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PERRON INTEGRABILITY VERSUS LEBESGUE INTEGRABILITY

BY
ARLO W. SCHURLE

ABSTRACT. The paper investigates the relationship between Perron–Stieltjes integrability and Lebesgue–Stieltjes integrability within the generalized Riemann approach. The main result states that with certain restrictions a Perron–Stieltjes integrable function is locally Lebesgue–Stieltjes integrable on an open dense set. This is then applied to show that a nonnegative Perron–Stieltjes integrable function is Lebesgue–Stieltjes integrable. Finally, measure theory is invoked to remove the restrictions in the main result.

1. **Introduction**. This paper is concerned with what has been called the generalized Riemann approach to integration. This approach originated with ideas of Henstock [2], Kurzweil [6] and McShane [8]. The reader should consult these references as well as [3], [4], [7] and [10] for more details than we provide here.

The paper was motivated by the problem of finding a proof that a Perron integrable nonnegative function is Lebesgue integrable directly from the generalized Riemann approach rather than from the classical results. Although we do not provide such a proof here we do present results which clarify the relationships between Perron and Lebesgue integrability and other properties which appear in the generalized Riemann and measure-theoretic approaches. Some of these results are already known in the classical Perron–Lebesgue theory as presented in Saks [11] or Natanson [9], for example, but it is of interest to find generalized Riemann proofs as well as to extend them to the Perron–Stieltjes case. Consequently, only Section 5 of this paper uses measure theory or any of the classical theory of integration. This section then clarifies the connection between the two approaches. See [1] and [5] for other work along these lines.

This problem and in fact this entire area of integration theory was brought to the author's attention in a seminar at the University of Petroleum and Minerals conducted by W. Pfeffer, to whom the author is grateful for several stimulating conversations.

2. **Background and terminology**. Our interest is in Perron–Stieltjes and Lebesque–Stieltjes integrals on real intervals so by an interval J we will always mean a nondegenerate closed bounded interval of real numbers and J^0 will denote its interior. Further all functions will be real-valued.

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By a partition P of an interval J we mean a set $\{I_1, I_2, \ldots, I_n; x_1, x_2, \ldots, x_n\}$ where $I_1, I_2, \ldots, I_n$ are nonoverlapping (their interiors are pairwise disjoint) intervals whose union is J and $x_1, x_2, \ldots, x_n$ are points of J. We call x_i the point of P corresponding to I_i. If each point x_i belongs to its corresponding interval I_i then we say that P is a Perron partition. If δ is a positive function on the interval J then we say that P is δ-fine if I_i is contained in $(x_i - \delta(x_i), x_i + \delta(x_i))$ for each i.

We use α to denote a nondecreasing function on the real numbers which we consider to be fixed throughout the paper. We set $\alpha([a,b]) = \alpha(b) - \alpha(a)$.

Let f be any function on an interval J and let P be a partition of J, say $P = \{I_1, I_2, \ldots, I_n: x_1, x_2, \ldots, x_n\}$. We denote the sum $\sum_{i=1}^{n} f(x_i)\alpha(I_i)$ by $\sigma(f, P)$. If there is a number A such that for each $\epsilon > 0$ there is a positive function δ on J for which $|\sigma(f, P) - A| < \epsilon$ whenever P is a δ-fine partition of J then we say that f is Lebesgue–Stieltjes integrable with respect to α (or just L-integrable) on the interval J. If we require only that $\sigma(f, P)$ is near A for P a δ-fine Perron partition of J, then we say that f is Perron–Stieltjes integrable with respect to α (or just P-integrable) on J. We then denote A by $L\int f\,d\alpha$ in the Lebesgue case and by $P\int f\,d\alpha$ in the Perron case.

Now suppose that f is any functionon an interval J. Then we define the functions f^+ and f^- by

$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) > 0 \\ 0 & \text{if } f(x) \leq 0 \end{cases} \quad \text{and} \quad f^-(x) = \begin{cases} -f(x) & \text{if } f(x) < 0 \\ 0 & \text{if } f(x) \geq 0 \end{cases}$$

and the set F^+ by $F^+ = \{x \mid f(x) \geq 0\}$. Further, the support of f is the set $\operatorname{supp}(f) = \{x \mid \text{every neighborhood of } x \text{ contains a point at which } f \text{ is not zero}\}$.

Finally, if g is any function on an interval J then we say that g has property ∞ on J if for each positive function δ on J the supremum of the sums $\sigma(g, P)$ for P a δ-fine Perron partition of J is $+\infty$.

