

# DO AVERAGE HAMILTONIANS EXIST?

S. FERRAZ-MELLO

*Instituto Astronômico e Geofísico, Universidade de São Paulo, Caixa Postal 3386, São Paulo, SP, Brasil, E-mail: sylvio@usp.br*

**Abstract.** The word “average” and its variations became popular in the sixties and implicitly carried the idea that “averaging” methods lead to “average” Hamiltonians. However, given the Hamiltonian  $H = H_0(J) + \epsilon R(\theta, J)$ , ( $\epsilon \ll 1$ ), the problem of transforming it into a new Hamiltonian  $H^*(J^*)$  (dependent only on the new actions  $J^*$ ), through a canonical transformation given by zero-average trigonometrical series has no general solution at orders higher than the first.

## 1. Introduction

Hamiltonian perturbation theories reached their apex in 1954, when one of their many versions was used by Kolmogorov for the construction of solutions of a perturbed Hamiltonian system. Hamiltonian perturbation theories are theories seeking for a canonical transformation able to transform the given Hamiltonian system into another one whose energy depends only on the new actions (i.e., a Hamiltonian independent of angles). They are used in Celestial Mechanics since the XIX<sup>th</sup> century. Examples are Delaunay’s theory of the motion of the Moon, the theory called “Lindstedt method” by Poincaré, and the Lie-series methods, introduced by Born and Hori in the study of Quantum Mechanics and Celestial Mechanics, respectively (Delaunay, 1868; Poincaré, 1893; Charlier, 1907; Born, 1926; Kolmogorov, 1954; Brouwer, 1959; Hori, 1966; Deprit, 1969).

The more ancient methods look for a classical Jacobian generating function  $S(\theta_i, J_i^*)$ , of the old angles  $\theta_i$  and new actions (or momenta)  $J_i^*$ , and the canonical transformation is written as

$$\theta_i^* = \frac{\partial S}{\partial J_i^*} \quad J_i = \frac{\partial S}{\partial \theta_i} \quad (i = 1, 2, \dots, N). \quad (1)$$

The more recent Lie-series methods look for a function  $W(\theta_i^*, J_i^*)$ , of the new angles and actions (the use of action-angle variables is not necessary, but we adopt them for sake of simplicity), and the canonical transformation is written, using Lie series, as

$$\theta_i = E_W \theta_i^* \quad J_i = E_W J_i^* \quad (\text{for each } i). \quad (2)$$

The Lie series  $E_W \phi^*$  of a given function  $\phi(\theta_i, J_i)$  is defined by

$$E_W \phi^* = \phi(\theta_i^*, J_i^*) + \{\phi, W\} + \frac{1}{2!} \{\{\phi, W\}, W\} + \frac{1}{3!} \{\{\{\phi, W\}, W\}, W\} + \dots \quad (3)$$

where  $\{, \}$  denote Poisson brackets. In both cases, the generating functions ( $S$  or  $W$ ) are periodic functions of the angles  $\theta_1, \theta_2, \dots, \theta_N$  (or  $\theta_1^*, \theta_2^*, \dots, \theta_N^*$ ) and are seek as zero-average Fourier series in these angles with coefficients that depend



only on the actions. One problem is considered as solved (at a given order) when a zero-average generating function ( $S$  or  $W$ ) and the resulting “average” Hamiltonian  $H^*(J_i^*)$  are found.

The word “average”, and its variations, became popular in the sixties and implicitly carried the idea that “averaging” methods lead to “average” Hamiltonians governing the secular variation of the given system. However, in at least one instance (Milani *et al.* 1987), the inverse transformation was explicitly calculated, to obtain the asymptotic (or formal) solutions of the given Hamiltonian and, for general disappointment, new non zero-average terms appeared in the solution! This is obvious when canonical transformations defined by generating functions are used. Indeed, in the case of methods using Lie Series, a glance at eqn. 3 is enough to see that even if  $W$  is a pure zero-average trigonometric series, the series terms of order 2, and higher, will involve products of derivatives of  $W$  among them, and constant terms will be generated. In the case of methods using Jacobian generating functions, the solutions given by eqns. 1 are in mixed form. To get them explicitly, say  $\theta_i = \theta_i(\theta_i^*, J_i^*)$ ,  $J_i = J_i(\theta_i^*, J_i^*)$ , we have to use an inversion procedure, and any procedure will involve products of derivatives of  $S$ , thus leading to constant terms.

