Can. J. Math., Vol. XXXVII, No. 1, 1985, pp. 17-28

INFINITELY PERIODIC KNOTS

ERICA FLAPAN

One aspect of the study of 3-manifolds is to determine what finite group actions a given manifold has. Some important questions that one can ask about these actions on a given manifold are: What periods could they have? and, what sets of points may be fixed by the action? In the case of periodic transformations of homology spheres, Smith [**18**] classified the types of fixed point sets which could occur. For homology 3-spheres the fixed point set will be $\emptyset$, S^0, S^1, or S^2. Fox [**4**] looked at periodic transformations of the three sphere which leave a knot invariant and, using Smith's classification of fixed point sets, determined that there were eight types of transformations according to how the fixed point set met the knot. For convenience we shall say a knot is *(a, b)-periodic* if there is a periodic transformation of S^3 leaving the knot invariant with fixed point set homeomorphic to a and with the fixed point set meeting the knot in a set homeomorphic to b. As Fox points out the possibilities for b are only $\emptyset$, S^0, and S^1. And by the recent proof of the Smith conjecture [**21**] we can rule out S^1 if the knot is non-trivial.

Fox asked which knots could have infinitely many periods of each type; in other words, which knots are infinitely (a, b)-periodic for each possible pair (a, b). Seifert [**16**] has shown that any knot that could be drawn on the surface of a torus is infinitely $\emptyset$-periodic (here we write $\emptyset$-periodic instead of $(\emptyset, \emptyset)$-periodic). Hartley [**8**] conjectured that torus knots are the only infinitely $\emptyset$-periodic knots. On the other hand, Murasugi [**14**] showed that any infinitely $(S^1, \emptyset)$-periodic knot must have trivial Alexander polynomial.

If a knot is infinitely periodic then it must either have an infinite number of orientation reversing periodic diffeomorphisms or an infinite number of orientation preserving periodic diffeomorphisms. Suppose $\{g_i\}$ is an infinite collection of periodic diffeomorphisms of (S^3, K) with distinct orders $\{p_i\}$. Now $\{g_i^2\}$ is an infinite collection of periodic diffeomorphisms which preserve the orientation of both S^3 and K; and order $(g_i^2) = p_i$ or $p_i/2$, so an infinite subcollection of $\{g_i^2\}$ have distinct orders. Now by Smith theory a periodic diffeomorphism of S^3 is orientation preserving if and only if it has a fixed point set which is empty

Received February 22, 1983, and in revised form December 20, 1983.

https://doi.org/10.4153/CJM-1985-002-4 Published online by Cambridge University Press

or is homeomorphic to S^1. So for each orientation reversing periodic diffeomorphism g_i, $\text{fix}(g_i) \neq \emptyset$. Hence $\text{fix}(g_i^2) \neq \emptyset$ and since g_i^2 is orientation preserving

$$\text{fix}(g_i^2) \cong S^1.$$

Thus if a certain knot has infinitely many orientation reversing periodic diffeomorphisms, then it is infinitely $(S^1, \emptyset)$-periodic or infinitely (S^1, S^0)-periodic. Now suppose g is a periodic diffeomorphism of (S^3, K) with

$$\text{fix}(g) \cap K \neq \emptyset.$$

Then $g|K$ is orientation reversing, and $g^2|K$ is the identity. But by the Smith conjecture [**21**] now g^2 itself is the identity. Now since $g_i^2|K$ is orientation preserving, $h_i^n|K$ will be orientation preserving. Thus by taking $\{h_i\} \subseteq \{g_i^2\}$ as our infinite collection of periodic diffeomorphisms, we assure that no power of any of the h_i fixes any point on K and both h_i and $h_i|K$ are orientation preserving.

We prove Hartley's conjecture that only torus knots are infinitely periodic, and further that no non-trivial knots are infinitely $(S^1, \emptyset)$-periodic. Thus no non-trivial knots have infinitely many diffeomorphisms of distinct order with non-empty fixed point set, and no non-trivial knots have infinitely many distinct order orientation reversing diffeomorphisms. Our basic strategy will be to split the knot complement along characteristic tori and to show that if a knot is infinitely periodic then all its characteristic simple and Seifert fibered components are also infinitely periodic. Then we prove that all the components are in fact Seifert fibered and we go through the remaining possibilities one at a time.

