BULL. AUSTRAL. MATH. SOC. 05B05
VOL. 33 (1986), 321-327.

A FAMILY OF BALANCED TERNARY DESIGNS WITH BLOCK SIZE FOUR

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This paper shows the existence of an infinite family of cyclic balanced ternary designs where the block size is 4, the index 2 and each block contains precisely one repeated element.

A balanced ternary design (BTD) is a collection of B multisets, called blocks, of size K, chosen from a set of V elements where any element may occur 0, 1 or 2 times in any one block. Furthermore each of the $\binom{V}{2}$ pairs of distinct elements must occur a constant number of times, Λ, (the index). Balanced ternary designs first arose in a paper by Tocher in 1952 [6]. In addition to the above restrictions each element must occur a constant number of times throughout the design. It follows that

(1) $$VR = BK .$$

Let ρ_ℓ denote the number of blocks in which an element occurs ℓ times (ℓ=1 or 2). Then

(2) $$R = \rho_1 + 2\rho_2 ,$$

and

(3) $$\Lambda(V-1) = R(K-1) - 2\rho_2 .$$

Received 12 July 1985. I wish to thank my supervisor Dr. Elizabeth J. Billington for suggesting the problem and for her comments.

Moreover ρ_1 and ρ_2 are constant and the parameters of a BTD can be written $(V,B;\ \rho_1,\rho_2,R;\ K,\Lambda)$.

Since 1952 a number of papers have been written on the existence of BTDs. In particular Saha and Dey [4], Saha [3], Billington [1] and Chandak [2] have produced papers on the existence of cyclic BTDs. Cyclic BTDs arise from a set of initial blocks which when developed modulo V yield all blocks of the design. These initial blocks may be derived from a family of supplementary difference sets. (For a definition, see Wallis, Street and Wallis [7], page 280.)

In this paper we prove the existence of an infinite family of cyclic BTDs with parameters $K = 4$, $\Lambda = 2$ and $B = V\rho_2$. In general, when $K = 4$ and $\Lambda = 2$ it is not possible to have blocks of the form $xxyy$, thus $B \geq V\rho_2$. By taking equation (1) and substituting $R = \frac{BK}{V}$ into equation (3), for $\Lambda = 2$ and $K = 4$ we obtain

$$2V - 2 = 12\frac{B}{V} - 2\rho_2 ,$$

but $\frac{B}{V} \geq \rho_2$ implies $V \geq 5\rho_2 + 1$. Therefore when considering the case $B = V\rho_2$ and fixing ρ_2 we prove the existence of a BTD with $K = 4$, $\Lambda = 2$ and minimal V , that is $V = 5\rho_2 + 1$.

The proof of this result uses a concept of pairing (see for example Stanton and Goulden [5].) We take $3\rho_2$ dots which represent the numbers $\rho_2 + 1$ to $V - (\rho_2+1)$. We draw an edge from the point, say y , to the point $y - x$ and write down the triple $\{x,y-x,y\}$. From this we obtain the initial block $[0,0,x,y]$ which gives rise to the differences $\pm x$, $\pm y$ (each twice) and $\pm(y-x)$. The following example illustrates this method. Consider a BTD with parameters $(46,394;\ 18,9,34;\ 4,2)$. In Figure 1 the 27 dots represent the numbers 10 to 36 :

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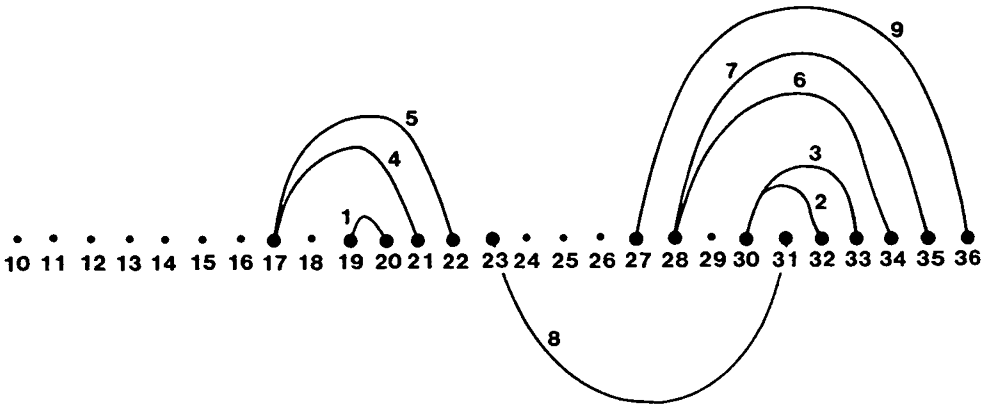

Figure 1.

