set such that*

$$\left| \mathrm{cl} \left[\bigcup_{k=0}^{\infty} (Q - \alpha_k) \right] \right| < \epsilon.$$

Then there is a positive number δ so that

$$\left| \mathrm{cl} \left[\bigcup_{k=0}^{\infty} (Q - [\alpha_k, \alpha_k + t]) \right] \right| < \epsilon$$

for $0 < t < \delta$.

Proof. Let

$$H_t = \mathrm{cl}\, U\left(\mathrm{cl} \bigcup_{k=0}^{\infty} (Q - \alpha_k); t \right) \quad (t > 0).$$

There is a $\delta > 0$ so that $|H_t| < \epsilon$ for $0 < t < \delta$ and since

$$\bigcup_{k=0}^{\infty} (Q - [\alpha_k, \alpha_k + t]) \subset H_t$$

for all t the lemma follows.

(6.5) LEMMA. *Let $\epsilon > 0$. Then there is a sequence of integers $0 = n_0 < n_1 < n_2 < \dots$, a sequence of positive numbers $\{\delta_k\}$, a sequence of intervals $\{I_i\}$, and a sequence of numbers $\{h_i\}$ such that*

(i) $0 < \delta_k < \alpha_{k-1} - \alpha_k$,
(ii) $\alpha_k < h_i < \alpha_k + \delta_k \quad (n_{k-1} < i \leqq n_k)$
(iii) $\bigcup_{i=n_{k-1}+1}^{n_k} I_i = [0, 1] \quad (k = 1, 2, 3, \dots)$,

and

(iv) *the measure of the set*

$$\bigcup_{i=n_{k-1}+1}^{n_k} \left[(I_i + h_i) \cup \bigcup_{m=k+1}^{\infty} [(I_i + h_i) - [\alpha_m, \alpha_m + \delta_m]] \right]$$

is smaller than $\frac{\epsilon}{2^k}$ for every $k = 1, 2, \dots$.

Proof. We define the sequences $\{n_k\}$ and $\{\delta_k\}$ inductively as follows. To begin we set $n_0 = 0$ and $\delta_1 = (\alpha_0 - \alpha_1)/2$. Let us suppose that k is a positive integer and that an integer n_{k-1} and a positive number δ_k have been defined. Using Lemma (6.3) with $\alpha = \alpha_k$ and $\beta = \alpha_k + \delta_k$ we can find an integer $n_k > n_{k-1}$, a sequence of numbers

$$h_{n_{k-1}+1}, h_{n_{k-1}+2}, \dots, h_{n_k}$$

from the interval $(\alpha_k, \alpha_k + \delta_k)$, and a sequence of closed intervals

$$I_{n_{k-1}+1}, I_{n_{k-1}+2}, \dots, I_{n_k}$$

such that statement (iii) of the present lemma is valid and such that the measure of the set

$$\bigcup_{i=n_{k-1}+1}^{n_k} \left[(I_i + h_i) \cup \bigcup_{m=0}^{\infty} (I_i + h_i - \alpha_m) \right]$$

is less than $\epsilon/2^k$.

Write

$$Q_k = \bigcup_{i=n_{k-1}+1}^{n_k} (I_i + h_i).$$

We may apply Lemma (6.4) now to find a positive number δ_{k+1} so that

$$0 < \delta_{k+1} < \min[\delta_k, \alpha_k - \alpha_{k+1}]$$

and

$$\left| Q_k \cup \bigcup_{m=0}^{\infty} (Q_k - [\alpha_m, \alpha_m + \delta_{k+1}]) \right| < \frac{\epsilon}{2^k}.$$

In this way we have defined inductively each of the sequences whose existence is asserted in the statement of the lemma and the properties (i), (ii), (iii), and (iv) may be readily verified.

