ANGULAR MEASURE AND INTEGRAL CURVATURE

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THE Gauss-Bonnet Theorem leads through well known arguments to the fact that the integral curvature[1] of a two-dimensional closed orientable manifold M of genus p equals $4\pi(1 - p)$. This implies, for instance, that the Gauss curvature[1] K can neither be everywhere positive nor everywhere negative, if M is homeomorphic to a torus.

The relations between the sign of K and the topological structure of M have been the subject of many investigations. Those of Cohn-Vossen [4, 5] are particularly interesting, because they are not restricted to closed manifolds.

Hadamard [6] showed that the condition $K < 0$ determines to a great extent the shape of the geodesics (on closed or open manifolds). The already mentioned papers of Cohn-Vossen show also how the condition $K > 0$ influences the behaviour of the geodesics.

All these investigations rest on the Gauss-Bonnet Theorem, which states in its most primitive form that the integral curvature of a geodesic triangle equals the spherical excess of the triangle. *Thus they depend ultimately on the concept of angular measure.* This concept is in turn derived from the local, that is the Euclidean geometry, where it means amount of rotation.

The Minkowskian geometry is the local geometry of non-Riemannian metric spaces. It does not permit general rotations. If the distance is symmetric, which will always be assumed here, the Minkowskian geometry permits reflection in a point, which in the Euclidean case is equivalent to rotation through π. Therefore no particular angular measure can be entirely natural in Minkowskian geometry. This is evidenced by the innumerable attempts to define such a measure, none of which found general acceptance.

Of course, it is generally agreed that *angular measure must be additive for angles with the same vertex.* In view of our previous observation, it is natural to add the requirement that *straight angles have measure π.* It will be shown here that *any angular measure with these two properties permits us to establish for general spaces most of the above quoted results on Riemann spaces,* provided we interpret conditions like $K > 0$ on M to mean that every non-degenerate small geodesic triangle on M has positive spherical excess. For some results it is necessary to add a condition, which is always satisfied by the ordinary angles

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[1]The English expression "total curvature" corresponds to the German "Gauss'sche Krümmung," whereas the German expression "Totalkrümmung" is "integral curvature" in English. In order not to confuse the reader interested in the original literature, the present paper avoids "total" altogether by using "Gauss curvature" corresponding to the German, and "integral curvature" corresponding to the English custom.

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in Riemann spaces, and which states essentially that, in a uniform way, an angle cannot be nearly straight without having a measure close to π.

The main point of the present paper is the tenet that *angular measure in Finsler spaces is*—contrary to the prevailing views—*a very fruitful concept, and that it becomes unnatural and barren only through insistence on particular measures.*

The extent of the material in the Riemannian case precluded its full discussion here. Except for a glance at the connection of excess with the theory of parallels (Sec. **2**) and the topological structure of compact manifolds (Sec. **3**) the paper concentrates on the work of Cohn-Vossen, whose arguments are partly reproduced here.

Hadamard's results are only briefly touched because the author showed recently in [3], although not in connection with angular measure, that they do not depend on the Riemannian character of the metric.

1. Angles in systems of plane curves. In the Euclidean plane E with the (Euclidean) distance xy let S be a system of curves with the following properties:

I. *Each curve is an open Jordan curve, that is, it has a representation* $q(t)$, $-\infty < t < \infty$ *where* $q(t)$ *is continuous and* $q(t_1) \neq q(t_2)$ *when* $t_1 \neq t_2$.

II. $q(t)q(0) \to \infty$ *when* $|t| \to \infty$.

III. *Any two distinct points of* E *lie on exactly one curve of* S.

The curve in S (S-curve) determined by the two points a, b will be denoted by $\mathfrak{g}(a, b)$. On $\mathfrak{g}(a, b)$ the points a, b bound an arc $\mathfrak{t}(a, b)$. The symbol (acb) means that a, b, c are three different points and that c lies on $\mathfrak{t}(a, b)$. We also put $\mathfrak{t}(a, a) = a$.

The S-curves satisfy all the axioms of order and connection of Hilbert, in particular the axiom of Pasch.[2] In addition, $a_\nu \to a$ and $b_\nu \to b$ implies $\mathfrak{t}(a_\nu, b_\nu) \to \mathfrak{t}(a, b)$ and, if $a \neq b$, also $\mathfrak{g}(a_\nu, b_\nu) \to \mathfrak{g}(a, b)$. The arrow indicates here that $\mathfrak{t}(a, b)$ or $\mathfrak{g}(a, b)$ is Hausdorff's closed limit of the sets $\mathfrak{t}(a_\nu, b_\nu)$ or $\mathfrak{g}(a_\nu, b_\nu)$.[2] A point p of an S-curve $\mathfrak{g}$ divides $\mathfrak{g}$ into two (closed) rays $\mathfrak{r}_1$, $\mathfrak{r}_2$, which we call opposite.

If $\mathfrak{r}_1$ and $\mathfrak{r}_2$ are two different rays with the same origin p, then $\mathfrak{r}_1 \cup \mathfrak{r}_2$ divides E into two (closed) domains D_1 and D_2. The sets of all rays with origin p in D_1 and D_2 respectively are *the two angles with legs* $\mathfrak{r}_1$ *and* $\mathfrak{r}_2$. They are called straight if $\mathfrak{r}_1$ and $\mathfrak{r}_2$ are opposite. Otherwise exactly one of the domains is S-convex[3] and we call the corresponding angle the convex angle $(\mathfrak{r}_1, \mathfrak{r}_2)^{\text{vex}}$, and the other the concave angle $(\mathfrak{r}_1, \mathfrak{r}_2)^{\text{cav}}$. It is convenient to complete this definition by letting $(\mathfrak{r}_1, \mathfrak{r}_1)^{\text{vex}}$ mean the set consisting of $\mathfrak{r}_1$ alone and $(\mathfrak{r}_1, \mathfrak{r}_1)^{\text{cav}}$ the set of all rays with origin p. If a, b, c are three points not on one S-curve, then $\angle\, abc$ means the convex angle whose legs are the rays from b through a and c.

[2]Proofs are found in [1, Sec. III.3].

[3]A set X is S-convex if $a, b \in X$ implies $\mathfrak{t}(a, b) \subset X$. Compare [1, Sec. III.3].

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We now assume that an *angular measure* $|D|$ has been defined for the angles D in S with the following properties:

1) $|D| \geqslant 0$.

2) $|D| = \pi$ *if and only if* D *is straight.*

3) If D_1 and D_2 *are two angles with a common leg but with no other common ray, then* $|D_1 \cup D_2| = |D_1| + |D_2|$.

We say that the angle D_ν tends to the angle D, if the legs of D_ν tend to the legs of D, and if $\mathfrak{r}_\nu \epsilon D_\nu$ and $\mathfrak{r}_\nu \rightarrow \mathfrak{r}$ implies $\mathfrak{r} \epsilon D$. We call the angular metric *continuous* if

4) $D_\nu \rightarrow D$ *implies* $|D_\nu| \rightarrow |D|$.

Some consequences of 1), 2), 3) are

a) $|D| = 0$ *if and only if the legs of* D *coincide.*

For if the legs $\mathfrak{r}_1$, $\mathfrak{r}_2$ of D coincide, let D' denote one of the two straight angles with $\mathfrak{r}_1 = \mathfrak{r}_2$ as one leg. Then $D \cup D' = D'$, therefore by 2) and 3)

$$\pi = |D \cup D'| = |D| + |D'| = |D| + \pi,$$

so that $|D| = 0$. Conversely let $|D| = 0$. Its legs $\mathfrak{r}_1$ and $\mathfrak{r}_2$ cannot be opposite by 2). Denote the opposite ray to $\mathfrak{r}_1$ by $\mathfrak{r}_3$. If $\mathfrak{r}_1$ and $\mathfrak{r}_2$ did not coincide and $D = (\mathfrak{r}_1, \mathfrak{r}_2)^{\text{cav}}$ then by 1), 2), 3) $|D| = \pi + |(\mathfrak{r}_3, \mathfrak{r}_2)^{\text{vex}}| > \pi$. If $D = (\mathfrak{r}_1, \mathfrak{r}_2)^{\text{vex}}$ then $\pi = |D| + |(\mathfrak{r}_2, \mathfrak{r}_3)^{\text{vex}}| = |(\mathfrak{r}_2, \mathfrak{r}_3)^{\text{vex}}|$ although $(\mathfrak{r}_2, \mathfrak{r}_3)^{\text{vex}}$ is not straight.

b) Convex angles have measure less than π and conversely.