3. **Statement of main results**. The bulk of the paper is concerned with the relationships between the following statements. (See Section 5 for Statement A.)

Statement B. If the function f is P-integrable with respect to α on the interval J but not L-integrable with respect to α on J, then f^+ has property ∞ on J.

Statement C. If f is P-integrable with respect to α on the interval J, F^+ is a G_δ-set and $O = \{x \mid f \text{ is } L\text{-integrable on a neighborhood of } x \text{ with respect to } \alpha\}$, then $O \cap \operatorname{supp}(f)$ is dense in $\operatorname{supp}(f)$.

Statement D. If f is a nonnegative function which is P-integrable with respect to α on an interval J, then f is L-integrable with respect to α on J.

In the classical theory of these integrals all these statements are true. However, the proofs seems to intimately involve measure theory. Our contribution is to prove the following theorems strictly within the generalized Riemann approach. Though Statement B trivially implies Statement D, we think Statement C has independent interest.

THEOREM 1. *Statement B implies Statement C.*

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THEOREM 2. *Statement C implies Statement D.*

We should remark that in the classical results the condition in Statement C that F^+ be a G_δ-set is not necessary. However, its removal necessitates an excursion into measure theory which it is our intention to avoid. See Section 5.

4. **The Proofs**. First we need two lemmas and then we proceed to proofs of the theorems stated above.

LEMMA 1. *Let f be any function defined on the interval $J = [a,b]$. Let $A = \lim_{t\to a^+} |f(a)|\alpha([a,t])$, $B = \lim_{t\to b^-} |f(b)|\alpha([t,b])$ and $M = A + B + 1$. If δ is a positive function defined on J for which any subinterval K of J satisfies $K^0 \subset \cup\{(x - \delta(x), x + \delta(x))|x \in K^0 \cap F^+\}$ and if P is any δ-fine Perron partition of J, then there is a δ-fine Perron partition P' of J for which $\sigma(f,P') \geq \sigma(f^+,P) - M$.*

PROOF. Let f, J, A, B, M, δ and P be as in the hypothesis. Suppose that $P = \{I_1, I_2, \ldots, I_n;\ x_1, x_2, \ldots, x_n\}$, where the intervals are in increasing order. Let $J_1, J_2, \ldots, J_k$ be the components of $\cup\{I_k|I_k$ is an interval of P with $f(x_k) < 0\}$. We can assume that the J's are also in increasing order. They are certainly pairwise disjoint closed intervals.

We adjust P in and near each J_i so as to obtain the desired partition P'. First, suppose that $J_1 = [a,r]$. Let the interval of P adjacent to J_1 be I_i. (If there is no interval of P adjacent to J_1 the argument is even simpler.) Choose a' such that $a < a' < r$, $a' < a + \delta(a)$ and $|f(a)|\alpha([a,a']) < A + 1/2$. Choose r' such that $a' < r' < r$ and $x_i - \delta(x_i) < r'$. Since $\{(x - \delta(x), x + \delta(x))|x \in J_1 \cap F^+\}$ covers J_1 it is easy to find a δ-fine Perron partition Q of an interval $[a'', r'']$ such that each point of Q lies in F^+ and $a < a'' \leq a' < r' \leq r'' < r$.

Let $I_i' = I_i \cup [r'', r]$. Let Q' be the partition formed by adding I_i' with point x_i to Q and let P' be the partition formed by the first i intervals of P with their corresponding points. Now $\sigma(f^+,P') = f(x_i)\alpha(I_i)$ since the value of f is negative at the first $i - 1$ points of P. By the condition relating a' and A we have that $f(a)\alpha([a,a'']) > -A - 1/2$ and at the other points of Q' the value of f is nonnegative. Consequently $\sigma(f,Q') \geq \sigma(f^+,P') - A - 1/2$. We then replace P' by Q' in the partition P.

Note that an exactly similar argument applies to J_k if $J_k = [w,b]$ as long as we replace A by B.

So we need only consider an interval $J_q = [c,d]$ contained in the interior of J. Suppose the interval of P adjacent to J_q on the left is I_i and the interval of P adjacent to J_q on the right is I_j. Choose c', d' such that $c < c' < x_i + \delta(x_i)$ and $\max\{c', x_j - \delta(x_j)\} < d' < d$. Since $\{(x - \delta(x), x + \delta(x))|x \in (c,d) \cap F^+\}$ covers (c,d), using points from F^+ we can form a δ-fine Perron partition of an interval $[c'', d'']$, where $c < c'' \leq c' < d' \leq d'' < d$. In the partition P replace the intervals $I_{i+1}, \ldots, I_{j-1}$ and their corresponding points by this partition; replace I_i by $I_i \cup [c, c'']$ and I_j by $I_j \cup [d'', d]$, keeping the same points x_i, x_j.