Now we may return to the proof of the theorem, making appropriate use of the lemmas. Let the sequence $\{\alpha_n\}$ be as given, and let $\epsilon > 0$. Suppose that we have obtained from Lemma (6.5) the sequences $\{n_k\}$, $\{\delta_k\}$, $\{I_i\}$, and $\{h_i\}$ having the properties (i), (ii), (iii), and (iv) of that lemma. Note firstly that for every index k and any index i for which

$$n_{k-1} < i \leqq n_k$$

the requirements (i) and (ii) of the lemma provide

$$h_i \in (\alpha_k, \alpha_{k-1}).$$

Define the following sets:

$$Q_k = \bigcup_{i=n_{k-1}+1}^{n_k} (I_i + h_i) \quad (k = 1, 2, \dots)$$

$$A = \bigcup_{k=1}^{\infty} \left[Q_k \cup \bigcup_{m=k+1}^{\infty} (Q_k - [\alpha_m, \alpha_m + \delta_m]) \right]$$

and

$$C = [0, 1]\backslash A.$$

By (iv) of the lemma the measure of the set C exceeds $1 - \epsilon$ and so we may select a closed subset $P \subset C$ whose measure also exceeds $1 - \epsilon$. We define the sets P_i $(i = 1, 2, \dots)$ by setting

$$P_i = P \cap I_i.$$

By statement (iii) of the lemma we see that every point $x \in P$ belongs to infinitely many of the sets P_i. Also, since for every index i

$$P_i + h_i \subset I_i + h_i \subset A \quad \text{and} \quad P \cap A = \emptyset,$$

we must have

$$P \cap [P_i + h_i] = \emptyset \quad \text{for each index } i.$$

Finally, suppose that for indices i, j, k, and m,

$$n_{k-1} < i \leqq n_k \leqq n_{m-1} < j \leqq n_m.$$

Then it must be the case that

$$[P_i + h_i] \cap [P + h_j] = \emptyset.$$

To see this observe that $h_j \in (\alpha_m, \alpha_m + \delta_m)$ so that

$$P_i + h_i - h_j \subset (I_i + h_i) - [\alpha_m, \alpha_m + \delta_m] \subset A$$

from which it follows that

$$P \cap (P_i + h_i - h_j) = \emptyset,$$

and hence the previous intersection is empty.

We have, accordingly, verified each of the required features of our construction and so the theorem has been proved.

7. Exact sequential and path derivatives. We have seen in the preceding sections that it is possible for a continuous function f to have, on a set of positive measure, an approximate derivative f'_{ap} that differs from the extreme sequential derivatives $\underline{f}'_{\mathbf{h}}$ and $\overline{f}'_{\mathbf{h}}$, and which differs from an exact path derivative $f'_{\mathbf{E}}$ even when the system of paths $\mathbf{E}$ is nonporous at each point.

It is not however possible for a function to have an approximate derivative and also to have an exact sequential derivative which differ on a set of positive measure. This is the content of our next theorem.

(7.1) THEOREM. *Let* $\mathbf{h}$ *be any sequence of nonzero numbers converging to zero and let* f *be a measurable function that has an approximate derivative everywhere on a measurable set* X. *Then*

$$f'_{\text{ap}}(x) = f'_{\mathbf{h}}(x)$$

at almost every point in X *at which the latter exists.*

Proof. If X has positive measure then, by (5.3), there is a subset $X_1 \subset X$ also of positive measure so that f has a derivative relative to the set X_1 at every point of X_1. By (5.2) we know that for almost every point $x \in X_1$, $x + h_n \in X_1$ for infinitely many indices n. Thus the derivatives

$$f'_{X_1}(x) = f'_{\mathbf{h}}(x)$$

must agree at almost every point x at which the latter derivative exists. From this statement the theorem evidently follows.

We see that the theorem requires for its proof only an appeal to the property (5.3) of approximate derivatives and to the measure-theoretic fact (5.2). Thus in order to obtain an extension of the theorem we may focus on this measure-theoretic property in order to obtain a version in a slightly enlarged context. The problem, if we express it for the moment in the context of path derivatives, is to find a condition on a system of paths $\mathbf{E}$ so that the existence everywhere on a set A on the two derivatives $f'_{\mathbf{E}}$ and f'_{ap} will require that they be almost everywhere equal on A. We know, because of Theorem (3.6), that this is not the case in general, but that if a sequence $\mathbf{h} = \{h_n\}$ may be selected so that

$$x + h_n \in E_x$$

at each point x, then this is the situation. We may ask instead for the selection of a sequence $h_n(x)$ that depends on the point x, i.e., so that

$$x + h_n(x) \in E_x$$

and so that some appropriate smoothness condition on the functions $x: \to h_n(x)$ is met allowing a similar result. It is this problem that we now address, expressing our results in the language of the sequential derivatives, but permitting a generalization in which the sequence $\mathbf{h}$ may vary from point to point.