Definition 1. A knot K in S^3 has a *free-symmetry* h of order q, if h is a fixed point free diffeomorphism of S^3 leaving K invariant, and h^q is the identity.

Definition 2. A knot K in S^3 has a *cyclic-symmetry* h of order q if h is a diffeomorphism of S^3 leaving K invariant and

$$\text{fix}(h) \cong S^1 \quad \text{and} \quad \text{fix}(h) \cap K = \emptyset,$$

and h^q is the identity.

Definition 3. h is said to be a *symmetry* of K if h is a free-symmetry or a cyclic-symmetry and $h|K$ is orientation preserving.

We have shown in the introduction that if K is infinitely periodic then K has an infinite number of symmetries of distinct order.

The following three lemmas are immediate from the work of Freedman, Haas and Scott [**5**] and Meeks and Scott [**13**].

LEMMA 1. [**5**]. *Let M be a Riemannian, Haken 3-manifold whose boundary is empty or has non-negative mean curvature. Let $\mathcal{S}$ be any collection of essential surfaces in M, no two of which are parallel. Then $\mathcal{S}$ is isotopic to a collection $\mathcal{T}$ of least area surfaces.*

LEMMA 2. [**5**]. *Let S_1 and S_2 be least area surfaces in a Riemannian, Haken 3-manifold M. If S_1 and S_2 can be homotoped to be disjoint then $S_1 \cap S_2 = \emptyset$ or $S_1 = S_2$.*

LEMMA 3. [**13**]. *Let M be a Haken 3-manifold with a $\mathbf{Z}_p$-action g. Then there exists a characteristic family of tori in M invariant under g.*

Remarks. 1) We use "characteristic" in the sense of Jaco and Shalen. Thus, if we remove the characteristic family from M we are left with Seifert fibered and simple components. Also if $f:M \to M$ is a diffeomorphism then $f(T)$ is ambiant isotopic to T.

2) We shall want to apply Lemmas 1 and 2 to a manifold M with a periodic diffeomorphism h by finding a Riemannian metric for M which makes h an isometry and makes ∂M have non-negative mean curvature. We do this as it is done in [**12**, p. 56].

LEMMA 4. *Let k be a knot, and let Q be its exterior in S^3. Further, let $\mathcal{T} = \{T_i\}$ be a characteristic family of tori in Q. If k is infinitely periodic then k has infinitely many symmetries leaving each T_i invariant.*

Proof. For each symmetry G_α of (S^3, k) we can define $g_\alpha:Q \to Q$ to be a periodic diffeomorphism. Let $\mathcal{S} = \{S_i\}$ be the characteristic family given by Lemma 3. Let f_α be an isotopy of Q taking $\mathcal{T}$ to $\mathcal{S}$. Thus

$$f_\alpha(T_i) = S_i \quad \text{for each } i.$$

Now

$$f_\alpha^{-1} \circ g_\alpha \circ f_\alpha(\mathcal{T}) = \mathcal{T}.$$

Let $h_\alpha = f_\alpha^{-1} \circ g_\alpha \circ f_\alpha$, then the order of h_α is the same as the order of g_α. Let the order of h_α be α. Since k is infinitely periodic, the α get arbitrarily large. Let m be the number of permutations there are of the elements of $\mathcal{T}$. Let h_α be a periodic diffeomorphism of Q leaving $\mathcal{T}$ invariant and such that $\alpha > m$. Then for some N_α with $0 < N_\alpha \leqq m + 1$, $h_\alpha^{N_\alpha}$ performs the identity permutation on $\mathcal{T}$. Define

$$H_\alpha = h_\alpha^{N_\alpha}.$$

Then the order of H_α is at least α/N_α.

Now since the h_α have arbitrarily large orders α, but the N_α are bounded by $m + 1$, the H_α must have arbitrarily large orders. Since h_α and $h_\alpha|k$ are orientation preserving, so are H_α and $H_\alpha|k$. Hence

$$\mathrm{fix}(H_\alpha) \cap k = \emptyset$$

since by the Smith conjecture $\mathrm{fix}(H_\alpha) \neq k$. Thus $\{H_\alpha\}$ is an infinite collection of periodic diffeomorphisms of Q, and

$$H_\alpha(T_i) = T_i \quad \text{for each } i.$$

Remark. Lemma 4 essentially says that if a knot is infinitely periodic then all its characteristic simple and Seifert fibered components are also infinitely periodic.