Performing all these replacements transforms P into a new δ-fine Perron partition P'. As we noted, replacing $[a,r]$ decreases $\sigma(f^+,P)$ by at most $A + 1/2$, replacing $[s,b]$

similarly decreases $\sigma(f^+, P)$ by at most $B + 1/2$, while replacing the intervals in the interior of J decreases $\sigma(f^+, P)$ not at all. Further, at each point z of P' the value of f is nonnegative so that $\sigma(f, P') = \sigma(f^+, P') \geq \sigma(f^+, P) - M$.

LEMMA 2. *Let f be any function on the interval I. Suppose that for each positive function δ on I there is a subinterval J of I such that*

(1) f^+ has property ∞ on J and

(2) if K is any subinterval of J then $K^0 \subset \cup\{(x - \delta(x), x + \delta(x)) | x \in K^0 \cap F^+\}$. Then f is not P-integrable on I.

PROOF. Suppose f is Perron integrable on I. Then there is a positive function δ from I into the reals such that whenever P and P' are δ-fine Perron partitions of I then $|\sigma(f, P) - \sigma(f, P')| < 1$ (see [10], pp. 7, 33). Let J be as given in the hypothesis for this δ and then let M be as given in Lemma 1 for this J. Let P be any δ-fine Perron partition of I for which J is a union of intervals of P and denote by $P1$ the partition of J thus induced from P. By condition (1) there is a δ-fine Perron partition $P2$ of J for which $\sigma(f^+, P2) > \sigma(f, P1) + M + 10$. By Lemma 1 there is a δ-fine Perron partition $P3$ of J for which $\sigma(f, P3) \geq \sigma(f^+, P2) - M > \sigma(f, P1) + 10$. If in the partition P we replace the portion $P1$ by $P3$ then we get a δ-fine Perron partition P' of I for which $\sigma(f, P') > \sigma(f, P) + 10$, a contradiction which establishes the lemma.

PROOF OF THEOREM 1. Let L be any interval containing a point of $\operatorname{supp}(f)$. We must show that L contains an interval whose interior contains a point of $\operatorname{supp}(f)$ and on which f is L-integrable. Let δ be any positive function on L. For each positive integer n let $Dn = \{x \in F^+ \cap \operatorname{supp}(f) \cap L | \delta(x) > 1/n\}$. Since both $\operatorname{supp}(f)$ and L are closed and F^+ is a G_δ-set we can apply the Baire Category Theorem to produce an integer m such that $\overline{Dm}$, the closure of Dm, contains a nonempty open subset of $F^+ \cap \operatorname{supp}(f) \cap L$. Let J be an interval with $J \subset L$ and $\phi \neq J^0 \cap F^+ \cap \operatorname{supp}(f) \cap L \subset \overline{Dm}$.

Let K be any subinterval of J. Let $z \in K^0$. If $z \in F^+$ then certainly $z \in (z - \delta(z), z + \delta(z))$. If $z \notin F^+$ then $f(z) < 0$. Take an interval H of length less than $m/2$ for which $z \in H^0 \subset H \subset K^0$. If f is L-integrable on H then we are done. If not, by Statement B f^+ has property ∞ on H, so there is certainly a point x in H^0 with $f(x) > 0$. This point x belongs to $\overline{Dm}$ so there is a point y in $H^0 \cap Dm$. But then $z \in (y - \delta(y), y + \delta(y))$ since $\delta(y) > 1/m$ and $|y - z| < m/2$. Consequently J satisfies condition (2) of Lemma 2.

For each positive function δ on L we have now constructed a subinterval J of L satisfying Condition (2) of Lemma 2. Since f is P-integrable on L, by Lemma 2 on one such J f^+ does not have property ∞. By Statement B f is L-integrable on this J.

PROOF OF THEOREM 2. Since f is nonnegative the set F^+ is certainly a G_δ-set, so Statement C applies. Let f_O be the function that agrees with f on the set O of Statement C and is zero elsewhere. Since O is an increasing union of sets, each of which is a finite union of intervals, and f is locally L-integrable on O, it is easy to construct an increasing sequence of L-integrable functions which converge to f_O. The integrals of these functions are bounded above by the P-integral of f itself so by the Monotone Convergence Theorem (see Pfeffer, [10], pp. 15, 33) f_O is L-integrable and consequently P-integrable.