(7.2) *Definition.* Let $\mathbf{h}$ be a nonvanishing real valued function defined on $\mathbf{N} \times [0, 1]$ so that

$$\lim_{n\to\infty} \mathbf{h}(n, x) = 0$$

for every $x \in [0, 1]$. By the $\mathbf{h}$-derivative of a function f at a point x we mean

$$f'_{\mathbf{h}}(x) = \lim_{n\to\infty} \frac{f(x + \mathbf{h}(n, x)) - f(x)}{\mathbf{h}(n, x)}$$

should this limit exist.

This definition allows a generalization of the sequential derivative and we may now state the conditions on the function $\mathbf{h}$ so that the result in Theorem (7.1) above remains valid.

(7.3) THEOREM. *Let* $\mathbf{h}$ *be a function such as appears in Definition* (7.2) *and which is Lipschitz in x uniformly in n, i.e.,*

$$|\mathbf{h}(n, x) - \mathbf{h}(n, y)| \leqq C\,|x - y| \quad (x, y \in [0, 1];\ n = 1, 2, \dots).$$

Let f be a measurable function that has an approximate derivative everywhere on a measurable set X. Then

$$f'_{\mathrm{ap}}(x) = f'_{\mathbf{h}}(x)$$

at almost every point in X at which the latter exists.

Proof. The proof is obtained, precisely as for the proof of Theorem (7.1), provided that we are able to show that for any measurable set H it must be the case that for almost every $x \in H$

$$x + \mathbf{h}(n, x) \in H$$

for infinitely many positive integers N.

We obtain this fact from the following three lemmas.

(7.4) LEMMA. *Let* $\mathbf{h}$ *be a real-valued function defined on the interval* $[a, b]$ *so that* $|h(x)| \leqq \delta$ *for every point x in that interval and*

$$|(h(x) - h(y)| \leqq \frac{1}{2}|x - y| \quad (x, y \in [a, b]).$$

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Then for every measurable set $H \subset [a, b]$,

$$|\{x \in H : x + h(x) \notin H\}| \leqq 4\delta + 2|[a, b] \setminus H|.$$

Proof. Let $B = \{x + h(x) : x \in H\}$ and write ψ for the function defined on B that is inverse to the function

$$x \to x + h(x) \quad (x \in H).$$

Observe that ψ has the property

(22) $$|\psi(s) - \psi(t)| \leqq 2|s - t| \quad (s, t \in B).$$

Now if $x \in H$ but $x + h(x) \notin H$ then

$$x + h(x) \in ([a, b] \setminus H) \cup [a - \delta, a] \cup [b, b + \delta],$$

and hence

$$\{x \in H : x + h(x) \notin H\}$$
$$\subset \psi(B \cap [([a, b] \setminus H) \cup [a - \delta, a] \cup [b, b + \delta]])$$

This set inclusion together with the estimate (22) on ψ provides the conclusion of the lemma.

(7.5) LEMMA. *Let H be a measurable set of finite measure. For every $\epsilon > 0$ there is a $\delta > 0$ such that if h is a real-valued function defined on the real line for which $\|h\| < \delta$ and satisfying a Lipschitz condition*

$$|h(x) - h(y)| \leqq \frac{1}{2}|x - y| \quad (x, y \in \mathbf{R}),$$

then necessarily

$$|\{x \in H : x + h(x) \in H\}| > |H| - \epsilon.$$

Proof. Let us choose a finite sequence of disjoint intervals

$$[a_i, b_i] \quad (i = 1, \ldots, N)$$

so that

$$\left| H \,\Delta \bigcup_{i=1}^{N} [a_i, b_i] \right| < \frac{\epsilon}{4}.$$

We verify the statement of the lemma for $\delta = \epsilon/16N$. Using the preceding lemma we compute

$$|\{x \in H : x + h(x) \notin H\}|$$
$$\leqq \left| H \setminus \bigcup_{i=1}^{N} [a_i, b_i] \right|$$

$$+ \sum_{i=1}^{N} |\{x \in H \cap [a_i, b_i]:x + h(x) \notin H \cap [a_i, b_i]\}|$$

$$\leqq \frac{\epsilon}{4} + \frac{4N\epsilon}{16N} + \sum_{i=1}^{N} 2|[a_i, b_i]\backslash H|$$

$$= \frac{\epsilon}{2} + 2\left|\bigcup_{i=1}^{N} [a_i, b_i]\backslash H\right| < \epsilon$$

as required to prove the lemma.