Definition 4. A 3-manifold is *simple* if it contains no essential torus, and a knot is *simple* if its exterior is simple.

LEMMA 5. *A simple non-torus knot is not infinitely periodic.*

Proof. If K is not a torus knot then K is not Seifert fibered. So by [**1**] the center of $\pi_1(S^3 - K)$ is trivial. Now by [**6**, Theorem] no symmetry of (S^3, K) is pairwise isotopic to the identity. So every symmetry of (S^3, K) induces an outer automorphism of $\pi_1(S^3 - K)$ of the same order. But by Johannson's finiteness Theorem [**11**, page 213] $\mathrm{Out}(\pi_1(S^3 - K))$ is finite. Hence (S^3, K) is not infinitely periodic.

Remark. The only Seifert fibered spaces in a knot exterior are a torus knot complement, a cable space, or a composing space, as shown by Jaco and Shalen [**10**, Lemma VI.3.4].

Definition 5. A *cable space* is a manifold obtained from a solid torus $S^1 \times D$ by removing an open regular neighborhood in $S^1 \times \mathring{D}$ of a simple closed curve C which lies in a torus $S^1 \times J$ where J is a simple closed curve in D and where C is non-contractible in $S^1 \times D$.

Definition 6. A *composing space* is a 3-manifold homeomorphic to $W \times S^1$ where W is a disk with n open cells removed, for $n \geqq 2$.

Definition 7. Let $K_2 \subseteq W \subseteq S^3$ be a knot which intersects every meridianal disk of a standardly embedded solid torus W in the 3-sphere. Let K_1 be another knot with a regular neighborhood V in S^3. Let $h: W \to V$ be a homeomorphism preserving preferred longitude and meridian, and let $k = h(K_2)$. Then we say K_1 is a *companion* of k and K_2 is a *presatellite*.

Definition 8. A *cable knot* is a knot with presatellite a torus knot.

Remark. A cable knot with companion a torus knot is one in which both K_1 and K_2 are torus knots.

LEMMA 6. *If there is a non-torus knot k which is infinitely periodic then either*:

1. *There is a cable knot with companion a torus knot which is infinitely periodic.*

2. *There is a composing space with incompressible boundary components $T_0, \ldots, T_n$, for $n \geqq 2$, in the complement Q of a knot k with $\partial Q = T_0$; and k has infinitely many symmetries h such that $h(T_i) = T_i$ for each i.*

Proof. Suppose k has infinitely many periods. Let Q be the complement of an open tubular neighborhood of k in S^3. Let $\{T_i\}$ be a characteristic set of tori for Q. By Lemma 4 Q has infinitely many periods leaving each T_i and hence each component X_l of $Q - \cup T_i$ invariant. Since Q is a knot complement every torus T_i separates. Let X_1 be a component of $Q - \cup T_i$ with only one boundary component. Since $\{T_i\}$ is characteristic each X_l is either simple or Seifert fibered. By [**10**, Lemma VI.3.4] the only Seifert fibered spaces in a knot exterior are a torus knot complement, a cable space, or a composing space. A cable space and a composing space each have more than one boundary component. So either X_1 is a torus knot complement or X_1 is simple. Now ∂X_1 is essential in Q and so by [**15**, Proposition 3.10], $S^3 - X_1 = V_1$, a solid torus with core K_1, which is a companion of k. Suppose h is any period of Q with order greater than 2. Then h takes a meridianal disk of V_1 to a meridianal disk. So we can define a periodic diffeomorphism $\hat{h}$ of K_1 by $\hat{h}|X_1 = h|X_1$ and $\hat{h}$ is defined in V_1 by extending h radially. Now the order of $\hat{h}$ must be equal to the order of h since X_1 cannot be the fixed point set of any periodic diffeomorphism of S^3. So the order of $\hat{h}$ is greater than 2. Thus

$$\operatorname{fix}(\hat{h}) \cap K_1 = \emptyset$$

and so $\hat{h}$ is actually a symmetry of K_1.