The above discussed drawbacks show that  $H^*$  is not an “average”. The actual solutions oscillate about the solutions of the Hamiltonian system defined by  $H^*$ , but with a non-zero average. This fact does not invalidate the classical perturbation theories (the zero average is not a necessary condition for their validity). It only conducts to some interpretation problems in Celestial Mechanics. The classical “secular theory” of Laplace and Lagrange is the construction of a first-order average Hamiltonian and the analysis of its solutions. The same is done for asteroids and serves to define “proper elements”. However, proper elements are not “average” elements: second-order proper elements differ from mean elements by second-order quantities.

In the following, we show that, in general, even if generating functions are not used, given the Hamiltonian  $H = H_0(J) + \epsilon R(\theta, J)$ , ( $\epsilon \ll 1$ ), it is not possible to transform it into a new Hamiltonian  $H^*(J^*)$  (dependent only on the new actions  $J^*$ ), through a canonical transformation given by zero-average trigonometrical series.

## 2. Perturbation Theory with a Direct Canonical Transformation

Let us consider an N-degrees of freedom, non degenerate, integrable Hamiltonian  $H_0(J)$ , a perturbation  $H_1 = \epsilon R(\theta, J)$ , ( $|\epsilon| \ll 1$ ), and one canonical transformation  $(\theta, J) \Rightarrow (\theta^*, J^*)$  defined explicitly through

$$\begin{aligned} \theta_i &= \theta_i^* + Q_1^i(\theta^*, J^*) + Q_2^i(\theta^*, J^*) + \dots \\ J_i &= J_i^* + P_1^i(\theta^*, J^*) + P_2^i(\theta^*, J^*) + \dots, \end{aligned} \tag{4}$$

where the functions  $Q_k^i(\theta^*, J^*)$ ,  $P_k^i(\theta^*, J^*)$  are zero-average Fourier series in  $\theta$ , of order  $\mathcal{O}(\epsilon^k)$ .

Let us adopt, for the canonical condition, the invariance of Poisson brackets:  $\{x, y\} = \{x^*, y^*\}$ ,  $x, y$  being any two of the canonical variables  $\theta_i, J_i$ . From eqns. 4, there follows:

$$\begin{aligned} \{x, y\} = & \{x^*, y^*\} + \{x^*, Y_1\} + \{X_1, y^*\} + \{x^*, Y_2\} + \{X_1, Y_1\} + \{X_2, y^*\} \\ & + \{x^*, Y_3\} + \{X_1, Y_2\} + \{X_2, Y_1\} + \{X_3, y^*\} + \dots \end{aligned}$$

where the letters  $X$  and  $Y$  were used instead of  $P^i, Q^i$ , since  $x$  and  $y$  can be any of the  $2N$  canonical variables; their meanings are immediate. Then, because of the canonical condition,

$$\begin{aligned} 0 = & \{x^*, Y_1\} + \{X_1, y^*\} + \{x^*, Y_2\} + \{X_1, Y_1\} + \{X_2, y^*\} \\ & + \{x^*, Y_3\} + \{X_1, Y_2\} + \{X_2, Y_1\} + \{X_3, y^*\} + \dots \end{aligned} \quad (5)$$

If we assume that the above equation is satisfied identically in  $\epsilon$ , it decomposes itself into

$$\begin{aligned} \{x^*, Y_1\} + \{X_1, y^*\} &= 0 & \mathcal{O}(\epsilon^1) \\ \{x^*, Y_2\} + \{X_1, Y_1\} + \{X_2, y^*\} &= 0 & \mathcal{O}(\epsilon^2) \\ \{x^*, Y_3\} + \{X_1, Y_2\} + \{X_2, Y_1\} + \{X_3, y^*\} &= 0 & \mathcal{O}(\epsilon^3) \\ \dots\dots\dots \end{aligned} \quad (6)$$

The generic equation, at order  $\mathcal{O}(\epsilon^k)$ , is

$$\{X_k, y^*\} = \{Y_k, x^*\} - \Gamma_{k,x,y}, \quad (7)$$

where  $\Gamma_{k,x,y}$  represents a known function of  $X_1, \dots, X_{k-1}, Y_1, \dots, Y_{k-1}$ .