From Figure 1 we can write down the nine triples and their corresponding initial blocks.

TRIPLES	INITIAL BLOCKS	TRIPLES	INITIAL BLOCKS
1,19,20	[0,0,1,20]	9,27,36	[0,0,9,36]
2,30,32	[0,0,2,32]	3,30,33	[0,0,3,33]
4,17,21	[0,0,4,21]	5,17 22	[0,0,5,22]
6,28,34	[0,0,6,34]	7,28,35	[0,0,7,35]
8,23,31	[0,0,8,31]		

It is easily checked that each of the numbers 1 to 45 occurs twice as a difference arising from the nine initial blocks, and so these blocks when taken modulo 46 generate a BTD with the above parameters.

We now state our main result.

THEOREM. *There exists an infinite family of cyclic BTDs with parameters* $(5\rho_2+1, \rho_2(5\rho_2+1);\ 2\rho_2, \rho_2, 4\rho_2;\ 4,2)$.

Proof. We consider four separate cases.

Case 1: Let $\rho_2 \equiv 0 \pmod 4$, so say $\rho_2 = 4m$ and $V = 20m + 1$ for any positive integer m . Consider the pairing diagram in Figure 2; from it we can write down $4m$ sets of triples and from these $4m$ initial blocks.

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TRIPLES	INITIAL BLOCKS
$1,10m+1,10m+2$	$[0,0,1,10m+2]$
$4m,10m+1,14m+1$	$[0,0,4m,14m+1]$
$4\ell,14m-2\ell+1,14m+2\ell+1$	$[0,0,4\ell,14m+2\ell+1]$ for $1\le\ell\le m-1$
$4\ell+1,14m-2\ell+1,14m+2\ell+2$	$[0,0,4\ell+1,14m+2\ell+2]$ for $1\le\ell\le m-1$
$4\ell+2,8m-2\ell-3,8m+2\ell-1$	$[0,0,4\ell+2,8m+2\ell-1]$ for $0\le\ell\le m-1$
$4\ell+3,8m-2\ell-3,8m-2\ell$	$[0,0,4\ell+3,8m+2\ell]$ for $0\le\ell\le m-1$

It can easily be checked that the numbers 1 to $20m$ occur twice as differences arising from the above initial blocks and so these blocks generate a cyclic BTD with parameters $(20m+1,80m^2+4m;\ 8m,4m,16m;\ 4,2)$.

Similar pairing diagrams, which have been omitted for brevity, yield the initial blocks stated in cases 2, 3 and 4.

Case 2: Let $\rho_2 \equiv 1 \pmod 4$ so that $\rho_2 = 4m+1$ and $V = 20m+6$ for any non-negative integer m . The initial blocks

$$[0,0,4m+1,16m+4]\ ;\ [0,0,1,8m+4]\ ;$$
$$[0,0,4\ell\ ,8m+2\ell+3]\ ,\ \text{for } 1\le\ell\le m-1;$$
$$[0,0,4\ell+1,8m+2\ell+4]\ ,\ \text{for } 1\le\ell\le m-1;$$
$$[0,0,4\ell+2,14m+2\ell+4]\ ,\ \text{for } 0\le\ell\le m-1;$$
$$[0,0,4\ell+3,14m+2\ell+5]\ ,\ \text{for } 0\le\ell\le m-1;$$
$$[0,0,4m,14m+3]\ ;$$

generate a cyclic BTD with parameters

$(20m+6,80m^2+44m+6;\ 8m+2,4m+1,16m+4;\ 4,2)$ for all non-negative integers m .