Suppose X_1 is not a torus knot complement. Then K_1 is a simple knot other than a torus knot. But every symmetry h of k induces a symmetry $\hat{h}$ of K_1 of the same order. Hence we contradict Lemma 5. So X_1 is a torus knot complement. Let $\partial X_1 = T_1$, and let X_2 be another component of $Q - \cup T_i$ with T_1 in its boundary. If no such X_2 exists then k would be a torus knot. Now let the components of $Q - \operatorname{Int} X_2$ be a collection $\{R_j\}$ and Y, where $\partial Q \subseteq \partial Y$ and $R_1 = X_1$ and $\partial R_j \subseteq \partial X_2$. Each j separates Q so we can let $\partial R_j = T_j$ and let $R = Q - Y$. Then $S^3 - R = V$ a solid torus with core K_1 by [**15**, Proposition 3.10].

By [**3**, Theorem 1] there is a homeomorphism of X_2 in S^3 such that

$$S^3 - \operatorname{Int}(X_2) = \cup_j W_j \cup V'$$

where each W_j and V' is a solid torus and $\partial W_j = T_j$. Let h be any symmetry of k leaving each X_1 and T_i invariant. Then h is orientation preserving and h takes a longitude of R_j to a longitude of R_j for homological reasons in R_j. Also h takes a meridianal disk of $S^3 - R_j$ to a meridianal disk of $S^3 - R_j$. Thus h takes any (p, q) curve on T_j to a $\pm(p, q)$ curve on T_j. So we can define a symmetry $\hat{h}$ of $(S^3, \widetilde{K})$ where $\widetilde{K}$ is the core of V'. Let $\hat{h}|X_2 = h|X_2$ then extend $\hat{h}$ radially to each W_j and V'. Now $\hat{h}(\widetilde{K}) = \widetilde{K}$, and the order of $\hat{h}$ is equal to the order of h.

Now X_2 was also either Seifert fibered or simple. Suppose X_2 is not Seifert fibered. Then X_2 is simple and contains no essential annulus. So $S^3 - \operatorname{Int} V'$ is simple and contains no essential annulus. Thus $\widetilde{K}$ is a simple non-torus knot. By Lemma 5, $\widetilde{K}$ has at most finitely many symmetries. But every symmetry of k induces a symmetry of $\widetilde{K}$ of the same order. This contradicts k having infinitely many symmetries. Hence X_2 is Seifert fibered. So X_2 is either a cable space or a composing space, since $|\partial X_2| \geqq 2$. If X_2 is a cable space then $R = X_l \cup X_2$, and $S^3 - R = V$ is a neighborhood of K, which is a cable knot with companion a torus knot. There are infinitely many symmetries of k taking a meridian of V to a meridian of V. So for each of these symmetries h we obtain a symmetry $\hat{h}$ of K. If X_2 is a composing space then X_2 has incompressible boundary components $T_0, \dots, T_n$; and T_0 separates S^3 into a solid torus and the complement Q, of a knot K. As when X_2 is a cable space we can obtain infinitely many symmetries h of K, and $h(T_i) = T_i$ for each i.

LEMMA 7. *Let X be a cable space $N_1 - \operatorname{Int} N_2$ where N_1 and N_2 are solid tori in S^3, and k is the core of N_2. Let h be a symmetry of (S^3, k) leaving X invariant. Let B be an essential annulus properly embedded in X with $\partial B \subset \partial N_2$. Then there exists a properly embedded annulus A in X with $\partial A \subset \partial N_2$ and $h(A) = A$ and A is properly isotopic to B in X.*

Remark. Recall, by our definition, if h is a symmetry then h and $h|K$ are orientation preserving.

Proof. Metrize X so that h is an isometry and the boundary of X has non-negative mean curvature. By Lemma 1, B is properly isotopic to a least area essential annulus A. Let l_i be a longitude and m_i a meridian for N_i. For homological reasons in N_i, $h(m_i)$ is isotopic to $\pm m_i$ on ∂N_i; and for homological reasons in $S^3 - N_i$, $h(l_i)$ is isotopic to $\pm l_i$ on ∂N_i. Now since h preserves orientation on k we must have $h(l_2)$ isotopic to $+l_2$. But h is orientation preserving, so in fact $h(m_2)$ is isotopic to $+m_2$. Thus if α is any curve on ∂N_2, $h(\alpha)$ is isotopic to α. In particular $h(\partial A)$ is isotopic to ∂A on ∂N_2. So we could isotope $h(A)$ in X to an essential annulus E, with ∂E contained in the interior of one component C of $\partial N_2 - \partial A$. Now E meets A in simple closed curves in the interior of both E and A.