Concave angles have measure greater than π and conversely.

c) Vertical angles are equal.

d) The sum of the measures of the angles in a triangle abc (set bounded by three segments $\mathfrak{t}(a, b)$, $\mathfrak{t}(b, c)$, $\mathfrak{t}(c, a)$, where a, b, c are not on one S-curve) is positive and less than 3π.

e) *If the angular metric is continuous and the points* a_ν, b_ν, c_ν *are not on an* S*-curve and tend to a point* p, *then the sum of the angular measures in the triangle* $a_\nu b_\nu c_\nu$ *tends to* π.

A proof follows immediately from the observation that $\mathfrak{g}(a_\nu, b_\nu)$, $\mathfrak{g}(b_\nu, c_\nu)$ and $\mathfrak{g}(c_\nu, a_\nu)$ may be assumed to converge. Then $\angle\, b_\nu a_\nu c_\nu$ and the vertical angles to $\angle\, a_\nu b_\nu c_\nu$ and $\angle\, a_\nu c_\nu b_\nu$ tend to three angles whose union is a straight angle.

Some of the preceding remarks extend in the usual way to degenerate triangles and will be used for such triangles.

The excess $\epsilon(abc)$ of the triangle abc is defined as

$$\epsilon(abc) = |abc| + |bca| + |cab| - \pi, \tag{1}$$

where $|abc| = |\angle\, abc|$.

Degenerate triangles have excess 0. If the triangle abc is decomposed (simplicially by S-curves) into the triangles $a_\nu b_\nu c_\nu$ then

$$\epsilon(abc) = \Sigma \epsilon(a_\nu b_\nu c_\nu).$$

If $a_1, \ldots, a_n$ are the vertices of a simple closed polygon P with sides $\mathfrak{t}(a_i, a_{i+1})$ and α_i is the measure of the angle at a_i measured inside the closed[4]

[4]In this paper domains bounded by geodesic polygons are always understood to be closed.

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domain G bounded by P then for any simplicial subdivision of G (by S-curves is always understood) into triangles $a_\nu b_\nu c_\nu$

(2) $$\epsilon(a_\nu b_\nu c_\nu) = 2\pi - \Sigma(\pi - \alpha_i) = \Sigma\alpha_i - (n-2)\pi.$$

Let $\mathfrak{r}_1$ and $\mathfrak{r}_2$ be two opposite rays with origin p determining the two straight angles D_1 and D_2. If $a_i \epsilon \mathfrak{r}_i$, $a_i \neq p$ and $q \epsilon D_1 - (\mathfrak{r}_1 \cup \mathfrak{r}_2)$ then a ray $\mathfrak{r}_x$ with origin p and through a point $x \epsilon \mathfrak{t}(a_1, q) \cup \mathfrak{t}(q, a_2)$ traverses monotonically all rays in D_1 as x traverses $\mathfrak{t}(a_1, q) + \mathfrak{t}(q, a_2)$. Therefore $|(\mathfrak{r}_1, \mathfrak{r}_x)^{\text{vex}}| = \phi(\mathfrak{r}_x)$ is a strictly increasing function with $\phi(\mathfrak{r}_1) = 0$, $\phi(\mathfrak{r}_2) = \pi$. The values of $\phi(\mathfrak{r}_x) = |(\mathfrak{r}_1, \mathfrak{r}_x)^{\text{cav}}|$ for $\mathfrak{r}_x \epsilon D_2$ are determined by 2). If D is any angle with vertex p which does not contain r_1 and has legs $\mathfrak{r}'$, $\mathfrak{r}''$, then

(3) $$|D| = |\phi(\mathfrak{r}') - \phi(\mathfrak{r}'')|.$$

If D contains $\mathfrak{r}_1$, then

(3′) $$|D| = 2\pi - |\phi(\mathfrak{r}') - \phi(\mathfrak{r}'')|.$$

Conversely, if in D_1 any strictly increasing function $\phi(\mathfrak{r}_x)$ with $\phi(\mathfrak{r}_1) = 0$, $\phi(\mathfrak{r}_2) = \pi$ is given, and $\phi(\mathfrak{r}_x)$ is determined in D_2 to satisfy 2), then (3) and (3′) determine an angular measure at p which satisfies 1), 2), 3).

2. Excess and parallels. The present section is concerned with the relation of the angular metric in a system S to the theory of parallels. It will not be needed later on but will elucidate the meaning of an angular metric.

If $\mathfrak{g}^+$ is an oriented S-curve and x traverses $\mathfrak{g}^+$ in the positive sense, then the line $\mathfrak{g}(p, x)$ converges for any fixed point p to a line $\mathfrak{a}$. If $\mathfrak{g}^+(a, b)$ denotes generally the line $\mathfrak{g}(a, b)$ with the orientation in which b follows a then $\mathfrak{g}^+(p, x)$ tends to an orientation $\mathfrak{a}^+$ of $\mathfrak{a}$. We call $\mathfrak{a}(\mathfrak{a}^+)$ the (oriented) asymptote to $\mathfrak{g}^+$ through p (for a proof of this and the next statements see [1, Sec. III.3]). The line $\mathfrak{a}$ does not intersect $\mathfrak{g}$. The asymptote to $\mathfrak{g}^+$ through any point $q \epsilon \mathfrak{a}$ is again $\mathfrak{a}$. But in general $\mathfrak{g}^+$ is not an asymptote to $\mathfrak{a}^+$, for an example see [1, Sec. III.5].

Let the *parallel axiom* hold, that means, through a given point p not on a given line $\mathfrak{g}$ there is exactly one line $\mathfrak{h}$ which does not intersect $\mathfrak{g}$. If we determine angular measure at one point p as at the end of the preceding section but with a continuous ϕ, and define measure for an arbitrary angle as equal to the corresponding angle at p with legs parallel to the given angle, then condition 4) is also satisfied and the excess of any triangle is 0.

(4) *A system S in which the parallel axiom holds possesses continuous angular metrics with excess* 0.

However, it is not true that zero excess implies the parallel axiom nor does the parallel axiom imply that every continuous angular metric has excess 0. The only statement which holds without further conditions on the angular metric is the following:

(5) *If the excess is non-positive and the angular metric is continuous then the parallel axiom implies zero excess.*

Let abc be a non-degenerate triangle. If (abx) and x traverses $\mathfrak{g}^+ = \mathfrak{g}^+(a, b)$

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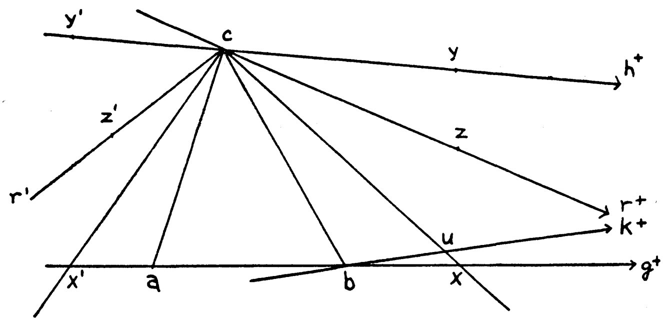
then $|bac| + |acx| < \pi$ and $\mathfrak{g}^+(c, x)$ tends to the asymptote $\mathfrak{h}^+$ to $\mathfrak{g}^+$. If y follows c on $\mathfrak{h}^+$ then $|acx| \to |acy| = |acb| + |bcy|$ so that

(6) $$|bac| + |acy| \leqslant \pi.$$



If (ycy') and (bax') then $\mathfrak{g}^+(c, y')$ is, because of the parallel axiom, the asymptote to $\mathfrak{g}^+(b, a)$ through c. As before we see

(6′) $$|x'ac| + |acy'| \leqslant \pi$$

and since $|x'ac| = \pi - |bac|$, $|acy'| = \pi - |acy|$ it follows from (6) and (6′) that $|bac| + |acy| = |x'ac| + |acy'| = \pi$ so that $|bac| = |acy'|$. For the same reason $|abc| = |bcy|$ and since $|acy'| + |acb| + |bcy| = \pi$ the theorem is proved.

