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A DIMENSION THEOREM FOR REAL PRIMES

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Introduction. Let k be a real closed field (see § 2 for a definition). Let $\bar{k}$ be an algebraic closure of k. An algebraic set defined over k is, as usual, a subset of $\bar{k}^n$ (n some integer greater than 0) which is the set of zeros of some polynomials in $k[X_1, \ldots, X_n]$. A variety is defined to be an absolutely irreducible algebraic set. We define the real points of an algebraic set X to be the points in $X \cap k^n$. One can then define X to be real if $I(X \cap k^n) = I(X)$. ($I(X)$ = the polynomials in $k[X_1, \ldots, X_n]$ which vanish on X.) By dimension of a variety we mean its usual dimension, e.g. the transcendence degree of its function field over the base field. We wish to prove:

THEOREM 2. *Let X be a real variety of dimension d. Let $W_1, \ldots, W_s$ be subvarieties of X of codimension at least 2. Then there exists a real variety W of codimension 1 in X and $W \supset W_1 \cup \ldots \cup W_s$.*

One application is the following: Let $\mathbf{R}$ be the real numbers. Let V be an algebraic surface in $\mathbf{R}^n$ (V irreducible). Let P be any point of V, even an isolated point. Then there exists an irreducible curve C on V passing through P.

The method of proof of Theorem 2 is similar to that of [**5**] where the result is proved for varieties. Namely, one uses Bertini's theorem. Of course, there is a problem of reality and this is taken care of by the criterion for reality given in Theorem 1. This criterion allows one to deduce that if a certain hyperplane section of a real variety is real, so are "nearby" sections.

1. Preliminary results on varieties. Let k be a field. We denote algebraic closure with a bar so $\bar{k}$ is the algebraic closure of k. An algebraic set defined over k is a subset of $\bar{k}^n$ which is the set of zeros, $V(I)$, of some ideal I of $k[X_1, \ldots, X_n] = k[X]$. A variety is an algebraic set which is absolutely irreducible, i.e., irreducible when considered over $\bar{k}$. All fields considered in this paper are of characteristic zero. We make this restriction since real fields are of characteristic zero and so there is no point in getting into separability questions.

LEMMA 1. *Let V be a variety of dimension $\geqq 2$ defined over k. Let $P_1, \ldots, P_m$ be points of V (and hence in $\bar{k}^n$). Let $k[x_1, \ldots, x_n] = k[x]$ be the coordinate ring of V.*

Then there exist $f_1, f_2 \in k[x]$ such that
(1) $f_i(P_j) = 0$ for all i, j, and
(2) tr. deg.$_k$ $k(f_1, f_2) = 2$.

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Proof. We can assume that $x_1, \ldots, x_d$, $d \geqq 2$, are independent transcendentals over k. There exists a finite extension K of k so that all $P_i \in K^n$. Thus $P_i = (a_1^{(i)}, \ldots, a_n^{(i)}) \in K^n$. Let

$$h_j = \prod_{i=1}^{n} (x_j - a_j^{(i)}), \quad j = 1, 2,$$

and

$$f_j = \prod_{\sigma \in G(K/k)} h_j^{\sigma}, \quad j = 1, 2, \qquad G(K/k) \text{ the Galois group}.$$

It is clear that f_1 and f_2 are in $k[x]$ and that $f_i(P_j) = 0$. Moreover, since f_1 is a polynomial in x_1 and f_2 is a polynomial in x_2, it is obvious that $\text{tr.}\deg_{\cdot k}(f_1, f_2) = 2$.

PROPOSITION 1. *Let V be a variety of dimension d, V defined over k. Let $W_1, \ldots, W_m$ be irreducible algebraic subsets of codimension $\geqq 2$ in V.*

Then there exist f_1, f_2 in $k[x] = k[x_1, \ldots, x_n]$, the coordinate ring of V, such that

(1) $f_i(W_j) = 0$ *all i, j, and*

(2) $\text{tr.}\deg_{\cdot k} k(f_1, f_2) = 2$.