Let us, now, write  $H_0$  and  $H_1$  in terms of the transformed variables. Limiting ourselves to the Taylor second-order terms, we have

$$H_0(J) = H_0(J^*) + \sum_i \nu_i P_1^i + \sum_i \nu_i P_2^i + \frac{1}{2} \sum_{i,j} \nu_{ij} P_1^i P_1^j + \mathcal{O}(\epsilon^3), \quad (8)$$

where

$$\nu_i = \frac{\partial H_0(J^*)}{\partial J_i^*}, \quad \nu_{ij} = \frac{\partial^2 H_0(J^*)}{\partial J_i^* \partial J_j^*}; \quad (9)$$

and

$$H_1(\theta, J) = H_1(\theta^*, J^*) + \sum_i \frac{\partial H_1(\theta^*, J^*)}{\partial J_i^*} P_1^i + \sum_i \frac{\partial H_1(\theta^*, J^*)}{\partial \theta_i^*} Q_1^i + \mathcal{O}(\epsilon^3) \quad (10)$$

The sequence follows the same steps of other Hamiltonian perturbation theories. We substitute the above expansions in the law of conservation of the Hamiltonian under a time-independent canonical transformation:

$$H(\theta, J) = H^*(\theta^*, J^*) \equiv H_0^* + H_1^* + H_2^* + \dots; \tag{11}$$

and identify in the powers of  $\epsilon$ . Then,

$$\begin{aligned} H_0^* &= H_0(J^*), \\ H_1^* &= \sum_i \nu_i P_1^i + H_1(\theta^*, J^*), \end{aligned} \tag{12}$$

$$H_2^* = \sum_i \nu_i P_2^i + \frac{1}{2} \sum_{i,j} \nu_{ij} P_1^i P_1^j + \sum_i \frac{\partial H_1}{\partial J_i} P_1^i + \sum_i \frac{\partial H_1}{\partial \theta_i} Q_1^i,$$

etc.

The above equations may be compacted in the *homological equation*

$$\sum_i \nu_i P_k^i = H_k^* - \Psi_k(\theta^*, J^*) \tag{13}$$

for all  $k \geq 1$ . In all cases, the function  $\Psi_k(\theta^*, J^*)$  is independent of  $P_k$  and is known if the equations for the previous subscripts were solved.

The homological equation has  $N$  unknowns  $P_k^i$ . Its indeterminacy is, however, only apparent, since the  $P_i$  must obey at the corresponding canonical condition given by eqn. 7. Let us transform eqn. 13 by composing it with the  $2N$  canonical variables, in Poisson brackets:

$$\begin{aligned} \sum_i \{\nu_i P_k^i, J_j^*\} &= \{H_k^*, J_j^*\} - \{\Psi_k, J_j^*\} \\ \sum_i \{\nu_i P_k^i, \theta_j^*\} &= \{H_k^*, \theta_j^*\} - \{\Psi_k, \theta_j^*\} \end{aligned} \tag{14}$$

or, decomposing the left-hand sides brackets,

$$\begin{aligned} \sum_i \nu_i \{P_k^i, J_j^*\} &= \{H_k^*, J_j^*\} - \{\Psi_k, J_j^*\} \\ \sum_i \nu_i \{P_k^i, \theta_j^*\} + \sum_i P_k^i \{\nu_i, \theta_j^*\} &= \{H_k^*, \theta_j^*\} - \{\Psi_k, \theta_j^*\} \end{aligned} \tag{15}$$

where we did take into account that  $\{\nu_i, J_j^*\} = 0$  because  $\nu_i$  is independent on the angles. We may, now, use eqn. 7, and transform the above set into

$$\begin{aligned} \sum_i \nu_i \{P_k^i, J_j^*\} + \sum_i \nu_i \Gamma_{k, J_i, J_j} &= \{H_k^*, J_j^*\} - \{\Psi_k, J_j^*\} \\ \sum_i \nu_i \{Q_k^i, J_j^*\} + \sum_i \nu_i \Gamma_{k, J_i, \theta_j} + \sum_i P_k^i \{\nu_i, \theta_j^*\} &= \{H_k^*, \theta_j^*\} - \{\Psi_k, \theta_j^*\}. \end{aligned} \tag{16}$$

After the computation of some elementary brackets, we obtain the homological system of equations:

$$\sum_i \nu_i \frac{\partial P_k^i}{\partial \theta_i^*} = \frac{\partial H_k^*}{\partial \theta_j^*} - \frac{\partial \Psi_k}{\partial \theta_j^*} - \sum_i \nu_i \Gamma_{k, J_i, J_j}, \tag{17}$$

$$\sum_i \nu_i \frac{\partial Q_k^i}{\partial \theta_i^*} = \sum_i \nu_{ij} P_k^i - \frac{\partial H_k^*}{\partial J_j^*} + \frac{\partial \Psi_k}{\partial J_j^*} - \sum_i \nu_i \Gamma_{k, J_i, \theta_j}. \tag{18}$$