It is clear that this argument and the additivity of the excess yield the following more general fact:

(7) *If the excess is non-positive and the metric is continuous and there is only one line* $\mathfrak{h}$ *through a point* p *not on* $\mathfrak{g}$ *which does not intersect* $\mathfrak{g}$, *then any triangle, whose vertices are in the closed strip bounded by* $\mathfrak{g}$ *and* $\mathfrak{h}$, *has excess 0.*

The arguments in the above proof could be reversed if $|cxb| \to 0$. The following examples will show that further progress is impossible without this property. A hemisphere H without the bounding great circle can be mapped on the Euclidean plane E in such a way that the arcs of great circles in H go into the Euclidean straight lines in E. If we assign to an angle in E the measure of the corresponding spherical angle (in the usual sense), then the excess in any triangle of E is positive in spite of the parallel axiom. With this measure the same holds for the straight line pieces in the interior of a circle in E. On the other hand the Euclidean angles in E may be used as angular measure for those same pieces. This means that both positive and zero excess are compatible with the hyperbolic parallel axiom.

We call an *angular metric* in a system S of curves *complete* if it is continuous and $|pxq| \to 0$ whenever x traverses a ray with origin p from p toward ∞.

(8) *In a complete angular metric the excess cannot always be positive.*

With the same notation as above positive excess would yield $\epsilon(x'cx) > \epsilon(abc) > \pi$. Because of $|cxx'| \to 0$ and $|cx'x| \to 0$ it would follow that for x and x' which are sufficiently far away $|x'cx| > \pi$ which is impossible.

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(9) *In a complete angular metric zero excess implies the parallel axiom.*

For then $|cax| + |acx| + |axc| = \pi$ and $|axc| \to 0$. Moreover $\mathfrak{t}(c, x)$ tends to a ray $\mathfrak{r}$ which lies on the asymptote $\mathfrak{r}^+$ through c to $\mathfrak{g}^+$. If $z \epsilon \mathfrak{r}$, $z \neq c$ then $|acx| \to |acz|$, hence $|cax| + |acz| = \pi$. Similarly $\mathfrak{t}(c, x')$ tends to ray $\mathfrak{r}'$ on the asymptote through c to $\mathfrak{g}^+(b, a)$ and if $z' \epsilon \mathfrak{r}'$, $z' \neq c$, then $|cax'| + |acz'| = \pi$. It follows from $|cax| + |cax'| = \pi$ that $|acz| + |acz'| = \pi$, so that $\mathfrak{r}$ and $\mathfrak{r}'$ are opposite rays, q.e.d.

(10) *In a complete angular metric with non-positive excess asymptotes are symmetric.*

If $\mathfrak{r}^+$ is an asymptote to $\mathfrak{g}^+$ and $\mathfrak{g}^+$ were not an asymptote to $\mathfrak{r}^+$ then the asymptote to $\mathfrak{r}^+$ through a point b of $\mathfrak{g}^+$ would be a line $\mathfrak{f}^+$ different from $\mathfrak{g}^+$ (see Figure). If $c \epsilon \mathfrak{r}^+$ and u follows b on $\mathfrak{f}^+$ then $\mathfrak{g}(c, u)$ intersects $\mathfrak{g}^+$ by the definition of asymptotes in a point x with (cux). Because the excess in bux is non-positive

$$|cub| \geqslant |ubx| + |bxu| > |ubx| > 0$$

but $|cub| \to 0$ when u traverses $\mathfrak{f}^+$ in the positive sense.

Example 1) in [1, Sec. III. 5] yields, with the ordinary Euclidean angles, a complete angular metric and non-symmetric asymptotes which shows that (10) would not hold without the assumption that the excess is non-positive.

(9) and (10) are first examples of statements which connect conditions on the excess with topological properties (in this case of the system S).

3. Angular measure for curve systems on two-dimensional manifolds. The word surface will be used here to denote a connected two-dimensional topological manifold.

As in Sec. **2** for the plane we consider on a given surface M a system S of curves with the topological properties of geodesics. The existence of such a system is guaranteed by the following two conditions.

1) *Every point p of M has a neighbourhood $U(p)$ homeomorphic to the plane, in which a system S_p of curves is distinguished with the properties* I, II, III *of Sec.* **1**.

2) *If a, b, c lie in $U(p) \cap U(q)$ then (abc) holds with respect to S_p if and only if it holds with respect to S_q.*

By 2) a segment $\mathfrak{t}\,(a, b)$ in S_p is also a segment in S_q. Therefore the notation $\mathfrak{t}(a, b)$ can be used without reference to a definite system S_p as long as a and b both lie in some $U(p)$.

The concept of a geodesic will actually not be used in the sequel. But since 1) and 2) are derived from this concept, we mention that an S-geodesic is to be defined as a continuous curve $x(t)$, $-\infty < t < \infty$ with the following property: if t_0 is given and $x(t_0) \epsilon U(p)$ then a suitable subarc $t_1 < t < t_2$ with $t_1 < t_0 < t_2$ of $x(t)$ represents a curve in S_p. The existence of geodesics can be established by the procedure of [2, Sec. II.5].

If $a \epsilon U(p) \cap U(q)$ then a ray $\mathfrak{r}_p$ with origin a in S_p will, in general, not be a ray in S_q, but by 2) the ray $\mathfrak{r}_p$ either contains, or is contained in, a ray $\mathfrak{r}_q$ with origin a of S_q, which is uniquely determined by $\mathfrak{r}_p$.

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If $\mathfrak{r}_p^1$, $\mathfrak{r}_p^2$ are two rays with origin a in S_p and $\mathfrak{r}_q^1$, $\mathfrak{r}_q^2$ are the corresponding rays in S_q, then 2) clearly implies that $\mathfrak{r}_p^1$ and $\mathfrak{r}_p^2$ are opposite if and only if $\mathfrak{r}_q^1$ and $\mathfrak{r}_q^2$ are. Also, if $\mathfrak{r}_p^1$ and $\mathfrak{r}_p^2$ are not opposite, then a ray in $(\mathfrak{r}_p^1, \mathfrak{r}_p^2)^{\text{vex}}$ or $(\mathfrak{r}_p^1, \mathfrak{r}_p^2)^{\text{cav}}$ corresponds to a ray in $(\mathfrak{r}_q^1, \mathfrak{r}_q^2)^{\text{vex}}$ or $(\mathfrak{r}_q^1, \mathfrak{r}_q^2)^{\text{cav}}$ respectively.

These facts lead to the following *formal definition of a ray* $\mathfrak{r}$ with origin a in S: $\mathfrak{r}$ is a set of rays in the local curve systems with these properties:

a) $\mathfrak{r}$ contains exactly one ray of every S_p for which $a \epsilon U(p)$ and no ray of any other S_p.

b) If $\mathfrak{r}_p \epsilon \mathfrak{r}$ and $\mathfrak{r}_q \epsilon \mathfrak{r}$ then either $\mathfrak{r}_p \supset \mathfrak{r}_q$ or $\mathfrak{r}_p \subset \mathfrak{r}_q$.