Proof. (as in [**5**] for instance) Let $m = d - 2$. Note we can assume dim $W_i = d - 2$ all i. Then let $k[a_{11}, \ldots, a_{ij}, \ldots, a_{mn}] = k[a]$ and $k[b_1, \ldots, b_m] = k[b]$ be polynomial rings in mn and m variables respectively. We use round brackets as usual to denote quotient fields, e.g., $k(a)$. Also let $k[a] \otimes_k k[b] = k[a, b]$, etc. Let L be an algebraic closure of $k(a, b)$.

Since V is a variety, we have k maximally algebraic in $k[x]$. By Zariski's theorem [**6**, p. 24, Proposition 1.61], we obtain that $k(a, b)$ is maximally algebraic in $k(a, b)[x]/(ax = b)$. We are letting $ax = b$ stand for the equations $\sum_{j=1}^{n} a_{ij}x_j = b_i$, $i = 1, \ldots, m$. If $I(V) =$ ideal of V in $k[X_1, \ldots, X_n] = k[X]$, it is clear that $k(a, b)[x]/(ax = b) = k(a, b)[X]/(I(V), aX = b)$. Also we have $\text{tr.}\deg_{\cdot k(a,b)} k(a, b)[x]/(ax = b)$ is 2. Thus $V(I(V), aX = b) \subset L^n$ and is a variety of dimension 2. Let $I(W_i) =$ ideal of W_i in $k[X]$. We have $\text{tr.deg}_{\cdot k(a,b)} k(a, b)[X]/(I(W_i), aX = b)$ is zero. Then $V(I(W_i), aX = b)$ is $\{P_{i1}, \ldots, P_{iN}\} \subset L^n$. Now by Lemma 1, we can get $f_1, f_2 \in k(a, b)[x]/(ax = b)$ so that $f_1(P_{ij}) = 0$, $f_2(P_{ij}) = 0$ for all i, j, and $\text{tr.}\deg_{\cdot k(a,b)} k(a, b, f_1, f_2) = 2$. Moreover we can multiply by constants in $k(a, b)$ so that f_1, f_2 are in $k[a, b][x]/ax = b$.

In $k[a, x]$, let

$$b_i = \sum_{j=1}^{n} a_{ij}x_j, \quad i = 1, \ldots, m.$$

Then we see that $k[a, x] = k[a, b][x]/ax = b$. Using this identification, we can embed $k[a, x]$ in $k(a, b)[x]/ax = b$. By the above work, we have $f_1, f_2 \in k[a, x]$.

The ideal $I(W_i)$ in $k[x]$ generates an ideal $J = (I(W_i), ax = b)$ in $k(a, b)[x]/(ax = b)$. Moreover $J = I(P_{i1}, \ldots, P_{in})$, so $f_1, f_2 \in J$. Now we claim

$$J \cap k[a, x] = k[a] \otimes I(W_i).$$

For if we order the monomials a^ν in the a_{ij}'s, we can write any $f \in k[a, x] \cap J$ as $f = \sum h_\nu(x)a^\nu$. Then for any $P \in W_i \subset \bar{k}^n$, we have $f(P) = \sum h_\nu(P)a^\nu$. But the a^ν are linearly independent over $\bar{k}$ and $h_\nu(P) \in \bar{k}$. Thus all $h_\nu(P) = 0$ and so all $h_\nu \in I(W_i)$. This implies $f_1, f_2 \in k[a] \otimes I(W_i)$.

We wish to specialize the a_{ij}. Let $\varphi : k[a] \to k$ be the specialization, and f^φ be the image of $f \in k[a, x]$ in $k[x]$. We want tr. $\deg._k k(f_1^\varphi, f_2^\varphi) = 2$. Note that tr. $\deg._{k(a)} k(a)[x] = d$. Thus we can extend f_1, f_2 to $f_1, f_2, \ldots, f_d$ a transcendence basis for $k(a)[x]$ over $k(a)$. Thus each x_i, $(i = 1, \ldots, n)$ is integral over $k(a)(f_1, \ldots, f_d)$. If we choose φ so that all the integrality equations are preserved and all f_i^φ well defined, then $k[x]$ will be integral over $k(f_1^\varphi, \ldots, f_d^\varphi)$. Then tr. $\deg._k k(f_1^\varphi, \ldots, f_d^\varphi) = d$ which implies tr. $\deg._k k(f_1^\varphi, f_2^\varphi) = 2$.