Case 3: Let $\rho_2 \equiv 2 \pmod 4$ so that $\rho_2 = 4m+2$ and $V = 20m+11$ for any non-negative integer m . If $m = 0$ the initial blocks $[0,0,1,7]$ and $[0,0,2,8]$ generate a cyclic BTD with parameters $(11,22;\ 4,2,8;\ 4,2)$. If $m = 1$ the initial blocks $[0,0,1,17]$, $[0,0,2,20]$, $[0,0,3,21]$, $[0,0,4,23]$, $[0,0,5,24]$ and $[0,0,6,22]$ generate a cyclic BTD with parameters $(31,186;\ 12,6,24;\ 4,2)$. If $m \ge 2$ the initial blocks

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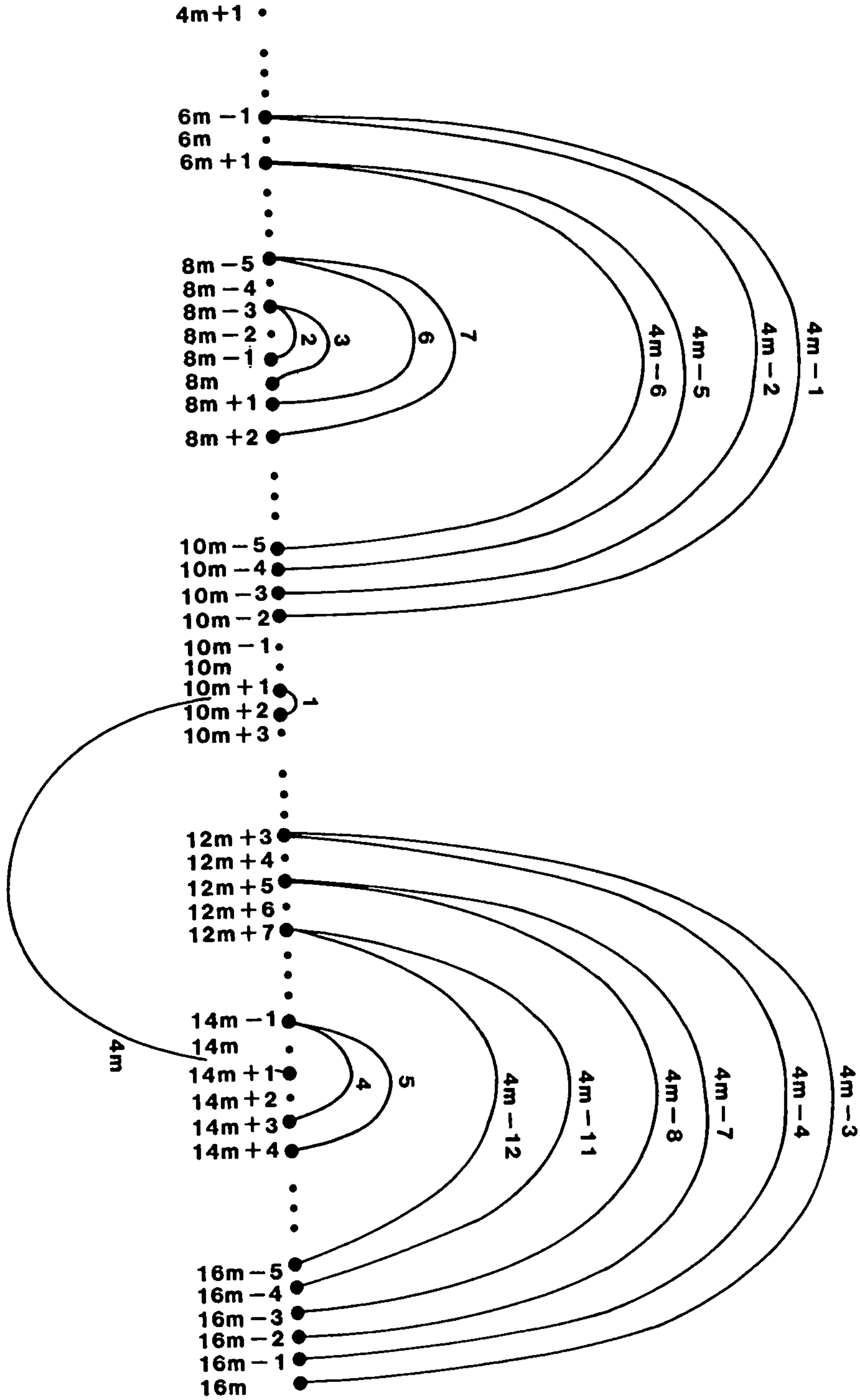
4m+1
6m − 1
6m
6m + 1
8m − 5
8m − 4
8m − 3
8m − 2
8m − 1
8m
8m + 1
8m + 2
10m − 5
10m − 4
10m − 3
10m − 2
10m − 1
10m
10m + 1
10m + 2
10m + 3
12m + 3
12m + 4
12m + 5
12m + 6
12m + 7
14m − 1
14m
14m + 1
14m + 2
14m + 3
14m + 4
16m − 5
16m − 4
16m − 3
16m − 2
16m − 1
16m
1
2
3
4
5
6
7
4m
4m − 1
4m − 2
4m − 3
4m − 4
4m − 5
4m − 6
4m − 7
4m − 8
4m − 11
4m − 12

