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EXISTENCE OF POSITIVE EVANESCENT SOLUTIONS TO SOME QUASILINEAR ELLIPTIC EQUATIONS

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Abstract

We establish that the elliptic equation

$$\Delta u + f(x, u) + g(|x|)x \cdot \nabla u = 0,$$

defined in an exterior domain of $\mathbb{R}^n$, $n \geq 3$, has a positive solution which decays to 0 as $|x| \to +\infty$ under quite general assumptions upon f and g.

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1. Introduction

Let us consider the quasilinear elliptic differential equation

$$\Delta u + f(x, u) + g(|x|)x \cdot \nabla u = 0, \quad x \in G_A, \tag{1}$$

where $G_A = \{x \in \mathbb{R}^n : |x| > A\}$, $A > 0$ and $n \geq 3$. The existence of positive solutions of equation (1), either bounded or decaying to 0 (a phenomenon called *evanescence*), has been investigated by several authors; see [1, 2, 4, 5, 9].

It has been established (see [1, 2]) that it is sufficient for the functions f, g to be Hölder continuous or continuously differentiable in order to analyze the asymptotic behavior of the solutions to equation (1) by the comparison method [7]. To set the general hypotheses, we assume that there exist the continuous functions $a : [A, +\infty) \to [0, +\infty)$ and $w : [0, +\infty) \to [0, +\infty)$ such that

$$0 \leq f(x, u) \leq a(|x|)w(u), \quad x \in G_A, u \in [0, \zeta], \tag{2}$$

for a certain $\zeta > 0$. We assume that w is monotone nondecreasing, $w(0) = 0$ and $w(u) > 0$ for any $u > 0$.

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In [1, p. 150], A. Constantin asked also that $w \in C^1([0,+\infty),[0,+\infty))$,

$$\int_1^{+\infty} \left(\frac{1}{w(s)}\right) ds = +\infty,$$

and g be bounded. Then, in the circumstances given by the condition

$$\int_A^{+\infty} s[a(s) + |g(s)|]\, ds < +\infty, \tag{3}$$

the existence of a positive solution to equation (1) was proved. In a further work [2, p. 335], it was established that this positive solution actually decays to 0 as $|x| \to +\infty$. By making use of the elementary inequality $w(u) \leq \sup\{|w'(s)| : s \in [0, \zeta]\} \cdot u$ for all $u \in [0, \zeta]$, the integral condition regarding w was successfully removed.

Another improvement was obtained by M. Ehrnström [4, Lemma 3.2] who concluded that, still for $w \in C^1$, the boundedness restriction upon g can be avoided. Moreover, if g takes only nonnegative values, hypothesis (3) reduces to

$$\int_A^{+\infty} sa(s)\, ds < +\infty; \tag{4}$$

see [4, Theorem 3.3].

It is important to remark at this point that by asking w to be merely continuous we could cover cases such as that where $w(u) = u^\lambda$ for any $u \in [0, 1]$ and $\lambda \in (0, 1)$. A result for simply continuous w's was obtained in [5, Theorem 1]; however, its main hypothesis is much more restrictive than (3) unless certain restrictions are imposed upon g, for example $\lim_{r\to+\infty} rg(r) > 0$; see [5, Remark 2].

Our aim in this note is to demonstrate a variant of Constantin's and Ehrnström's results in the case where w is only continuous and g is nonnegative-valued.

2. A special ordinary differential equation

In the spirit of [1, 4, 9], the heart of our proof relies upon a result concerning the positive solutions to the ordinary differential equation

$$v'' + b(t)w\left(\frac{v}{t}\right) = 0, \qquad t \geq t_0 \geq 1. \tag{5}$$

PROPOSITION 1. *Assume that the function $b : [t_0, +\infty) \to [0, +\infty)$ is continuous and such that*

$$\int_{t_0}^{+\infty} b(t)\, dt < +\infty. \tag{6}$$

Then, given $c > 0$, *equation* (5) *has a solution* $v(t)$ *defined in* $[t_0, +\infty)$ *that verifies the inequalities*

$$v(t) \geq c, \qquad v'(t) \leq \frac{v(t) - c}{t} \quad \textit{for all } t \geq t_0,$$

and is developable as

$$v(t) = o(t) \quad \textit{when } t \to +\infty.$$

PROOF. Suppose that $t_0 \geq 1$ is large enough so that

$$\int_{t_0}^{+\infty} b(\tau)\, d\tau \leq \frac{c}{w(2c)}. \tag{7}$$

We introduce the set

$$D = \{v \in C([t_0, +\infty), \mathbb{R}) \mid c \leq v(t) \leq 2ct \text{ for every } t \geq t_0\}.$$