PROPOSITION 2. *Let $V, W_1, \ldots, W_m$ be as in Proposition* 1. *Then there exist $f_1, \ldots, f_s$ in $k[x]$, the coordinate ring of V such that*

(1) $f_i(W_j) = 0$ *all* i, j;
(2) tr. $\deg._k k(f_1, \ldots, f_s) \geqq 2$;
(3) $f_1, \ldots, f_s$ *have no common components.*

Proof. We can take the f_1, f_2 found in Proposition 1 and add more f_i's to achieve (3). To do this note f_1 and f_2 have at most a finite number of common components. If Z is such a component, since $\dim Z > \dim W_i$ for all i, we can find $P \in Z$, $P \notin W_i$ for any i. Then there exists $f_Z \in k[x]$ such that $f_Z(P) \neq 0$, $f_Z(W_i) = 0$. Then f_1, f_2, f_Z no longer have Z as a common component.

2. Real varieties. The main properties of real fields can be found in [**4**]. Recall that a field is real if it can be ordered. A real field is real closed if no algebraic extension is real. We need to know that a real closed field k has the following property. Let $f(x) \in k[x]$, and $a < b \in k$. Then if $f(a) < 0, f(b) > 0$, there exists c in (a, b) with $f(c) = 0$, This is easy to prove using the fact that, in $k[x]$, every polynomial $f(x)$ factors into a product of irreducible linear and quadratic polynomials. The quadratic factors do not change sign in $[a, b]$ and so one of the linear factors does and f has a root in (a, b).

Definition. Let k be a real closed field. Let X be an algebraic set in $A^n(\bar{k}) = \bar{k}^n$, affine n space. We let $X_k = X \cap k^n$. A variety X is real if $I(X_k) = I(X)$ in $k[X_1, \ldots, X_n)$.

It follows from the real nullstellensatz [**1**] that a variety X is real if and only if the coordinate ring $\Gamma[X]$ of X is orderable over k. We wish to give another criterion for reality which will be very useful in this paper.

Let X be an affine variety. Using Noether's normalization theorem, we can find $x_1, \ldots, x_d \in \Gamma[X]$ so that $x_1, \ldots, x_d$ are independent transcendentals over k and $\Gamma[X] = k[x_1, \ldots, x_n]$ is integral over $k[x_1, \ldots, x_d]$. Moreover the inclusion $k[x_1, \ldots, x_d] \subset k[x_1, \ldots, x_n]$ induces a map $\pi : X \to A^d(\bar{k})$ of varieties. Moreover π is a surjection and π induces a map of sets $\pi_k : X_k \to k^d$ (by restriction).

THEOREM 1. *Let X, π etc. be as above. Then X is a real variety if and only if $\pi_k(X_k)$ contains a "sphere" $S_{P,\epsilon}$. We let $S_{P,\epsilon} = \{Q \in k^d \mid |Q - P| < \epsilon\}$ where if $Q = (a_1, \ldots, a_d)$, $P = (b_1, \ldots, b_d)$ then*

$$|Q - P| = \left(\sum_{i=1}^{n} (a_i - b_i)^2 \right)^{\frac{1}{2}}$$

and ϵ is a positive element of k.

Proof. First suppose X is real. Let $B = \Gamma[X] = k[x_1, \ldots, x_n]$. Then choose $y \in B$ so that $k[x_1, \ldots, x_d, y]$ has the same quotient field as B. Let

$$f(Y) = \sum_{i=0}^{m} a_i(x_1, \ldots, x_d) Y^i$$

be the primitive irreducible polynomial for y over $k(x_1, \ldots, x_d)$ so that f is irreducible in $k[x_1, \ldots, x_d, Y]$.

Now

$$x_{d+1} = \sum_{j=0}^{m-1} \frac{b_{ij}(x_1, \ldots, x_d)}{c_{ij}(x_1, \ldots, x_d)} y^j$$

where b_{ij} and $c_{ij} \in k[x_1, \ldots, x_d]$. Let $U = \{P' \in X | a_m(P') \neq 0$, all $c_{ij}(P') \neq 0$, and $(\partial f/\partial z)(P') \neq 0\}$. Then U is Zariski open in X. Since X is real, $X_k \cap U$ is non-empty and we can choose $P' \in U \cap X_k$. We let $P = \pi(P')$. We now know that $y(P')$ is a real simple root of $f_P(Y) = \sum_{i=0}^{m} a_i(P) Y^i$.