(7.6) LEMMA. *Let* **h** *be a function with the properties described in the assertion of the theorem, and let* H *be a measurable set that has finite measure. Then as* $n \to \infty$,

$$|\{x \in H:x + \mathbf{h}(n, x) \in H\}| \to |H|.$$

Proof. We may rescale the real line so that the function **h** satisfies a uniform Lipschitz condition with constant 1/2, i.e., so that for $n = 1, 2, \ldots$

$$|\mathbf{h}(n, x) - \mathbf{h}(n, y)| \leqq \frac{1}{2}|x - y| \quad (x, y \in [a, b]).$$

This implies that the pointwise convergence in

$$\lim_{n\to\infty}\mathbf{h}(n, x) = 0$$

is, in fact, uniform on $[a, b]$. Hence we may assume that for any given $\delta > 0$,

$$|\mathbf{h}(n, x)| \leqq \delta \quad (n = 1, 2, \ldots).$$

Clearly then this lemma is an immediate consequence of the preceding lemma.

Now the proof of the theorem is complete since it is obvious, in view of this last lemma, that for any measurable set H of positive measure it must be the case that, for almost every $x \in H$,

$$x + \mathbf{h}(n, x) \in H$$

for infinitely many positive integers n.

Let us conclude with an example to show that the hypothesis of this theorem may not be removed.

(7.7) *Example.* Let P be any perfect nowhere dense subset of [0, 1] of positive measure. We shall produce a continuous function f on [0, 1] which vanishes on P and a function **h** such as appears in Definition (7.2) and which has the properties

(i) $\lim_{n\to\infty}\mathbf{h}(n, x) = 0$ uniformly for $x \in [0, 1]$,
(ii) $x \to \mathbf{h}(n, x)$ is continuous for each n, and
(iii) $f'_{\mathbf{h}} = 1$ a.e. in P.

This gives us, in contrast to the situation in (7.3),

$$f'_{\mathrm{ap}}(x) < f'_{\mathbf{h}}(x)$$

at almost every point of P.

We again employ, for this purpose, the construction used in the proof of Theorem (3.6). Let $\{I_n\}$, $\{P_n\}$, $\{n_k\}$, and f be as in the proof of that theorem. Let R_k be the union of those components of P_{n_k} that have the same right hand endpoint as a component of $P_{n_{k-1}}$. Let R denote the set of points in P that are in infinitely many of the sets R_k; it may be checked that R has measure zero. This follows from the inequalities

$$|R_k| \leqq n_{k-1}\|P_{n_k}\| \leqq \frac{n_{k-1}\epsilon_{k-1}}{k^2} \leqq \frac{1}{k^2}.$$

For any index k and any point x in $[0, 1]$ we shall define $\mathbf{h}(k, x)$ as follows. If x is in $P_{n_k}\setminus R_k$ then there is a first interval I_i, $n_{k-1} < i \leqq n_k$ to the right of x. We choose $\mathbf{h}(k, x)$ to be $y - x$ where y is chosen as the first point in that interval I_i at which $f(y) = y - x$. For remaining x in $[0, 1]$ we define the function by extending it so as to be linear and continuous on the remaining intervals. Note that this function must satisfy everywhere in $[0, 1]$ the inequalities

$$0 < \mathbf{h}(k, x) \leqq \|P_{n_k}\| + \max\{\, |I_i| : n_{k-1} < i \leqq n_k\}.$$

It may now be readily verified that this construction has the required properties. At every point x in $P\setminus R$ the derivative $f'_{\mathbf{h}}(x)$ clearly exists and is 1.

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