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A GLOBAL EXISTENCE AND UNIQUENESS THEOREM FOR ORDINARY DIFFERENTIAL EQUATIONS OF GENERALIZED ORDER

BY

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ABSTRACT. We extend the Picard's theorem to ordinary differential equation of generalized order α, $0<\alpha\leq 1$, and prove a global existence and uniqueness theorem by using the Banach contraction principle.

1. **Introduction.** Al-Abedeen [3] and Bassam [3] proved an existence and uniqueness theorem in a local sense for the differential equation

$$y^{(\alpha)}(x)=f(x, y), \tag{1.1}$$

satisfying the initial condition

$$y^{(\alpha-1)}(a)=y_0, \tag{1.2}$$

where a, y_0 are real numbers and α is a real number satisfying $0<\alpha\leq 1$.

Chu and Diaz [4] have found that the contraction principle can be applied to operator or functional equations and even partial differential equations if the metric of the underlying function space is suitably changed. The ideas of Chu and Diaz [4] were applied by Derrick and Janos [5] to prove a global existence and uniqueness theorem for the differential equation

$$y'(x)=f(x, y),$$

where $f(x, y)$ is a continuous function from $(-a, a)xE^k$ into E^k, $0<a\leq\infty$ and which satisfies the global Lipschitz condition.

The purpose of this paper is to apply the ideas of Chu and Diaz [4] to prove a global existence and uniqueness theorem for (1.1) which satisfies (1.2).

2. **Definitions and lemmas.** In this section, we set forth definitions and lemmas to be used in the proof of the main theorem. For reference see [2].

DEFINITION 2.1. Let f be a function which is defined a.e. (almost everywhere) on $[a, b]$. For $\alpha>0$, we define

$$\overset{b}{\underset{a}{\mathrm{I}}}{}^{\alpha} f=\frac{1}{\Gamma(\alpha)}\int_a^b f(t)(b-t)^{\alpha-1}\,dt,$$

provided this integral (Lebesgue) exists. Γ, is the Gamma function.

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DEFINITION 2.2. If $\alpha \le 0$, and n is the smallest positive integer such that $\alpha + n > 0$, we define

$$\overset{b}{\underset{a}{\mathrm{I}}}{}^{\alpha} f = D_x^n \overset{x}{\underset{a}{\mathrm{I}}}{}^{\alpha+n} f \quad \text{at} \quad x = b, \quad \text{provided} \quad \overset{x}{\underset{a}{\mathrm{I}}}{}^{\alpha+n} f$$

and its first $(n-1)$ derivatives exist in a segment, $|b-x|<h$, and the nth derivative exists at $x=b$.

LEMMA 2.1. *Let* $\alpha, \beta \in R$, $\beta > -1$. *If* $x > a$, then

$$\overset{x}{\underset{a}{\mathrm{I}}}{}^{\alpha} \frac{(t-a)^{\beta}}{\Gamma(\beta+1)} = \begin{cases} \dfrac{(x-a)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}, & \alpha+\beta \ne \text{negative integer} \\ 0 \qquad , & \alpha+\beta = \text{negative integer.} \end{cases}$$

Proof. The proof is straightforward.

LEMMA 2.2. *If* $\alpha > 0$ *and* $f(x)$ *is continuous on* $[a, b]$, *then*

$$\overset{x}{\underset{a}{\mathrm{I}}}{}^{\alpha} f \text{ exists.}$$

Proof. For proof see ([2], p. 531).

LEMMA 2.3. *Let* α, $M > 0$. *If* f *is continuous and* $|f| \le M$, *for all* $x \in (a, b]$, *then*

$$\lim_{x \to a^+} \overset{x}{\underset{a}{\mathrm{I}}}{}^{\alpha} f = 0.$$

Proof. By lemma 2.2, $\overset{x}{\underset{a}{\mathrm{I}}}{}^{\alpha} f$ exists as a Lebesgue integral. The rest of the proof is straightforward.