The meaning of angles in S, of symbols like $(\mathfrak{r}_1, \mathfrak{r}_2)^{\text{vex}}$, and of convergence of angles is now obvious.

An *angular measure for the angles in S* is then characterized by the properties 1), 2), 3) of Sec. **1** and a continuous angular measure by the additional property 4). Through the natural one-to-one correspondence between the angles with vertex a in S and the angles with vertex a in S_p, $U(p) \supset a$, an angular measure in S induces an angular measure in S_p.

Whenever the word triangle is used it is understood that its vertices lie in one $U(p)$. The excess of a triangle is still defined by (1). A geodesic polygon P on M is a curve of the form $\bigcup\limits_{i=1}^{n} \mathfrak{t}(a_i, a_{i+1}),\ a_i \neq a_{i+1}$. Some of the angles of P may be straight, that is the segments $\mathfrak{t}(a_{i-1}, a_i)$ and $\mathfrak{t}(a_i, a_{i+1})$ may belong to opposite rays with origin a_i.

We then call a_i an *improper vertex* of P, otherwise a *proper vertex*. If all vertices of P are improper then P is a *geodesic arc*. If in addition $a_1 = a_{n+1}$ and the angle at a_1 is straight, we call P a *closed geodesic*.

Let G be a compact domain of finite genus on M which is bounded by n simple closed mutually non-intersecting geodesic polygons. If G is simplicially divided into triangles, then the number of vertices minus the number of sides plus the number of triangles is an integer $X(G)$ which depends only on G and not on the choice of simplicial division. According to the terminology prevailing in topology, *$X(G)$ is the negative Euler characteristic of G*.[5]

Any simplicial division of G into triangles $a_\nu b_\nu c_\nu$ satisfies the following fundamental relation

$$\sum_{\nu} \epsilon(a_\nu b_\nu c_\nu) = 2\pi X(G) - \sum_{i} (\pi - \alpha_i), \tag{11}$$

where α_i are the angular measures of the angles at the vertices of the boundary of G measured in G.[6] It is immaterial whether α_i traverses the angles at all or only the proper vertices.

If M is compact and has finite genus p we find that for any simplicial decomposition of M into triangles $a_\nu b_\nu c_\nu$

[5]Compare Kerekjarto [8] and Seifert-Threlfall [9]. Cohn-Vossen [4] calls $X(G)$ (and not $-X(G)$) the characteristic of G.

[6]A modification of the topological proof for (11) which is adapted to the present conditions is found in [4, p. 120].

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$$\sum_{\nu} \epsilon(a_\nu b_\nu c_\nu) = 2\pi X(M) = \begin{cases} 4\pi(1-p) \text{ if } M \text{ is orientable,} \\ 2\pi(2-p) \text{ if } M \text{ is not orientable.} \end{cases} \tag{12}$$

The number $\Sigma\epsilon(a_\nu b_\nu c_\nu)$ in (11) or (12) which is independent of the simplicial division is called the integral curvature $C(G)$ of G or $C(M)$ of M. We say that M or a domain G on M has positive, negative, non-positive, non-negative, or zero curvature if for every non-degenerate triangle abc in M or G

$$\epsilon(abc) > 0, \; < 0, \; \leqslant 0, \; \geqslant 0, \text{ or } = 0 \text{ respectively.}$$

If the curvature of a two dimensional Riemann space R is non-negative, non-positive, zero, positive, or negative in the usual sense then R has the same property in the present sense. The converse is true in the first three cases, but not always in the last two. If the Gauss curvature of R is positive (negative) except on some curves or isolated points, R has still positive (negative) curvature in the present sense. The existence part of the following theorem follows therefore from well-known facts regarding Riemann spaces, the remainder is a consequence of (12).

(13) THEOREM. *A compact surface M can be provided with a system S of geodesics and an angular measure such that curvature is*:

a) *non-negative, if and only if M is homeomorphic to the sphere, torus, one-sided torus* (also called Klein-Bottle), *or the projective plane.*

b) *non-positive, if and only if M is not homeomorphic to the sphere or the projective plane.*

c) *positive, if and only if M is homeomorphic to the sphere or the projective plane.*

d) *negative, if and only if M is not homeomorphic to the sphere, torus, one-sided torus or the projective plane.*

A torus or one-sided torus with non-positive or non-negative curvature has curvature 0.

4. Two dimensional metric manifolds. No statements which approach (13) in completeness seem to be possible for non-compact surfaces unless the curves in S are really geodesics in the metric sense, and not only curves with the topological properties of geodesics. That M is a space with metric geodesics is expressed by the following conditions:

A. *M is a metric space with distance xy.*

B. *M is finitely compact*, or a bounded sequence has an accumulation point.

The fact that the three points a, b, c are different and satisfy the relation $ab + bc = ac$ will be written as (abc).

C. *M is convex*, that is, for any two distinct points a, c a point b with (abc) exists.

D. *Prolongation is locally possible*, or for every point p there is a $\rho(p) > 0$ such that for any two different points a_1, a_2 with $a_i p < \rho(p)$ a point d with $(a_1 a_2 d)$ exists.

E. *Prolongation is unique*, or, if $(a_1 a_2 d')$, $(a_1 a_2 d'')$ and $a_2 d' = a_2 d''$ then $d' = d''$.

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These axioms guarantee the existence of geodesics (compare [2]). In the present case we add

F. *M has dimension* 2 (in the sense of Menger-Urysohn).

It can be proved that M is a connected topological manifold or a surface (for this and the following statements see [1, Sec. I.4]). A space which satisfies Axioms A to F will be called a *G-surface.*

A *metric segment* is an isometric map of a Euclidean segment. If $U(p)$ is the interior of a sufficiently small geodesic triangle on M, then the open metric segments in $U(p)$ with endpoints on the boundary of $U(p)$ form a curve system S_p with properties 1) and 2) of Sec. **3**.

Since any two points of M can be connected by a metric segment, only those metric segments are segments in the previous sense, which lie entirely in one $U(p)$. But since every metric segment can be divided into a finite number of metric segments each of which lies in one $U(p)$, and the angles at the points of division are straight, the distinction between the two kinds of segments turns out to be immaterial and will therefore be dropped.

If M has finite connectivity it can be represented topologically as a compact manifold $\overline{M}$ of finite genus which has been punctured at a finite number of points $z_1, \ldots, z_k$. Let P be a simple closed geodesic polygon which bounds on $\overline{M}$ a simply connected closed domain T which contains exactly one z_i, say z_{i_0}. Because of B the set $T - z_{i_0}$ appears on M as a set which looks like a half cylinder and extends to ∞. We call T a *tube* (Fluchtgebiet in the terminology of Cohn-Vossen [4]).

The tubes are the new feature of non-compact M as compared to compact surfaces. The study of non-compact M must therefore be based on the properties of tubes. *The remainder of this section investigates tubes.*

With the above notation, consider on T the class $C(u)$, $u > 0$, of all curves C which are homotopic to P on T and have distance at most u from P. Whether this distance is measured on M or on T is immaterial. For if measured on M then a segment connecting a point of P to a point of C exists whose length equals the distance of P and C on M. This segment cannot contain a second point of P and lies therefore entirely in T.