Claim. We can homotop E off A.

Proof of Claim. Isotop E so that E and A are in general position and meet in a minimal number of components. Suppose some component J of $A \cap E$ bounds a disk D in A. Then by the incompressibility of E, J also bounds a disk D' in E. By the irreducibility of X, $D \cup D'$ bounds a ball. Hence we could isotop E to remove J, and thus contradict minimality. Similarly if J had bounded a disk in E. Thus if $E \cap A \neq \emptyset$, then $E \cap C$ consists of incompressible annuli. But C is homeomorphic to $T^2 \times I$ or to a solid torus and in either case every incompressible annulus in C is parallel into ∂C. Let B be a component of $E \cap C$. Then B could be pushed into ∂C then further into $C - A$. Again contradicting minimality. Thus we can homotop E off A.

Hence we could properly homotop $h(A)$ off A. Now since h is an isometry, $h(A)$ is also of least area. So by Lemma 2 either $h(A) = A$ or $h(A) \cap A = \emptyset$.

We show $h(A) = A$ by assuming $h(A) \cap A = \emptyset$ and deriving a contradiction. Let W be a solid torus and let Q be the torus knot complement

$$Q = W \cup (N_1 - \text{Int } N_2),$$

where W is sewn to ∂N, longitude to meridian. Then by [**15**, Lemma 3.1] A is an essential annulus properly embedded in Q. Let U and V be the closed components of $Q - A$. Then U and V are solid tori. Let $g:Q \to Q$ be $h|N_1 - \text{Int } N_2$ extended radially to W.

Case 1. Suppose there is an $i < p$ such that $g^i(U) \cap U \neq \emptyset$. Then either

$$g^i(U) \subseteq \text{Int } U \quad \text{for some } i,$$

or

$$U \subseteq \text{Int } g^i(U) \quad \text{for some } i.$$

If $g^i(U) \subseteq \text{Int } U$ then $g^{p \cdot i}(U)$. But this is a contradiction since $g^{p \cdot i}$ is the identity. Similarly, we could not have

$$U \subseteq \text{Int } g^i(U).$$

Thus Case 1 does not occur.

Case 2. For every $i < p$,

$$g^i(U) \cap U = \emptyset.$$

Let $U_i = g^i(U)$ for $i = 0, \dots, p - 1$. Then the U_i are solid tori. Let

$$Y = V \bigcup_{i=1}^{p-1} U_i.$$

Let $B = \partial U - A$, then $B \subseteq \partial Q$. Since $g(\partial Q) = \partial Q$ we have $g(B) \subseteq \partial Q$. Now let $C = \partial V - A$. Since we are assuming that $g(A) \cap A = \emptyset$ we must have $g(B) \subseteq C$. In fact,

$$\partial V \cap \partial U_i = g^i(B) \quad \text{for } i = 1, \dots, p - 1.$$

Now since V is a solid torus, Y is a solid torus. Now $A = Y \cap U$ is injective in Y and U, but not surjective in U since A is essential in Q. Let y generate $\pi_1(Y)$ and u_i generate $\pi_1(U_i)$. Now since g cyclically permutes the U_i, by the Van Kampan Theorem

$$\pi_1(Q) = \langle y, u_0, \dots, u_{p-1} | y^n = u_0^m = \dots = u_{p-1}^m \rangle$$

where $n \geqq 1$ and $m \geqq 2$. Thus there is an epimorphism from $\pi_1(Q)$ to $\bigoplus_p \mathbf{Z}_m$. But $S^3 \cong W \cup N_1$ so

$$Q = \mathrm{cl}(S^3 - f(N))$$

where f is a reembedding of N_2, and so Q is a knot space. Hence

$$H_1(Q) \cong \mathbf{Z}.$$

So we have a contradiction. Thus Case 2 also does not occur.

Hence $h(A) = A$ as desired.

Remark. The proof of Case 2 is similar to that of Lemma 7.1 of [**15**].