A partial order on D is given by the usual pointwise order '$\leq$', that is, we say that $v_1 \leq v_2$ if and only if $v_1(t) \leq v_2(t)$ for all $t \geq t_0$, where $v_1, v_2 \in D$. It is not hard to see that $(D, \leq)$ is a complete lattice.

For the operator $V : D \to C([t_0, +\infty), \mathbb{R})$ with the formula

$$V(v)(t) = c + t \int_t^{+\infty} \frac{1}{s^2} \int_{t_0}^{s} \tau b(\tau) w\left(\frac{v(\tau)}{\tau}\right) d\tau\, ds, \qquad v \in D,\ t \geq t_0,$$

we may write

$$\begin{aligned} c \leq V(v)(t) &\leq c + w(2c) \cdot t \int_t^{+\infty} \frac{1}{s^2} \int_{t_0}^{s} \tau b(\tau)\, d\tau\, ds \\ &= c + w(2c) t \left[\frac{1}{t} \int_{t_0}^{t} \tau b(\tau)\, d\tau + \int_t^{+\infty} b(\tau)\, d\tau\right] \\ &\leq c + \left[w(2c) \int_{t_0}^{+\infty} b(\tau)\, d\tau\right] t \leq 2ct, \end{aligned}$$

leading to $V(D) \subseteq D$. We also notice that the operator V is *isotone*, that is, $V(v_1) \leq V(v_2)$ whenever $v_1 \leq v_2$.

Since $c \leq V(c)$, by applying the Knaster–Tarski fixed point theorem [3, p. 14], we deduce that the operator V has a fixed point $\tilde{v}$ in D. This is the pointwise limit of the sequence of functions $(V^n(c))_{n \geq 1}$, where $V^1 = V$ and $V^{n+1} = V^n \circ V$.

Finally, we deduce that

$$\begin{aligned} \tilde{v}'(t) = [V(\tilde{v}) - c]' &= \frac{\tilde{v}(t) - c}{t} - \frac{1}{t} \int_{t_0}^{t} b(\tau) w\left(\frac{\tilde{v}(\tau)}{\tau}\right) d\tau \\ &\leq \frac{\tilde{v}(t) - c}{t}, \qquad t \geq t_0. \end{aligned}$$

The proof is complete. □

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REMARK 1. It is not necessary that the solution $\tilde{v}$ of equation (5) obtained in Proposition 1 be bounded. In fact, if we replace (6) with the stronger hypothesis

$$\int_{t_0}^{+\infty} t b(t)\, dt < +\infty,$$

then L'Hospital's rule yields

$$\lim_{t\to+\infty} \tilde{v}(t) = \lim_{t\to+\infty} V(\tilde{v})(t) = c + \int_{t_0}^{+\infty} \tau b(\tau) w\left(\frac{\tilde{v}(\tau)}{\tau}\right) d\tau$$
$$\leq c + w(2c) \int_{t_0}^{+\infty} \tau b(\tau)\, d\tau < +\infty.$$

The ordinary differential equation

$$v'' + \frac{1}{t^2\sqrt{\ln t}}\sqrt{v} = 0, \qquad t \geq t_0 = e^2,$$

has the unbounded solution $\tilde{v}(t) = \ln t$ in $[t_0, +\infty)$ that verifies the conclusion of Proposition 1. Here, $b(t) = t^{-3/2}(\ln t)^{-1/2}$, $w(u) = \sqrt{u}$ and $c = 1$.

REMARK 2. Proposition 1 is in perfect agreement with the conclusion of [1, Lemma 2]. The latter result, however, uses in an essential manner the fact that $w \in C^1$. We mention that Constantin's theorem provides an interesting complement to old contributions by Staikos and Philos; see [8].

3. Positive solution to equation (1)

The following result is needed in our investigation.