$C(u)$ contains curves of finite length (for instance P). Since T, considered as space, satisfies B and every member of $C(u)$ contains a point whose distance from P is at most u, there is a shortest curve $R(u)$ in $C(u)$ (for a proof compare [1, p. 10] and [2, p. 234]). The length $\lambda(u)$ of $R(u)$ is obviously a non-increasing function of u and the triangle inequality yields easily that $\lambda(u)$ is continuous (see [4, §16]). We represent $R(u)$ with the arc length t as parameter in the form $x(t)$, $0 \leqslant t \leqslant \lambda(u)$, $x(0) = x(\lambda(u))$. Notice first

(14) *If $x(t_0)$ is not a vertex of P and has either distance greater than u from P or is not the only point of $R(u)$ whose distance from P is at most u then the subarc*[7] $t_0 - \delta \leqslant t \leqslant t_0 + \delta$ *of $x(t)$ is a segment for sufficiently small $\delta > 0$.*

[7]This inequality is to be replaced by the two inequalities $0 \leqslant t \leqslant \delta$ and $\lambda(u) - \delta \leqslant t \leqslant \lambda(u)$ if $t_0 = 0$ or $t_0 = \lambda(u)$.

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For otherwise the subarc $t_0 - \delta \leqslant t \leqslant t_0 + \delta$ can be replaced by a segment with the same endpoints. If $\delta > 0$ is small enough, the new curve R' will still lie in T, even when $x(t_0)$ lies on P, but is not a vertex of P. Moreover R' will still be homotopic to P and have distance $< u$ from P. But the length of R' would be less than $\lambda(u)$ which contradicts the definition of $R(u)$.

(14) implies that $R(u)$ is a geodesic polygon. Moreover, if $R(u)$ contains points with distance $< u$ from P then $R(u)$ contains infinitely many such points and none of them can be a vertex of $R(u)$. Therefore we see

(15) *$R(u)$ is either a closed geodesic, or all its vertices are vertices of P, or $R(u)$ has exactly one vertex and its distance from P equals u, whereas all other points of $R(u)$ have greater distance from P than u.*

We show next that $R(u)$ *is a Jordan curve.* Since $R(u)$ is homotopic to P and T is homeomorphic to a halfcylinder, $R(u)$ must contain subpolygon R' which is a Jordan curve and homotopic to P. If $R \neq R'$ then R' cannot have distance $\leqslant u$ from P, otherwise R' would belong to $C(u)$ and be shorter than $R(u)$. Therefore $R(u) - R'$ contains a point r with distance $\leqslant u$ from P; r may be chosen as $x(0)$. Then R' is a subarc of $x(t)$ of the form $0 < \alpha \leqslant t \leqslant \beta < \lambda(u)$ with $x(\alpha) = x(\beta)$. The arcs $0 \leqslant t \leqslant \alpha$ and $\beta \leqslant t \leqslant \lambda(u)$ of $x(t)$ must have the same length, otherwise replacing the longer by the shorter would yield a curve in $C(u)$ with smaller length than $\lambda(u)$.

Replacing the arc $\beta \leqslant t \leqslant \lambda(u)$ by the arc $0 \leqslant t \leqslant \alpha$, that is defining $y(t) = x(t)$ for $0 \leqslant t \leqslant \beta$ and $y(t + \beta) = x(\alpha - t)$ for $0 \leqslant t \leqslant \alpha = \lambda(u) - \beta$, yields again a curve R^* in $C(u)$ of length $\lambda(u)$. Statement (14) would then apply to R^*, hence for small $\delta > 0$ the arcs $\alpha - \delta \leqslant t \leqslant \alpha + \delta$ and $\beta - \delta \leqslant t \leqslant \beta + \delta$ would be segments. By construction the arcs $\alpha - \delta \leqslant t \leqslant \alpha$ and $\beta \leqslant t \leqslant \beta + \delta$ coincide. The uniqueness of the prolongation E would imply that the arcs $\alpha \leqslant t \leqslant \alpha + \delta$ and $\beta - \delta \leqslant t \leqslant \beta$ also coincide, but then R' would not be a Jordan curve.

Since $R(u)$ is a simple closed geodesic polygon homotopic to P it bounds a subtube $T(u)$ of T. If a vertex r of $R(u)$ is a vertex of P, then the angle of $R(u)$ at r measured in $T(u)$ cannot be convex, otherwise $R(u)$ could, because of $u > 0$, be shortened without violating the conditions for belonging to $C(u)$.

Finally it will be proved that in case $R(u)$ has exactly one vertex q with distance u from P, the angle at q measured in $T(u)$ must be convex. Let $\mathfrak{t}$ be a segment of length u connecting q to a point d on P. Then $\mathfrak{t}$ cannot contain other points of either P or $R(u)$ because the distance of $R(u)$ from P would then be smaller than u, contrary to (15). If the angle D at q in $T(u)$ were concave, let c_1, c_2 be points on the legs of D and close to q. Then the interior I of the triangle qc_1c_2 would lie outside of $T(u)$. Also, $I \cup T(u)$ contains a neighbourhood of q. The segment $\mathfrak{t}$ connects d to q without entering $T(u)$. It must therefore cross $\mathfrak{t}(c_1, c_2)$ at a point q' and q' has distance $u' < u$ from P. If then the arc $\mathfrak{t}(c_1, q) \cup \mathfrak{t}(q, c_2)$ of $R(u)$ is replaced by $\mathfrak{t}(c_1, c_2)$, the length decreases so that $\lambda(u') < \lambda(u)$, which is impossible.

Thus we have proved the Theorem of Cohn-Vossen:

(16) *$R(u)$ is a simple closed geodesic polygon. It is either a closed geodesic, or all*

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its proper vertices are also vertices of P and the corresponding angles measured in $T(u)$ are concave, or $R(u)$ has exactly one proper vertex q, which is the only point on $R(u)$ with distance u from P and the angle at q in $T(u)$ is convex.

5. Angular metric and structure of non-compact metric surfaces. We now assume that an angular measure has been defined for the system of geodesics of a G-surface M of finite connectivity. With the notations of the preceding section we associate with the points z_i a set of k mutually disjoint tubes T_i each bounded by a geodesic polygon P_i.

Let $u_i > 0$. By Cohn-Vossen's Theorem T_i contains a subtube $T_i(u_i)$ bounded by a geodesic polygon $R_i(u_i)$ such that $R_i(u_i)$ is either a closed geodesic, or all angles of $R_i(u_i)$ measured in $T_i(u_i)$ are concave, or $R_i(u_i)$ has exactly one convex angle whose measure in $T_i(u_i)$ is not zero because $R_i(u_i)$ is a Jordan curve. Let k' ($\leqslant k$) denote the number of the $R_i(u_i)$ with a convex angle.

Call G the compact domain on M bounded by the $R_i(u_i)$. Since concave (convex) angles of $R_i(u_i)$ measured in $T_i(u_i)$ are convex (concave) when measured in G, the relation (11) yields

$$C(G) \leqslant 2\pi X(G) + k'\pi, \tag{17}$$

where the equality sign holds only when all $R_i(u_i)$ are closed geodesics.

It is well-known (see [8, pp. 145, 147]) that

$$X(G) = \begin{cases} 2 - (2p + k),\ p \geqslant 0 \text{ if } M \text{ is orientable,} \\ 2 - (p + k),\ p \geqslant 1 \text{ if } M \text{ is non-orientable,} \end{cases}$$

where p is the genus of M or G.

Hence for non-compact M (that is $k \geqslant 1$) and $C(G) \geqslant 0$ only the following cases are possible. If M is orientable, then $p = 0$ and 1) $k = 1$, $k' = 0, 1$; 2) $k = 2, k' = 0, 1, 2$; 3) $k = k' = 3$. If M is not orientable then $p = 1$, $k = 1$, $k' = 0, 1$.

Taking first only k into account we find in addition to Theorem (13):

(18) THEOREM. *A non-compact G-surface with non-negative curvature is homeomorphic to a plane, a cylinder, a sphere with three holes, or a Moebius strip.*

This agrees again with the known facts regarding Riemann spaces, except for the sphere with three holes. It may therefore be of interest to discuss this exception in some detail.