THEOREM 1. *Let k be a cable knot with companion a torus knot K. Then (S^3, k) is not infinitely periodic.*

Proof. Let $N(k)$ be a regular neighborhood of k and let

$$Q = S^3 - \mathrm{Int}\ N(k).$$

Let V be a regular neighborhood of K containing $N(k)$, and let

$$R = S^3 - \mathrm{Int}\ V.$$

Since $V - \mathrm{Int}\ N(k)$ is a cable space it contains an essential annulus a with $\partial a \subset \partial N(k)$.

Now ∂a separates $\partial N(k)$ into b_1 and b_2. Without loss of generality $a \cup b_1$ bounds a solid torus W and $a \cup b_2$ is parallel to ∂V. Let (p, q) be the unoriented isotopy class of ∂a in $a \cup b_2$. Let

$$R' = Q - \mathrm{Int}\ W.$$

Then R' is isotopic to R. So R' is a torus knot complement, hence contains an essential annulus B with

$$\partial B \subset \partial R' = a \cup b_2.$$

Let (r, s) be the unoriented isotopy class of ∂B in $a \cup b_2$.

Claim. $(p, q) \neq (r, s)$.

Proof of Claim. Suppose $(p, q) = (r, s)$. Then we could properly isotope B in R' so that $\partial B \cap a = \emptyset$. Since $a \subseteq \partial R'$ in fact $B \cap a = \emptyset$; and thus B is actually properly embedded in Q. Now B and a are disjoint essential annuli in Q. So by [**19**, Lemma 2.b] they are parallel. But B was essential in R' and hence could not be parallel into $\partial R'$. Hence $(p, q) \neq (r, s)$, as claimed.

Now suppose $\tilde{g}$ is a symmetry of (S^3, k). If (S^3, k) is infinitely periodic, then without loss of generality we can assume that the order of $\tilde{g}$ is not 2. Now let $g = \tilde{g}^2$. Then $g|k$ must be orientation preserving. Now $\{\partial V\}$ is a characteristic family for Q. So by Lemma 4 we can assume that $g(V) = V$ for an infinite class of such symmetries g of (S^3, k). Hence $g(R) = R$ and

$g(Q) = Q$. Let X be the cable space $V - \text{Int } N(k)$. Then $g(X) = X$. Now by Lemma 7 there is an essential annulus A properly embedded in X with $\partial A \subset \partial N(k)$ and $g(A) = A$ and A is properly isotopic to a in X. Let f_t realize this proper isotopy of X. We can extend f_t to an isotopy $\hat{f}_t$ of Q. Now $\hat{f}_1(a) = A$. Let $C = \hat{f}_1(B)$, and let $R'' = \hat{f}_1(R')$. Let $Q' = Q - R''$. Then the components of $Q - A$ are the solid torus W' and the torus knot complement R''. Now $g(Q) = Q$ and $g(A) = A$ and $W \not\cong R''$. Hence

$$g(W') = W' \text{ and } g(R'') = R''.$$

So C is an essential annulus in the invariant torus knot complement R''. Now C is in some cable space inside R'', so by Lemma 7 we can obtain an essential annulus D properly embedded in R'' with $g(D) = D$ and D is properly isotopic to C in R''.

Observe that (p, q) remains the unoriented isotopy class of ∂A in $\partial R''$, and (r, s) remains the unoriented isotopy class of ∂D in $\partial R''$. Since $g(\partial R'') = \partial R''$, we can let $h = g|\partial R''$. Let $l = \text{order}(g)$. By assumption we can find symmetries g so that l is arbitrarily large, so we can take g^2 if necessary to assure that h does not switch the boundary components of A or those of D; and

$$h(\partial A) = +\partial A \quad \text{and} \quad h(\partial D) = +\partial D.$$

Then h is an order l diffeomorphism of a torus $\partial R''$ which fixes setwise both a (p, q) curve $\alpha \subset \partial A$ and an (r, s) curve $\beta \subset \partial D$. In fact α and β are simple closed curves.

Claim. There are only finitely many possible numbers l.

Proof of Claim. Subdivide β into alternating arcs γ_i and δ_i such that:
1) The boundaries of γ_i and δ_i are in $\alpha \cap \beta$.
2) Each γ_i together with an arc of α is null homotopic in $\partial R''$.
3) Each γ_i is maximal with respect to the above properties.