LEMMA 2.4. *Let* α, $M > 0$. *If* f *is continuous and* $|f| \le M$ *for all* $x \in (a, b]$, then

$$\overset{x}{\underset{a}{\mathrm{I}}}{}^{-\alpha} \overset{t}{\underset{a}{\mathrm{I}}} f = f(x), \; x \in (a, b].$$

Proof. From theorem 1.5. [2], we have

$$\overset{x}{\underset{a}{\mathrm{I}}}{}^{-\alpha} \overset{t}{\underset{a}{\mathrm{I}}}{}^{\alpha} f = f, \text{ a.e. on } (a, b].$$

Let $g(x) = \overset{x}{\underset{a}{\mathrm{I}}}{}^{-\alpha} \overset{t}{\underset{a}{\mathrm{I}}}{}^{\alpha} f$, $x \in (a, b]$. then g is absolutely continuous (see [6], p. 106) and hence continuous on $(a, b]$. Now if x is any point in $(a, b]$ such that $f(x) \ne g(x)$, then there exists a neighbourhood of x in which $f(x) - g(x) \ne 0$ and this neighbourhood has a positive measure. This contradicts the fact that $f(x)=g(x)$ a.e. Hence $f(x)=g(x)$, $x\in(a,b]$. This completes the proof.

DEFINITION 2.3. If $\alpha\in R$, f is defined a.e. on $[a,b]$, we define $f^{(2)}(x)=\overset{x}{\underset{a}{\mathrm{I}}}{}^{-\alpha}f$, for all $x\in(a,b]$, provided $\overset{x}{\underset{a}{\mathrm{I}}}{}^{-\alpha}f$ exists.

3. **The Main Theorem.** In this section, we state and prove the main existence and uniqueness theorem. Consider the initial value problem (P)

$$(P)\quad \begin{aligned} &y^{(\alpha)}(x)=f(x,y), && (3.1)\\ &y^{(\alpha-1)}(c)=y_0,\ y_0\in E,\ a<c<b, && (3.2)\end{aligned}$$

where $f(x,y)$ is a continuous function from $(a,b)\times E$ into E, a, $b\in R$, and E is Euclidean space. Let $f(x,y)$ satisfies the global Lipschitz condition

$$|f(x,y_2)-f(x,y_1)|\le z(x)\,|y_2-y_1|, \tag{3.3}$$

for all $x\in(a,b)$, y_1, $y_2\in E$, and some non-negative continuous function $z(x)$ defined on (a,b). Here $|\cdot|$ denotes the usual norm in E.

Let $\{I_n \mid n\ge 1\}$ be an increasing family of compact intervals which contain c, $a<c<b$ such that $U_n(I_n)=(a,b)$. Denote by $C(I_n)$ the Banach space of continuous functions $g: I_n\to E$, with norm

$$\|g\|_{(n,\lambda)}=\operatorname*{Sup}_{x\in I_n}\left\{\exp\left(-\lambda\left|\int_c^x (x-t)^{\alpha-1}z(t)\,dt\right|\right)|g(x)|\right\}, \tag{3.4}$$

where λ is an arbitrary parameter. If $\lambda=0$, the space $C(I_n)$ have the usual sup norm $\|\cdot\|$ on I_n.

THEOREM 3.1. *Let the right hand side $f(x,y)$ of the differential equation* (3.1) *satisfies the condition* (3.3), *then there exists a unique and continuous function $y(x)$ which is the solution of the initial value problem* (P).

Proof. Let I be a compact sub-interval containing c of (a,b). Let the norm of $g\in C(I)$, a complete subspace of $C(a,b)$, be denoted by $\|g\|_\lambda$ and defined as in (3.4). Let the restriction of the operator F on $C(a,b)$ to $C(I)$ be denoted by T and defined as

$$Tg(x)=\frac{y_0(x-c)^{\alpha-1}}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha)}\int_c^x (x-t)^{\alpha-1}f(t,g(t))\,dt. \tag{3.5}$$

Now we claim that T is a contraction mapping, that is

$$\|Tg_2-Tg_1\|_\lambda\le\frac{1}{\lambda\Gamma(\alpha)}\|g_2-g_1\|, \tag{3.6}$$

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for all $g_1, g_2 \in C(I)$ and $\lambda > 1/\Gamma(\alpha) > 0$. It is easy to verify that the identity

$$\left| \int_c^x \exp\left(\lambda \left| \int_c^t (x-s)^{\alpha-1} z(s)\, ds \right| \right) (x-t)^{\alpha-1} z(t)\, dt \right| = \frac{1}{\lambda}\left(\exp\left(\lambda \left| \int_c^x (x-t)^{\alpha-1} z(t)\, dt \right| \right) - 1 \right), \tag{3.7}$$

is valid for every $x \in (a, b)$.