PROPOSITION 2 (see [1, 4]). *If there exist a nonnegative subsolution v_1 and a positive supersolution v_2 to equation (1) in G_A, such that $v_1(x) \leq v_2(x)$ for $x \in \overline{G}_A$, then (1) has a solution u in G_A such that $v_1 \leq u \leq v_2$ throughout $\overline{G}_A$. In particular, $u = v_2$ on $|x| = A$.*

Our main contribution here is given next.

THEOREM 1. *Assume that there exists $\alpha \in (0, 1)$ such that $f \in C^\alpha(M \times J, \mathbb{R})$ for every compact set $M \subset G_A$ and every compact interval $J \subset \mathbb{R}$, and $g \in C^1([A, +\infty), [0, +\infty))$. Suppose further that (2) and (4) hold true.*

Then equation (1) has a positive solution u, defined in G_B for some $B > A$, such that $\lim_{|x|\to+\infty} u(x) = 0$.

PROOF. Consider the positive, twice continuously differentiable functions given by

$$U(x) = y(r) = \frac{\tilde{v}(t)}{t}, \qquad t \geq t_0,$$

where

$$r = |x| = \beta(t) = \left(\frac{t}{n-2}\right)^{1/(n-2)} \quad \text{and} \quad t_0 \geq \max\{1, (n-2)A^{n-2}\}.$$

Here, $\tilde{v}$ is the solution of equation (5) obtained in Proposition 1 for $c = \zeta/2$.

By a straightforward computation,

$$t\beta'(t) = \frac{1}{n-2}\beta(t) \tag{8}$$

and

$$\begin{cases} \dfrac{d\tilde{v}}{dt} = y + t\beta'(t)\dfrac{dy}{dr}, \\ \dfrac{d^2\tilde{v}}{dt^2} = \dfrac{n-1}{n-2}\beta'(t)\dfrac{dy}{dr} + \dfrac{\beta(t)\beta'(t)}{n-2}\dfrac{d^2y}{dr^2}. \end{cases} \tag{9}$$

Further, taking into account (8) and (9),

$$\begin{aligned} &r^{n-1}(\Delta U + f(x, U) + g(|x|)x \cdot \nabla U) \\ &\quad = \frac{d}{dr}\left(r^{n-1}\frac{dy}{dr}\right) + r^{n-1}f(x, U) + r^n g(r)\frac{dy}{dr} \\ &\quad = \frac{n-2}{\beta(t)\beta'(t)}[\beta(t)]^{n-1}\Bigg[\tilde{v}''(t) + \frac{1}{n-2}\beta(t)\beta'(t)f(x, U) \\ &\qquad + \beta(t)\beta'(t)g(\beta(t))\left(\tilde{v}'(t) - \frac{\tilde{v}(t)}{t}\right)\Bigg], \end{aligned}$$

for any $t \geq t_0$.

We have obtained that

$$\begin{aligned} &|x|^{n-1}(\Delta U + f(x, U) + g(|x|)x \cdot \nabla U) \\ &\quad \leq \frac{n-2}{\beta(t)\beta'(t)}[\beta(t)]^{n-1}\left[\tilde{v}''(t) + b(t)w\left(\frac{\tilde{v}(t)}{t}\right)\right] = 0, \end{aligned}$$

where $b(t) = \beta(t)\beta'(t)a(\beta(t))$.

Now, U is a positive super-solution of (1). Also, the trivial solution of equation (1) is its (nonnegative) sub-solution. According to Proposition 2 (see [1, 4]) there exists a nonnegative solution u to (1), defined in $\overline{G_B}$ for $B > A$ large enough (recall (7)). Since

$$(\Delta + g(|x|)x \cdot \nabla)(-u) = f(x, u) \geq 0,$$

the strong maximum principle [6] can be applied to $-u$. This means that the function $-u$ cannot attain a nonnegative maximum at a point of G_B unless it is constant. Since $-u$ is negative on $\{x : |x| = B\}$ and $-u(x) \leq 0$ throughout $\overline{G_B}$ as u is confined between 0 and a positive super-solution U, it follows that $-u$ cannot have zeros.

We conclude that u is a positive solution of (1) that decays to 0 when $|x| \to +\infty$. The proof is complete. □

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