Let N be the minimal number of points of intersection of a (p, q) curve and an (r, s) curve. Then

$$N = \sum_i |\text{Int } \delta_i \cap \alpha|.$$

Observe that by property 2) above, $h(\{\gamma_i\}) = \{\gamma_i\}$ and hence

$$h(\{\delta_i\}) = \{\delta_i\}.$$

So h permutes the set $\cup_i \text{Int } \delta_i \cap \alpha$. Note that since $(p, q) \neq (r, s)$, $N > 0$. Thus either l divides N or there is an $i < l$ such that

$$h^i(x) = x \quad \text{for some } x \in \cup \text{ Int } \delta_i \cap \alpha.$$

Recall that $h|\alpha$ and $h|\beta$ preserve orientation. So, in fact, $h^i|(\alpha \cup \beta)$ is the identity. But now g^i is an orientation preserving periodic diffeomorphism

https://doi.org/10.4153/CJM-1985-002-4 Published online by Cambridge University Press

of S^3 yet

$$\alpha \cup \beta \subseteq \mathrm{fix}(g^i).$$

This contradicts Smith Theory. Thus l divides N and so there can be only finitely many l's.

Thus (S^3, k) has only a finite number of symmetries.

LEMMA 8. *Let X_2 be a composing space with incompressible boundary components $T_0, T_1, \ldots, T_n$ in Q, the complement of a knot k, with $\partial Q = T_0$ and $n \geqq 2$. Then K has no symmetry h of order greater than 2 such that $h(T_i) = T_i$ for each i.*

Remark. The reader should recall that by our definition if h is a symmetry, then h and $h|K$ are orientation preserving.

Proof of Lemma 8. Assume there is such a symmetry h. Let $g = h^2$, and let l_i and m_i be a longitude and meridian of T_i, respectively. Further, let T_i separate S^3 into a solid torus W_i and a knot complement R_i. Then g takes a surface in R_i bounded by l_i to a surface in R_i bounded by $g(l_i)$. Since $g = h^2$ we can in fact assure that $g(l_i)$ is isotopic to $+l_i$ on T_i. Now if $n \geqq 3$ extend g radially within a solid torus U_i replacing R_i for $i = 3, \ldots, n$. The original knot K was a composite knot with n components. By replacing these $R_3, \ldots, R_n$ by $U_3, \ldots, U_n$ we have created a new manifold, X, which is the complement of a composite knot k with two components. Now g is a symmetry of $W \times S^1$ where W is a disk with two holes. Metrize $X = W \times S^1$ so that h is an isometry and ∂X has non-negative mean curvature. Let A be an essential annulus properly embedded in X with both boundary components in T_1. By Lemma 1 find a least area surface B isotopic to A in X. Now B separates X into components Y_1 and Y_2 which are each homeomorphic to $S^1 \times S^1 \times I$, and $T_i \subseteq Y_i$. Now Y_i contains no essential annulus with both boundaries in $T_0 \cup B$, so we can homotop $g(B)$ disjoint from B. So by Lemma 2,

$$g(B) = B \quad \text{or} \quad g(B) \cap B = \emptyset.$$

We show $g(B) = B$. Suppose $g(B) \cap B = \emptyset$. Then

$$g(Y_1) \subseteq \mathrm{Int}\ Y_1 \quad \text{or} \quad Y_1 \subseteq \mathrm{Int}\ g(Y_1)$$

since $T_2 = g(T_2) \subseteq g(Y_1)$. But if $g(Y_1) \subseteq \mathrm{Int}\ Y_1$ and let $\rho = \text{order}\ (g)$ then

$$g^\rho(Y_1) \subseteq \mathrm{Int}\ Y_1$$

and this implies that $Y_1 \subseteq \mathrm{Int}\ Y_1$. By this contradiction we conclude that $g(B) = B$ as desired.

Now by [**9**, Theorem VI.34] B is saturated in some Seifert fibration of X. So by [**15**, Lemma 6.4] each component b_i of the boundary of B is a meridian of the solid torus $V = S^3 - Q$ with core K. Now since $p > 2$, g is not the identity, and since $g - h^2$, then $g(b_i) = b_i$ for each component b_i of ∂B. But b_i is a meridian of V, so $g(b_i) = b_i$ implies that g fixes a point on k. But $g|k = h^2 = k$ is orientation preserving. Thus g can only fix a point of k if $g|k$ is the identity. But this contradicts the Smith Conjecture [**21**]. So we could not have had such a symmetry h to begin with.