For that and other purposes we divide the tubes, following Cohn-Vossen, into three categories. Let T be a tube bounded by the simple closed geodesic polygon P, β the greatest lower bound of the length of all curves homotopic to P on T. We call minimal sequence a sequence of curves on T homotopic to P whose length tends to β.

If there is no bounded minimal sequence, we call T *contracting*.

If no subtube of T is contracting we call T *expanding*.

If T is neither contracting nor expanding we call T *bulging*.[8]

[8]Cohn-Vossen calls a contracting tube a Schaft, and uses Kelch for both bulging and expanding tubes. The latter are distinguished as "eigentliche Kelche."

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The following facts are obvious (Compare [4, §18]):

(19) A subtube of a contracting tube is contracting.

(20) A subtube of an expanding tube is expanding.

(21) A subtube of a bulging tube which is sufficiently far away is contracting.

An expanding or bulging tube contains a bounded minimal sequence. This sequence contains a converging subsequence which tends to a curve R homotopic to P of length β (see [1, Sec. I.1]). If the distance of R from P is u' then $R(u) = R$ for every $u > u'$. By the Theorem of Cohn-Vossen $R(u)$ is either a closed geodesic or all its angles measured in $T(u)$ are concave.

In the preceding discussion k' may therefore be interpreted as the number of contracting tubes and we see:

(22) *A sphere with three holes and non-negative curvature has only contracting tubes* and the angle of at least one $R_i(u_i)$ measured in $T_i(u_i)$ must be less than $\pi/3$.

Cohn-Vossen proves that u can be chosen such that the angle of an $R(u)$ on a contracting tube is as close to π as desired. This is not true for general angular metrics.

An instructive *example* can be obtained as follows: In the ordinary space consider the surface M of revolution $z = (x^2 + y^2)^{-1}$. It is homeomorphic to a cylinder or a sphere with two holes, one corresponding to $z = \infty$, the other to $z = 0$. If P_1 and P_2 are two simple closed geodesic polygons associated with those holes as in the beginning of this section, say P_1 to $z = \infty$ and P_2 to $z = 0$, and T_i is the tube bounded by P_i, then P_1 is contracting and P_2 is expanding. Well-known facts on geodesics on surfaces of revolution yield readily that the $R_1(u)$ have all exactly one convex angle $D(u)$ in $T_1(u)$ whose vertex $q(u)$ has distance u from P. Because M is a surface of revolution and the meridians are geodesics the $q(u)$ either lie, or can be assumed to lie, on one meridian.

Let $\alpha(u)$ be the ordinary radian measure of $D(u)$; by Cohn-Vossen's already mentioned result $\alpha(u) \to \pi$ for $u \to \infty$. We now define an angular measure at $q(u)$ as follows. If $D(u) = (\mathfrak{r}_1, \mathfrak{r}_2)^{\text{vex}}$ let D be the straight angle of the form $(\mathfrak{r}_1, \mathfrak{r}'_1)$ that contains $D(u)$. For any $\mathfrak{r} \epsilon D$ let $\alpha(\mathfrak{r})$ be the ordinary radian measure of $(\mathfrak{r}_1, \mathfrak{r})^{\text{vex}}$, so that $\alpha(\mathfrak{r}_2) = \alpha(u)$ and define $\phi(\mathfrak{r})$ by

$$\phi(\mathfrak{r}) = \begin{cases} \delta\alpha(\mathfrak{r}) \text{ for } \mathfrak{r} \epsilon D(u), \ 0 < \delta < 1, \\ \delta\alpha(u) + (\alpha(\mathfrak{r}) - \alpha(u)) \cdot (\pi - \delta\alpha(u)) \cdot (\pi - \alpha(u))^{-1} \\ \qquad\qquad \text{for } \mathfrak{r} \ \epsilon D - D(u). \end{cases}$$

We use $\phi(\mathfrak{r})$ as at the end of Sec. **1** to define an angular measure at the point $q(u)$.

For points on the same parallel circle as $q(u)$ we define angular measure in an obvious way by rotation of $q(u)$ about the z-axis. On the remainder of M we use the ordinary angular metric. Then the new angular metric is continuous on M except on the parallel circle corresponding to $u \to 0+$. It can easily be smoothed out there.

Then $D(u) = \delta\pi$ for all $u > 0$, so that $D(u)$ does not approach π for $u \to \infty$.

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By the same method a sphere with three contracting tubes can be constructed for which the angles of all $R_i(u_i)$ are less than $\pi/3$, so that (22) cannot be improved without a new condition on the angular metric. The example shows also in which direction such a condition has to go:

An angular metric is called *uniform* on a subset G of M if two positive functions $\delta(\epsilon)$ and $\rho(p, \epsilon)$, where $0 < \epsilon < 1$ and $p \epsilon G$, exist, such that the relations $0 < a_1 p = a_2 p < \rho(p, \epsilon)$ and $a_1 a_2/(a_1 p + p a_2) \geqslant 1 - \delta(\epsilon)$ imply for $p \epsilon G$ that $|a_1 p a_2| \geqslant \pi - \epsilon$.

The uniformity is contained in the requirement that $\delta(\epsilon)$ is independent of p. The usual angular metric of a Riemann space is uniform, because $|a_1 p a_2| \to 2 \arccos [(1 - \delta)/2]$ for $a_i \to p$ and $a_1 a_2/(a_1 p + p a_2) = 1 - \delta$. According to Cohn-Vossen (16) may be completed by

(23) THEOREM. *If the angular metric on the tube T is uniform, then for a suitable $u_0 > 0$ the curve $R(u_0)$ is either a closed geodesic, or all angles of $R(u_0)$ measured in $T(u_0)$ are concave, or the angle of $R(u_0)$ at its only vertex q is at least $\pi - \epsilon$.*

Proof. Consider the function

$$f(u) = \lambda(u) + 2\delta(\epsilon)u, \quad u \geqslant 1.$$

Since $\lambda(u)$ is non-negative and continuous, $f(u)$ reaches its minimum at some value u_0 ($\geqslant 1$). Therefore

$$\lambda(u_0 + h) + 2\delta(\epsilon)(u_0 + h) \geqslant \lambda(u_0) + 2\delta(\epsilon)u_0, \quad \text{for } h > 0,$$

$$\text{(23a)} \qquad 2\delta(\epsilon)h \geqslant \lambda(u_0) - \lambda(u_0 + h), \quad \text{for } h > 0.$$

If $R(u_0)$ is not a closed geodesic or its angles are not concave, let q_0 be the vertex of $R(u_0)$. If $\mathfrak{t}(q_0, a^*_1)$, $\mathfrak{t}(q_0, a^*_2)$ are proper segments on the legs of the angle at q_0, let $(q_0 a_i a_i^*)$, $i = 1, 2$, with $h = q_0 a_i < \rho(q_0, \epsilon)$.

Consider the curve R' originating from $R(u_0)$ by replacing $\mathfrak{t}(a_1, q_0) \cup \mathfrak{t}(q_0, a_2)$ by $\mathfrak{t}(a_1, a_2)$. The distance of R' from P is at most $u_0 + h$. Therefore $\lambda(u_0 + h) \leqslant \lambda' =$ length of R' and

$$\text{(24)} \qquad \lambda(u_0) - \lambda(u_0 + h) \geqslant \lambda(u_0) - \lambda' = 2h - a_1 a_2 = 2h(1 - a_1 a_2/2h),$$

and (23a) and (24) yield

$$\delta(\epsilon) \geqslant 1 - a_1 a_2 (a_1 q_0 + q_0 a_2)^{-1};$$

hence $|a_1 q_0 a_2| \geqslant \pi - \epsilon$ by the definition of $\delta(\epsilon)$.