Now using the definition of $\|g\|_\lambda$, (3.3) and (3.7), we have

$$\|Tg_2 - Tg_1\| = \frac{1}{\Gamma(\alpha)} \left\| \int_c^x (x-t)^{\alpha-1} (f(t, g_2) - f(t, g_1))\, dt \right\|$$

$$= \frac{1}{\Gamma(\alpha)} \operatorname*{Sup}_{x\in I} \left\{ \exp\left(-\lambda \left| \int_c^x (x-t)^{\alpha-1} z(t)\, dt \right| \right) \left| \int_c^x (x-t)^{\alpha-1} (f(t, g_2) - f(t, g_1))\, dt \right| \right\}$$

$$\leq \frac{1}{\Gamma(\alpha)} \operatorname*{Sup}_{x\in I} \left\{ \exp\left(-\lambda \left| \int_c^x (x-t)^{\alpha-1} z(t)\, dt \right| \right) \left(\int_c^x (x-t)^{\alpha-1} z(t)\, |g_2 - g_1|\, dt \right) \right\}$$

$$\leq \frac{1}{\Gamma(\alpha)} \|g_2 - g_1\| \operatorname*{Sup}_{x\in I} \left\{ \exp\left(-\lambda \left| \int_c^x (x-t)^{\alpha-1} z(t)\, dt \right| \right) \left| \int_c^x (x-t)^{\alpha-1} z(t) \exp\left(\lambda \left| \int_c^t (x-s)^{\alpha-1} z(s)\, ds \right| \right) dt \right| \right\}$$

$$\leq \frac{1}{\lambda\Gamma(\alpha)} \|g_2 - g_1\| \operatorname*{Sup}_{x\in I} \left\{ 1 - \exp\left(-\lambda \left| \int_c^x (x-t)^{\alpha-1} z(t)\, dt \right| \right) \right\}$$

$$\leq \frac{1}{\lambda\Gamma(\alpha)} \|g_2 - g_1\|.$$

This establishes (3.6). Thus T is a contraction mapping and hence it has one and only one fixed point. We have

$$Ty(x) = y(x), \quad \text{for all} \quad x \in (a, b).$$

Next we prove that $y(x)$ satisfies the problem (P). From (3.5) and definition 2.1, we have

$$y(x) = \frac{y_0(x-c)^{\alpha-1}}{\Gamma(\alpha)} + \overset{x}{\underset{c}{\mathrm{I}}}{}^{\alpha} f. \tag{3.8}$$

From (3.8) we obtain

$$\overset{x}{\underset{c}{\mathrm{I}}}{}^{-\alpha} y = \overset{x}{\underset{c}{\mathrm{I}}}{}^{-\alpha} \frac{y_0(t-c)^{\alpha-1}}{\Gamma(\alpha)} + \overset{x}{\underset{c}{\mathrm{I}}}{}^{-\alpha} \overset{t}{\underset{c}{\mathrm{I}}}{}^{\alpha} f. \tag{3.9}$$

Now using lemmas 2.1 and 2.4 and definition 2.3, we get

$$y^{(\alpha)}(x) = f(x, y).$$

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From (3.8), we have

$$\overset{x}{\underset{c}{\mathrm{I}}}{}^{1-\alpha}\, y = \overset{x}{\underset{c}{\mathrm{I}}}{}^{1-\alpha}\, \frac{y_0(t-c)^{\alpha-1}}{\Gamma(\alpha)} + \overset{x}{\underset{c}{\mathrm{I}}}{}^{1-\alpha}\, \overset{t}{\underset{c}{\mathrm{I}}}{}^{\alpha} f.$$

Using lemmas 2.1, 2.3 and 2.4 and definition 2.3, we get

$$y^{(\alpha-1)}(c) = y_0.$$

This completes the proof.

We make the following remarks in conclusion.

(a) The theorem 3.1 can easily be extended to the case when $f(x, y)$ is a continuous function from $(a, b)\times E^k$ into E^k, k being positive integer >1.

(b) We have proved the theorem 3.1 for $\alpha \in (0, 1]$. The case $\alpha > 1$ will be taken up later on.

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