THEOREM 2. *No knots other than torus knots could have infinitely many symmetries with distinct orders.*

Proof. By Lemma 6 if there were an infinitely periodic knot other than a torus knot then either

1) There is a cable knot with companion a torus knot which is infinitely periodic, or

2) There is a composing space X_2 with incompressible boundary components $T_0, \ldots, T_n$, $n \geqq 2$; and X is in the complement Q of a knot k with $\partial Q = T_0$. Further K has a symmetry h of order at least 3 and $h(T_i) = T_i$ for each i.

However, case 1) is ruled out by Theorem 1 and Case 2 is ruled out by Lemma 8. Thus neither case can occur.

THEOREM 3. *The only knots with infinitely many distinct order free-symmetries are torus knots. No non-trivial knots have infinitely many distinct order cyclic-symmetries.*

Proof. Conner [**2**, Theorem 4.3] showed that torus knots have only finitely many cyclic-symmetries. Whereas, Seifert [**16**] proved that torus knots have infinitely many free-symmetries.

Hence, as pointed out in the introduction, we have also shown that no non-trivial knots have infinitely many orientation reversing periodic diffeomorphisms, and further no non-trivial knots have infinitely many periodic diffeomorphisms with non-empty fixed point set.

REFERENCES

1. G. Burde and H. Zieschang, *Eine Kennzeichnung der Torus knoten*, Math. Ann. *167* (1966), 169-176.

2. P. E. Conner, *Transformation groups on a* $K(\pi, 1)$, *II*, Michigan Math. J. *6* (1959), 413-417.

3. R. H. Fox, *On the imbedding of polyhedra in* 3-*space*, Ann. of Math. *49* (1948), 462-470.

4. ——— *Knots and periodic transformations*, Proc. The Univ. of Georgia Inst. (Prentice-Hall, Englewood Cliffs, N.J., 1961), 120-167.

5. M. Freedman, J. Haas and P. Scott, *Lease area incompressible surfaces in* 3-*manifolds*, to appear in Inventiones Mathematicae.

6. C. H. Giffen, *On transformations of the* 2-*sphere fixing in a knot*, Bull. Amer. Math. Soc. *73* (1967), 913-914.

7. R. I. Hartley, *Knots and involutions*, Math. Z. *171*, (1980), 175-185.

8. ——— *Knots with free period*, Can. J. Math. *33* (1981), 91-102.

9. W. Jaco, *Lectures on three-manifold topology*, Memoirs AMS *43* (1980).

10. W. Jaco and P. Shalen, *Seifert fibered spaces in* 3-*manifolds*, Memoirs AMS (1979).

11. K. Johannson, *Homotopy equivalences of* 3-*manifolds with boundaries*, Lecture Notes in Mathematics *761* (1979).

12. W. Meeks, *A survey of the geometric results in the classical theory of minimal surfaces*, Bol. Soc. Bras. Mat. *12* (1981), 29-86.

13. W. Meeks and P. Scott, *Finite group actions on* 3-*manifolds*, preprint.

14. K. Murasugi, *On periodic knots*, Comment. Math. Helv. *46* (1971), 162-174.

15. R. Myers, *Companionship of knots and the Smith conjecture*, Trans. Amer. Math. Soc. *259* (1980), 1-32.

16. H. Seifert, *Topologie dreidimensionalen gefaserter Räume*, Acta Math. *60* (1933), 147-238.

17. J. Simon, *An algebraic classification of knots in* S^3, Ann. of Math. *97* (1973), 1-13.

18. P. A. Smith, *Transformations of finite period II*, Ann. of Math. *40* (1939), 690-711.

19. G. A. Swarup, *P. A. Smith conjecture for cable knots*, Quart. J. Math. Oxford *31* (1980), 105-108.

20. F. Waldhausen, *Gruppen mit Zentrum und dreidimensionale Mannigfaltigkeiten*, Topology *6* (1967), 505-517.

21. Proceedings of the 1979 Conference on the Smith Conjecture at Columbia University, to appear.

Rice University,
Houston, Texas