Now we let $x = (x_1, \ldots, x_d)$ and abbreviate to $f(x, Y)$. We can change f so that $P = (0, 0) = 0$ and $y(P') = 0$. Then we wish to show as in [**4**] that there exists ϵ so that $|Q| < \epsilon$ implies there exists $\alpha \in k$ with $(Q, \alpha) = 0$. We have $f(0, 0) = 0$ and we can normalize to get $(\partial f/\partial y)(0, 0) = 1$. Then $f(x, y) = y(1 + h(x, y)) + g(x)$ where $h(0, 0) = 0$ and $g(0) = 0$. We can find δ so that $|Q| \leqq \delta$, $|\alpha| \leqq \delta$ implies $|h(Q, \alpha)| \leqq 1/2$. Then $1/2 \leqq 1 + \mathrm{h}(Q, \alpha) \leqq 3/2$ if $|Q| \leqq \delta$, $|\alpha| \leqq \delta$. Thus

$$\begin{matrix} \delta/2 \leqq \delta(1 + h(Q, \delta)) \leqq 3\delta/2 \\ -3\delta/2 \leqq -\delta(1 + h(Q, -\delta)) \leqq -\delta/2 \end{matrix} \quad \text{if} \quad |Q| \leqq \delta.$$

Choose ϵ so that $0 < \epsilon < \delta$ and $|Q| \leqq \epsilon$ implies $|g(Q)| \leqq \delta/4$. Then if $|Q| \leqq \epsilon$, we have $f(Q, \delta) > 0$, $f(Q, -\delta) < 0$ and so there exists $\alpha \in k$ with $-\delta < \alpha < \delta$ and $f(Q, \alpha) = 0$.

Now let $Q' = (x_1(Q), \ldots, x_d(Q), x_{d+1}(Q, \alpha), \ldots, x_n(Q, \alpha))$. We claim that Q' is a point of X_k with $\pi(Q') = Q$. To see this, note that f irreducible in $k[x_1, \ldots, x_d, Y]$ implies (f) is prime in this ring. And, we have $Y \to y$ induces $k[x_1, \ldots, x_d, Y] \to k[x_1, \ldots, x_d, y]$. This map is onto and since the kernel will be a prime of $k[x_1, \ldots, x_d, y]$, it will be a minimal prime; but as it contains (f), it must be (f).

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Next note that $x_i \to x_i(Q)$, $i = 1, \ldots, d$, and $Y \to \alpha$ now induces a homomorphism $k[x_1, \ldots, x_d, y] \to k$. This extends to a homomorphism $B \to k$ since all $c_{ij}(Q) \neq 0$. And this is just the point Q'. We now have shown $\pi(X_k) \supset S_{P,\epsilon}$.

Next assume X is not real. Then $X_k \subset W$ where W is an algebraic set properly contained in X. Then $\dim W < \dim X = d$. So $\pi(W)$ is an algebraic set of dimension less than d. But $\pi_k(X_k) \subset \pi(W)$. And if $\pi(W)$ contains a sphere $S_{P,\epsilon}$, then $\pi(W) = A^d(\bar{k})$ by the following lemma.

LEMMA. *If $f \in k[X_1, \ldots, X_d]$ and $f(S_{P,\epsilon}) = 0$, then $f = 0$.*

Proof. Use induction on d. For $d = 1$, since a non-zero polynomial can have at most a finite number of roots, the result is clear.

For $d > 1$, let $P = (0, \ldots, 0)$ and suppose $|a| < \epsilon$ implies $f(a) = 0$ (for $a = (a_1, \ldots, a_d)$). Now fix a_d with $|a_d| < \epsilon$, and let $g(x_1, \ldots, x_{d-1}) = f(x_1, \ldots, x_{d-1}, a_d)$. Then $g(a_1, \ldots, a_{d-1}) = 0$ if $|(a_1, \ldots, a_{d-1})| < \epsilon - |a_d|$, so by induction $g = 0$. So if $|a_d| < \epsilon$ we have $f(a_1, \ldots, a_{d-1}, a_d) = 0$. Thus, fix $a_1, \ldots, a_{d-1}$ and apply the same argument to $h(x) = f(a_1, \ldots, a_{d-1}, x)$.