Let $\bar{f} = f - f_O$. Then $\bar{f}$ is a nonnegative P-integrable function which is 0 on O. If $O_1 = \{x \mid \bar{f}$ is L-integrable on a neighborhood of $x\}$ then by Statement C $O_1 \cap \mathrm{supp}(\bar{f})$ is dense in $\mathrm{supp}(\bar{f})$. If $\mathrm{supp}(\bar{f})$ is not empty, then there is a point x such that $\bar{f}(x) > 0$ and $\bar{f}$ is L-integrable on a neighborhood N of x, which we may take to be an interval. But on N the function $\bar{f}$ and f_O are both L-integrable so that $f = \bar{f} + f_O$ is also L-integrable on N. Hence x must belong to O in which case $\bar{f}(x) = 0$. This contradiction establishes the theorem.

5. **Measure theory and the classical approach**. In the classical theory the function α induces a measure space on the interval in question and in turn this is used to construct the classical Lebesgue–Stieltjes and Perron–Stieltjes integrals with respect to α. To distinguish them from the integrals constructed using the generalized Riemann approach we use the notation CL and CP for the classical integrals. For more information the reader is referred to Saks [11] and Natanson [9]. The relationship between the two approaches rests on the following. We use $\alpha(A)$ to denote the α-measure of a set A.

THEOREM 3. *Let g be the characteristic function of an α-measurable set A in an interval. Then $L\int g\, d\alpha = \alpha(A)$.*

PROOF. Let ϵ be a positive number. There are a closed set K and an open set O such that $K \subset A \subset O$ and $\alpha(A) - \epsilon < \alpha(K) \leq \alpha(O) < \alpha(A) + \epsilon$. Define the function δ on the interval I by

$$\delta(x) = \begin{cases} \frac{1}{2} d(x, I\backslash O) & \text{if } x \in K \\ \min\{\frac{1}{2} d(x, I\backslash O),\ \frac{1}{2} d(x,K)\} & \text{if } x \in O - K \\ \frac{1}{2} d(x,K) & \text{if } x \notin 0 \end{cases}$$

By the definition of δ any interval of a δ-fine partition P which meets K must have its corresponding point in K and any interval which has its corresponding point in A must lie in O. It is then easy to check that $\alpha(A) - \epsilon < \sigma(g, P) < \alpha(A) + \epsilon$. The theorem follows.

We should remark that a proof like the preceding one shows that changing a function's values on a set of α-measure 0 does not affect its integrability. Using this fact and standard measure-theoretic results one can remove the G_δ-set condition in Statement C.

Now consider the following three statements.

Statement A. Any function which is P-integrable with respect to α is α-measurable.

Statement A$'$. Any function which is L-integrable with respect to α is α-measurable.

Statement A$''$. Any function is L-integrable with respect to α if and only if it is CL-integrable with respect to α.

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THEOREM 4. *Statement A implies Statement A′ and Statement A′ is equivalent to Statement A″.*

PROOF. It is easy to see from the definitions that any L-integrable function is P-integrable so the first part of the theorem is trivial. For the second part we use (1) a function f is L (CL) integrable if and only if both f^+ and f^- are L (CL)-integrable, (2) the Monotone Convergence Theorems for both L and CL and (3) Theorem 3 above. For proofs of (1) and (2) in the L-case see Pfeffer [10].

THEOREM 5. *Statement A implies Statement B.*

PROOF. Suppose f is a function which is P-integrable but not L-integrable on an interval I. By Statement A f is α-measurable and hence so are the functions f^+ and f^-. Consequently either $\mathrm{CL}\int f^+ d\alpha = +\infty$ or f^+ is CL-integrable. In the latter case one can show using the fact that f is P-integrable that f^- is also CL-integrable. Hence f would be CL and consequently by Statement A″ L-integrable, a contradiction.

So we have that $\mathrm{CL}\int f^+ d\alpha = +\infty$. Then given any number M there is a simple function $\phi \leq f^+$ such that $\mathrm{CL}\int \phi d\alpha > M$. But by Theorem 3 and the fact that L-integrable implies P-integrable this means that $P\int \phi d\alpha > M$ so for any positive function δ on the interval there must be a δ-fine Perron partition P for which $\sigma(\phi, P) > M$. Since $\phi \leq f^+$ we have $\sigma(f^+, P) > M$ so that f^+ has property ∞ on I.

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DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF PETROLEUM AND MINERALS
DHAHRAN 31261, SAUDI ARABIA

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