From (22) and (23) we find

(25) *A sphere with three holes and uniform angular metric cannot have non-negative curvature.*

Other well-known theorems can be proved under these general conditions. We mention only one example from Hadamard's theory (see [6]):

(26) *On a G-surface M with negative curvature a class of freely homotopic curves contains at most one closed geodesic.*

The universal covering space of a G-surface M is again a G-surface M' (see [2, Sec. 13]). An angular metric on M induces an angular metric on M'. If M has negative curvature, then M' has negative curvature with respect to this induced metric.

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If M contained two freely homotopic closed geodesics $\mathfrak{g}_1$ and $\mathfrak{g}_2$, draw a segment $\mathfrak{t}$ from a point p_1 of $\mathfrak{g}_1$ to a point p_2 of $\mathfrak{g}_2$. The figure consisting of $\mathfrak{g}_1$, $\mathfrak{g}_2$ and $\mathfrak{t}$ is image of a quadrangle in M' whose angle sum is 2π. Because of the additivity of the excess M' must contain arbitrarily small non-degenerate triangles with non-negative excess, but then M would contain such triangles.

6. The integral curvature of non-compact surfaces. A polygonal region G is the closure of an open set on M whose boundary B (if any) is locally a simple geodesic polygon. That means: if p is any point on B then a geodesic triangle abc exists which contains p in its interior I and such that the intersection of B with the closure of I decomposes I and consists of two segments $\mathfrak{t}(p, x)$, $\mathfrak{t}(p, y)$.

For compact G the integral curvature $C(G)$ was defined in Sec. **3**. For general G we proceed as follows: Let $G_1 \subset G_2 \subset \ldots$ be a sequence of compact polygonal regions with $\bigcup G_i = G$ and the further property that a sequence of points $P_i \epsilon G_{n_{i+1}} - G_{n_i}$, where $\{n_i\}$ is any increasing sequence of positive integers, has no accumulation point. If $\lim C(G_n)$ exists ($\pm\ \infty$ admitted) it is independent of the particular sequence $\{G_n\}$ and is called the *integral curvature* of G.

The condition that $\{P_i\}$ has no accumulation point implies for compact G that $G_n = G$ for large n, so that the present definition of $C(G)$ agrees with the previous one. It is necessary to add some such condition because $G = \bigcup G_\nu$ implies $C(G) = \lim C(G_\nu)$ in general only if $C(G)$ can be extended to a completely additive set function (compare Sec. **7**).

If M is a G-surface of finite connectivity and the tubes T_i are defined as in the beginning of Sec. **5** and H is the compact domain on M bounded by the T_i, then $X(H)$ is independent of the choice of the T_i and $-X(H)$ is called the *characteristic* $-X(M)$ of M.

If $C(M)$ exists, it may be evaluated as follows: Let T_i^n be a sequence of subtubes of T_i with $T_i^n \subset T_i^{n-1}$ and $\bigcap_n T_i^n = 0$. If H^n denotes the compact domain on M bounded by $T_i^n, \ldots, T_k^n$ then

$$C(M) = \lim C(H^n).$$

Since any tube, in particular T_i^n, contains a subtube bounded by a polygon $R(u)$ as constructed in (16), it follows from (11) that

(27) $\qquad C(M) \leqslant 2\pi X(M) + k\pi,$

provided $C(M)$ exists. The discussion preceding (22) yields

(28) $\qquad C(M) \geqslant 2\pi X(M)$ if M has no expanding tubes.

An application of (27) is

(29) *A non-compact surface with non-negative, but not identically vanishing curvature is homeomorphic to a plane.*

For there is a triangle on M with positive excess. This triangle contains then a triangle abc with positive excess which is so small that the images of abc on the universal covering surface M' of M are disjoint. If M were not a plane M' would have infinitely many sheets, and in each a copy of abc. The

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integral curvature of M', which exists because M' has non-negative curvature, is therefore ∞. But this contradicts (27).

Finer results than (27) can be obtained if the angular metric on M is uniform:

(30) *If M has an integral curvature and a uniform angular metric then*

$$C(M) \leqslant 2\pi X(M).$$

The equality sign holds if M possesses no expanding tubes.

For if the previous notations are used, then T_i^n carries a polygon $R_i^n(u_i)$ with the properties described in (23). If H^n is the compact domain bounded by the $R_i^n(u_i^n)$ then by (11)

$$C(H^n) \leqslant 2\pi X(M) + k'\epsilon.$$

The remark about the equality sign follows from (28).

Theorem (23) yields also

(31) *If the tube T with boundary P has an integral curvature and a uniform angular metric then*

$$C(T) \leqslant -\Sigma(\pi - \alpha_i),$$

where α_i are the angles of P measured in T.

If $C(T)$ exists, then every subtube of T has an integral curvature. If T is contracting or bulging, then it contains $T(u)$ bounded by an $R(u)$ which has one vertex q with a convex angle in T or is a closed geodesic. (31) yields then $C(T(u)) \leqslant 0$. Therefore

(32) *A tube with positive curvature and a uniform angular metric is expanding.*

Cohn-Vossen proves this for tubes with non-negative curvature, but the Riemannian character of his metric is essential for this refinement.

We next prove a theorem which is similar to (31) and is found in the paper [5] of Cohn-Vossen.

(33) *On M let Q be an open Jordan curve of the form $\bigcup_{i=-\infty}^{\infty} \mathrm{t}(a_i, a_{i+1})$, $a_i \neq a_{i+1}$, and such that only a finite number of its angles are not straight. Assume, moreover, that Q bounds on M a domain G homeomorphic to a halfplane, and that each subarc of Q is a shortest connection of its endpoints in G. If $\alpha_1, \ldots, \alpha_n$ are the angles at the proper vertices of Q in G and G has a uniform angular metric and an integral curvature, then*

$$C(G) \leqslant -\sum_{i=1}^{n} (\pi - \alpha_i).$$

Proof. Let G' be any simply connected domain in G bounded by a subarc Q' of Q which contains all n vertices of Q and a simple geodesic polygon Q'' in G connecting the two endpoints of Q'. Let $p \epsilon Q'$ and let $q_1(t)$, $q_2(t)$ be the two points of Q for which the subarcs from p to $q_i(t)$ of Q have length t. For a proper choice of t' the points $q_i(t)$ will lie on $Q - Q'$ for $t \geqslant t'$. Let $p(t)$ be a shortest connection of length $\lambda(t)$ of $q_1(t)$ and $q_2(t)$ in $(G - G') \cup Q''$. By the minimum property of Q

(34) $$\lambda(t) \geqslant 2t.$$

As in the proof of (16) it is seen that $p(t)$ is a simple geodesic polygon whose

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proper vertices, if any, coincide with vertices of Q'' and such that the corresponding angles are convex if measured in the domain $G(t)$ bounded by $p(t)$ and the subarc from $q_1(t)$ to $q_2(t)$ of Q. By (11)

$$C(G(t)) \leqslant 2\pi - (\pi - \beta_1(t) - (\pi - \beta_2(t)) - \Sigma(\pi - \alpha_i),$$

where $\beta_i(t)$ is the angle of $p(t)$ and Q at q_i measured in $G(t)$. Due to the arbitrariness of G' the theorem is proved if a $t_0 \geqslant t'$ exists for which $\beta_i(t_0) \leqslant \epsilon$. Let $k = 2\delta(\epsilon)$, where $\delta(\epsilon)$ is the function entering the definition of a uniform angular metric. Then because of (34)

$$\lambda(t) - 2t + kt \to \infty, \quad \text{for } t \to \infty. \tag{35}$$

The triangle inequality implies that $\lambda(t)$ is continuous, therefore the left side of (35) reaches a minimum at some value $t_0 \geqslant t'$. Then

$$\lambda(t_0 + h) - 2(t_0 + h) + k(t_0 + h) - \lambda(t_0) + 2t_0 - kt_0 \geqslant 0, \quad \text{for } h > 0, \text{ or}$$

$$\lambda(t_0 + h) - \lambda(t_0) \geqslant h(2 - k), \quad \text{for } h > 0. \tag{36}$$