Remark. Note that we have actually proven more in the second part of the proof, namely: Let $k[x_1, \ldots, x_d] \subset \Gamma[X]$ define $\pi : X \to A^d(k)$. Then if $\dim X = d$ and $\pi_k(X_k)$ contains a sphere $S_{P,\epsilon}$ in k^d, then X is real.

3. The main result. We are now ready to state our main theorem.

THEOREM 2. *Let X be a real variety defined over a real closed field k. Let $W_1, \ldots, W_s$ be subvarieties of X on codimension at least 2 in X. Then there exists a real subvariety W of X such that $W \supset W_1 \cup \ldots \cup W_s$ and W is of codimension 1 in X.*

Proof. We choose $\Gamma[X] = k[x_1, \ldots, x_n]$ as before so that $x_1, \ldots, x_d$ are independent transcendentals and $x_1, \ldots, x_n$ are integral over $k[x_1, \ldots, x_d]$. As before, $k[x_1, \ldots, x_d] \subset k[x_1, \ldots, x_n]$ induces a surjective morphism $\pi : X \to A^d(\bar{k})$.

Let $\pi(W_i) = Z_i$. We know Z_i is a subvariety of $A^d(\bar{k})$ of codimension at least 2. Moreover, by Theorem 1, $\pi_k(X_k)$ contains $S_{Q,\epsilon}$ a sphere in k^d. There exists $P \in S_{Q,\epsilon}$, $P \notin Z_1 \cup \ldots \cup Z_s$ by the lemma in section 2. Thus we can find another ϵ so that $\pi_k(X_k) \supset S_{P,\epsilon}$ and $S_{P,\epsilon} \cap (Z_1 \cup \ldots \cup Z_s) = \emptyset$.

By Proposition 2, we can find $g_1, \ldots, g_r \in k[x_1, \ldots, x_d]$ so that

(1) $g_i(P) = 0$ all i, and $g_i(Z_j) = 0$ all i, j,

(2) $g_1, \ldots, g_r$ have no common components,

(3) tr. deg.$_k$ $k(g_1/g_1, \ldots, g_r/g_1) \geqq 2$.

We want to find $g_{r+1} \in k[x_1, \ldots, x_d]$ so that $g_{r+1}(P) = 0$, $g_{r+1}(Z_j) = 0$ for all j, and $(\partial g_{r+1}/\partial x_d)(P) \neq 0$.

To do this we can suppose $P = (0, \ldots, 0)$. Then find h so that $h(P) \neq 0$, $h(Z_j) = 0$ all j. Finally let $g_{r+1} = x_d h$. One easily checks that this works.

Consider the linear system $\sum_{i=1}^{r+1} a_i g_i = g$ on X. If we let $a_1, \ldots, a_{r+1}$ be

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independent transcendentals, we obtain from the theorem of Zariski [**6**] quoted in the proof of Proposition 1, that $k(a_1, \ldots, a_{r+1})$ is maximally algebraic in the field $K = k(a_1, \ldots, a_{r+1}, x_1, \ldots, x_n)$ where

$$a_1 = -\sum_{i=2}^{r+1} a_i \frac{g_i}{g_1}.$$

Since $g_1, \ldots, g_{r+1}$ have no common component in $A^d(\bar{k})$, they have none in X, and so K is the quotient field of $k(a_1, \ldots, a_{r+1})[x_1 \ldots x_n]/(g)$. The map $k[a_1, \ldots, a_{r+1}] \subset k[a_1, \ldots, a_{r+1}, x_1 \ldots x_n]/(g)$ induces a morphism of varieties $\sigma : Y \rightarrow A^{r+1}(\bar{k})$. Moreover, at the generic point μ of $A^{r+1}(\bar{k})$ (generic in the sense of Grothendieck), the fiber Y_μ is geometrically integral (which is just the statement that $k(a_1, \ldots, a_{r+1})$ is maximally algebraic in K). Therefore, there exists a Zariski open set U in $A^{r+1}(\bar{k})$ so that if $(b_1, \ldots, b_{r+1}) \in U$, then Y_b is geometrically integral over k, i.e., Y_b is a variety. The proof of this theorem should appear in [**2**] and a proof was given in [**3**].