Figure 2.

$$[0,0,4m+2,14m+8]\ ;\ [0,0,1,10m+7]\ ;$$
$$[0,0,4\ell,14m+2\ell+7]\ ,\quad \text{for } 1\le\ell\le m\ ;$$
$$[0,0,4\ell+1,14m+2\ell+8]\ ,\ \text{for } 1\le\ell\le m\ ;$$
$$[0,0,4\ell+2,8m+2\ell+5]\ ,\quad \text{for } 0\le\ell\le m-2\ ;$$
$$[0,0,4\ell+3,8m+2\ell+6]\ ,\quad \text{for } 0\le\ell\le m-2\ ;$$
$$[0,0,4m-2,14m+6]\ ;\ [0,0,4m-1,14m+7]\ ;$$

generate a cyclic BTD with parameters

$(20m+11, 80m^2+84m+22;\ 8m+4, 4m+2, 16m+8;\ 4,2)$ for $m \ge 2$.

Case 4: Let $\rho_2 \equiv 3 \pmod 4$ and so $\rho_2 = 4m + 3$ and $V = 20m + 16$ for any non-negative integer m . The initial blocks

$$[0,0,4m+2,14m+12]\ ;\ [0,0,1,10m+7]\ ;$$
$$[0,0,4\ell,14m+2\ell+11]\ ,\quad \text{for } 1\le\ell\le m\ ;$$
$$[0,0,4\ell+1,14m+2\ell+12]\ ,\ \text{for } 1\le\ell\le m\ ;$$
$$[0,0,4\ell+2,8m+2\ell+6]\ ,\quad \text{for } 0\le\ell\le m-1\ ;$$
$$[0,0,4\ell+3,8m+2\ell+7]\ ,\quad \text{for } 0\le\ell\le m-1\ ;$$
$$[0,0,4m+3,14m+11]\ ;$$

generate a cyclic BTD with parameters

$$(20m+16, 80m^2+124m+48;\ 8m+6, 4m+3, 16m+12;\ 4,2)\ .$$

This completes the proof. □

References

[1] Elizabeth J. Billington, "Balanced n-ary designs: A combinatorial survey and some new results" *Ars Combinatoria* 17A (1984), 37-72.

[2] M.L. Chandak, "On construction of balanced n-ary designs", *Acta Ciencia Indica* VII(m) No 3 (1981), 205-208.

[3] G.M. Saha, "On construction of balanced ternary designs", *Sankhya* 37(B) (1975), 220-227.

[4] G.M. Saha and A. Dey, "On construction and uses of balanced n-ary designs", *Ann. Inst. Statist. Math.* 25 (1973), 439-445.

[5] R.G. Stanton and I.P. Goulden, "Graph factorization, general triple systems and cyclic triple systems", *Aequationes Math.* 22 (1981), 1-28.

[6] K.D. Tocher, "The design and analysis of block experiments", *J. Roy. Statist. Soc. Ser. B* 14 (1952), 45-100.

[7] W.D. Wallis, Anne Penfold Street and Jennifer Seberry Wallis, *Combinatorics: Room squares, sum-free sets, Hadamard matrices* (Lecture Notes in Mathematics, 292. Springer-Verlag, Berlin, Heidelberg, New York, 1972).

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https://doi.org/10.1017/S0004972700003889 Published online by Cambridge University Press