On the other hand if $a'_i, a''_i, a'_i \subset p(t_0), a''_i \subset Q(t_0)$, lie on the legs of the angle in $(G - G(t_0)) \cup p(t_0)$ at $q_i(t_0)$ (that means $|a'_i q_i(t_0) a''_i| = \pi - \beta_i(t_0)$) and satisfy the relations

$$a'_i q_i(t_0) = a''_i q_i(t_0) = h < \min_{1,2} \rho(q_i(t_0), \epsilon)$$

then $p(t_0 + h)$ is at most as long as the polygon originating from $p(t_0)$ by replacing $\mathrm{t}(q_i(t_0), a'_i)$ by $\mathrm{t}(a'_i, a''_i)$. Therefore

$$\lambda(t_0 + h) \leqslant \lambda(t_0) + (a'_1 a''_1 - h) + (a'_2 a''_2 - h)$$

which yields together with (36)

$$h(2 - k) \leqslant a'_1 a''_1 - h + a'_2 a''_2 - h, \quad \text{or}$$

$$(1 - a'_1 a''_1/2h) + (1 - a'_2 a''_2/2h) \leqslant k/2 = \delta(\epsilon).$$

Since $1 - a'_i a''_i/2h \geqslant 0$ it follows that $1 - a'_i a''_i/2h \leqslant \delta(\epsilon)$ and from the definition of $\delta(\epsilon)$ that

$$\pi - \beta_i(t_0) = |a'_i q_i(t_0) a''_i| \geqslant \pi - \epsilon \quad \text{q.e.d.}$$

If in addition to the assumptions of (33) Q is a straight line (see [3, p. 232]) then there are no corners, hence $C(G) \leqslant 0$. Therefore

(37) *A plane with positive curvature does not contain a straight line.*

If the assumption that every subarc of Q is a shortest connection in G is omitted, Cohn-Vossen proves that

$$C(G) \leqslant \pi - \Sigma(\pi - \alpha_i). \tag{38}$$

In general spaces the inequality $C(G) \leqslant 2\pi - \Sigma(\pi - \alpha_i)$ is trivial, but the refinement from 2π to π rests on the fact that in Riemannian geometry perpendicular directions form the angle $\pi/2$. This fact has no analogue in general Finsler spaces, no matter how the angular metric is defined, because perpendicularity is not symmetric. Consequently there is no reason to believe that (38) holds with a suitable definition of angular measure, unless perpendicularity is symmetric, although the author did not try to construct an example because this would obviously be very laborious.

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We conclude the analysis of the validity of Riemannian methods in general spaces, which could be continued almost *ad libitum*, by mentioning that the proofs of the following two interesting results of Cohn-Vossen [5] hold without any change:

Let M be a plane with positive curvature and uniform angular metric. Then every point of M lies on at least one geodesic without multiple points. If a geodesic $\mathfrak{g}$ has multiple points, then it contains exactly one 1-gon P, moreover $\mathfrak{g} - P$ lies in the exterior of P and consists of two branches without multiple points (*but the two branches may intersect each other*).

7. The integral curvature as set function. On a surface M with a system S of geodesics and an angular metric as defined in Sec. **3**, let F_0 be the collection of the following sets: the empty set, the points, the segments without end-points (1-cells), the interiors of the non-degenerate triangles (2-cells). The *excess* is called *completely additive* if for any representation of a 2-cell abc as union of a countable number of disjoint 2-cells $a_\nu b_\nu c_\nu$ and points and 1-cells on the boundaries of the $a_\nu b_\nu c_\nu$

$$\epsilon(abc) = \sum_\nu \epsilon(a_\nu b_\nu c_\nu).$$

The unions σ of a finite number of disjoint elements in F_0 form a field F_1. If $a_\nu b_\nu c_\nu$ are the two cells of a given set $\sigma \epsilon F_1$ we put

$$C(\sigma) = \Sigma \epsilon(a_\nu b_\nu c_\nu).$$

If $C(\sigma)$ is bounded on every bounded subset of M and the excess is completely additive, then $C(\sigma)$ can be extended to a completely additive set function on the σ-field F of all Borel sets on M. Moreover the extended set function is bounded on every bounded subset of M with the same bounds as the old function.[9]

Let a measure m be defined on M for which segments have measure 0, and such that every bounded measurable set on M has finite measure. We are going to prove the theorem

(39) *If for every bounded set B on M a number $\beta(B)$ exists such that for any 2-cell abc in B*

$$|\epsilon(abc)| \leqslant \beta(B) m(abc)$$

then ϵ is completely additive, $C(\sigma)$ is bounded on every bounded subset of M and absolutely continuous on F with respect to m.

Proof. Let σ be a set in F_1 which lies in the given set B and $a_\nu b_\nu c_\nu$ the 2-cells of σ. Then

$$C(\sigma) \leqslant \Sigma|\epsilon(a_\nu b_\nu c_\nu)| < \beta(B) \cdot \Sigma m(a_\nu b_\nu c_\nu) \leqslant \beta(B) m(\sigma) \leqslant \beta(B) m(B),$$

so that $C(\sigma)$ is bounded in B for $\sigma \epsilon F_1$.

If abc is the union of the disjoint 2-cells $a_\nu b_\nu c_\nu$, $\nu = 1, 2, \ldots$ and points and

[9]The arguments which lead to these conclusions are implicitly contained in many modern treatments of set functions. For those who are able to read Danish an unusually clear exposition is available in Jessen [7, part 3] which also determined the present formulation.

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1-cells δ_i on the boundaries of the $a_\nu b_\nu c_\nu$, then $m(abc) = \Sigma m(a_\nu b_\nu c_\nu)$ because $m(\delta_i) = 0$. For a given $\epsilon > 0$ we can therefore find an $n(\epsilon)$ such that

$$m(abc) - \sum_{\nu=1}^{n(\epsilon)} m(a_\nu b_\nu c_\nu) < \epsilon/\beta(abc).$$

Then $abc - \sum_{\nu=1}^{n(\epsilon)} a_\nu b_\nu c_\nu$ is the sum of a finite number of 2-cells $a'_i b'_i c'_i$, $i = 1, \ldots, m$ and a finite number of points and 1-cells. Therefore

$$\sum_{i=1}^{m} \epsilon(a'_i b'_i c'_i) \leqslant \beta(abc) \sum_{i=1}^{m} m(a'_i b'_i c'_i) = \beta(abc) m(abc - \sum_{\nu=1}^{n(\epsilon)} a_\nu b_\nu c_\nu) \leqslant \epsilon$$

which shows that $\epsilon(abc) = \sum_{\nu=1}^{\infty} \epsilon(a_\nu b_\nu c_\nu)$ or that ϵ is completely additive.

By the preceding remarks and the first part of this proof $C(\sigma)$, $\sigma \epsilon F_1$, can be extended to a completely additive function on F with the same bounds. It then follows that

$$|C(\sigma)| \leqslant \beta(B) m(\sigma) \text{ for } \sigma \epsilon F \text{ and } \sigma \subset B.$$

Therefore $m(\sigma) = 0$ implies $C(\sigma) = 0$ so that $C(\sigma)$ is absolutely continuous.

Under the hypotheses of the theorem, $C(\sigma)$ is therefore the indefinite integral $\int_\sigma f(p)$ of a function $f(p)$ with respect to the measure m. This does not yet assign a definite value to $f(p)$ at any given point since $f(p)$ can be changed at will in a set of measure 0. This indefiniteness can be eliminated if sufficient restrictions on the angular metric and on the measure guarantee that there is at least (and then exactly) one continuous $f(p)$. But it seems more worthwhile to discuss these questions in connection with a specific angular measure in a Finsler space.

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