Now

$$\Gamma[Y_b] = k[x_1, \ldots, x_n]/\sum_{i=1}^{r+1} b_i g_i$$

and Y_b is a subvariety of X. Our problem now is to choose the b's so that Y_b will also be real.

We have $(\partial g_{r+1}/\partial x_{d+1})(P) \neq 0$ so there exists λ so that $g_{r+1}(0, \ldots, 0, \lambda) > 0$, $g_{r+1}(0, \ldots, 0, -\lambda) < 0$ and $|\lambda| < \epsilon$. By choosing $b_1, \ldots, b_r$ small enough, we have that if $g = g_{r+1} + \sum_{i=1}^{r} b_i g_i$, then

$$g(0, \ldots, 0, \lambda) > 0, \quad g(0, \ldots, 0, -\lambda) < 0.$$

We let $V(g) = Y_b$ where $g = \sum_{i=1}^{r} b_i g_i + g_{r+1}$. We claim that we can find $b_1, \ldots, b_r$ small enough so that $(b_1, \ldots, b_r, 1) \in U$. This is because we can really consider $(b_1, \ldots, b_{r+1})$ as projective coordinates since $V(g)$ depends only on $(b_1, \ldots, b_{r+1})$ up to scalar multiples. For such $b_1, \ldots, b_r$, we claim $V(g)$ is a real subvariety of X. To see this, note that $(b_1, \ldots, b_r, 1) \in U$ implies $V(g)$ is a variety.

For reality, note there exists δ with $0 < \delta \leqq \epsilon - |\lambda|$ so that, if $|(a_1, \ldots, a_{d-1})| \leqq \delta$, then $g(a_1, \ldots, a_{d-1}, \lambda) > 0$, $g(a_1, \ldots, a_{d-1}, -\lambda) < 0$.

We are still assuming $P = (0, \ldots, 0)$. But then there exists a_d in $(-\lambda, \lambda)$ so that $g(a_1, \ldots, a_d) = 0$. Then we note $(a_1, \ldots, a_d) \in S_{P,\epsilon}$ and so there exists $a_{d+1}, \ldots, a_n \in k$ so that $(a_1, \ldots, a_n) \in X$. This implies that, if τ is the morphism: $V(g) \rightarrow A^{d-1}(\bar{k})$ induced by

$$k[x_1, \ldots, x_{d-1}] \subset k[x_1, \ldots, x_n]/\sum_{i=1}^{r+1} b_i g_i = g,$$

then $\tau_k(V(g)) \supset S_{0,\delta}$ in k^{d-1}. By the remark at the end of section 2, this implies $V(g)$ is real and we are done; for, just let $W = V(g)$.

Definition. Let k be an ordered field, A a k-algebra and $\mathfrak{p}$ a prime ideal of A. Then $\mathfrak{p}$ is a real prime if $A/\mathfrak{p}$ is orderable over k.

COROLLARY. *Let k be a real closed field. Let A be a finitely generated k algebra. Assume A is an integral domain and that K, the quotient field of A, is orderable over k.*

Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$ be primes of A of height $\geqq 2$. Then there exists a real prime $\mathfrak{q}$ of height 1 in A with $\mathfrak{q} \subset \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_m$.

Proof. Let $A = k[x_1, \ldots, x_n]$. Then $A \cong k[X_1, \ldots, X_n]/I$ for some real prime I of $k[X_1, \ldots, X_n]$. Then the algebraic set $V(I) \subset A^n(\bar{k})$ defined by I will be a variety if k is maximally algebraic in K. So let α be algebraic over k, and $\alpha \in K$. Then, since $k(\alpha) \subset K$, $k(\alpha)$ is orderable, but k real closed implies $k = k(\alpha)$ and $\alpha \in k$. Thus $V(I)$ is a variety and it is real by the real nullstellensatz [**1**].

Each $\mathfrak{q}_i$ defines an algebraic set $V(\mathfrak{q}_i)$ in $A^n(\bar{k})$ and we can let $V(\mathfrak{q}_i) = \bigcup_j W_{ij}$ a union of varieties. Now apply Theorem 2 to obtain $W \subset V(I)$, where W is of codimension 1 in $V(I)$ and $W \supset \bigcup_{i,j} W_{ij}$.

Let $\mathfrak{q} = I(W)$ and $\mathfrak{q}$ is real, again using the real nullstellensatz [**1**].

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