### 3. Non-Existence of Average Hamiltonians

Let us consider, in this section, the question title of this article, and search for solutions of eqns. 17 and 18 such that  $\langle P_k^i \rangle = \langle Q_k^i \rangle = 0$ . The condition for the existence of such solutions is that the right-hand sides of the two equations have zero averages, that is,

$$\left\langle \frac{\partial H_k^*}{\partial \theta_j^*} \right\rangle - \sum_i \nu_i \langle \Gamma_{k, J_i, J_j} \rangle = 0 \quad (19)$$

and

$$- \left\langle \frac{\partial H_k^*}{\partial J_j^*} \right\rangle + \frac{\partial \langle \Psi_k \rangle}{\partial J_j^*} - \sum_i \nu_i \langle \Gamma_{k, J_i, \theta_j} \rangle = 0 \quad (20)$$

These equations show that, for  $k > 1$ , it is not possible to find a solution of the given problem such that we have simultaneously (for all  $i$ )  $\langle P_k^i \rangle = 0$ ,  $\langle Q_k^i \rangle = 0$ , and  $H_k^*$  independent of  $\theta^*$ . Indeed, in this case,  $\left\langle \frac{\partial H_k^*}{\partial \theta_j^*} \right\rangle = 0$ , and eqn. 19 can only be generally satisfied when  $\langle \Gamma_{k, J_i, J_j} \rangle = 0$ , what is true, in general, only if  $k = 1$ .

#### 3.1. FIRST-ORDER AVERAGE HAMILTONIAN

For  $k = 1$ , since  $\Gamma_{1, J_i, J_j} = 0$ , eqns. 19 and 20 become

$$\left\langle \frac{\partial H_1^*}{\partial \theta_j^*} \right\rangle = 0 \quad (21)$$

and

$$- \left\langle \frac{\partial H_1^*}{\partial J_j^*} \right\rangle + \frac{\partial \langle H_1 \rangle}{\partial J_j^*} = 0 \quad (22)$$

which have the trivial solution  $H_1^* = \langle H_1 \rangle$ . It is worth recalling that, to this order, the methods using the generating functions ( $S$  or  $W$ ) also give this same result.

The calculations were done following a constructive scheme, but it is easy to make a verification using a reversed reasoning and prove that, indeed, this solution satisfies the condition given by the first of eqns. 6, for the components of  $P_1$ .

This result means that it is possible to obtain a first-order “average” Hamiltonian. This fact certainly played a role in the introduction of the word “average” and its variations in the study of the construction of asymptotic (or formal) solutions of perturbed systems.

#### 4. Conclusion

The conclusion of the above sections is the following: Given the Hamiltonian  $H = H_0(J) + \epsilon R(\theta, J)$ , ( $\epsilon \ll 1$ ), the problem of transforming it into a new Hamiltonian  $H^*(J^*)$  (dependent only on the new actions  $J^*$ ), through a canonical transformation given by zero-average trigonometrical series has no general solution at orders higher than the first. It is worth mentioning that a general solution cannot be found even in the particular case, usual in Celestial Mechanics, in which the disturbing potential  $R(\theta, J)$  is a cosine series.

#### Acknowledgements

This investigation was supported by CNPq.

#### References

- Born, M.: 1926, *Problems of Atomic Dynamics*, M.I.T., Cambridge (US).
- Brouwer, D.: 1959, Solution of the problem of artificial satellite theory without drag, *Astron. J.*, **64**, 378-397.
- Charlier, C.V.L.: 1907, *Die Mechanik des Himmels*, De Gruyter, Leipzig, Vol. II.
- Delaunay, C.: 1868, *Mémoire sur la Théorie de la Lune*, Acad. Sc., Paris.
- Deprit, A.: 1969, Canonical transformation depending on a small parameter, *Cel. Mech. & Dyn. Astr.*, **1**, 12-30.
- Hori, G.-I.: 1966, Theory of General Perturbations with Unspecified Canonical Variables, *Publ. Astron. Soc. Japan*, **18**, 287-296.
- Kolmogorov, A.N.: 1954, Preservation of conditionally periodic movements with small change in Hamiltonian function, *Dokl. Akad. Nauk*, **98**, 527-530.
- Milani, A.; Nobili, A.M. and Carpino. M.: 1987, Secular variations of the semimajor axes: theory and experiments, *Astron. Astrophys.*, **172**, 265-279.
- Poincaré, H.: 1893, *Les Méthodes Nouvelles de la Mécanique Celeste*, Gauthier-Villars, Paris, Vol. II.