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Finite group actions on shifts of finite type

R. L. ADLER, B. KITCHENS AND B. H. MARCUS

IBM T. J. Watson Research Center, P.O. Box 218, *Yorktown Heights, NY* 10598, *USA*; *IBM Research*, 5600 *Cottle Road, San Jose, CA* 95193, *USA* and *Mathematics Department, University of North Carolina, Chapel Hill, North Carolina* 27514, *USA*

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Abstract. A continuous $\mathbb{Z}\otimes_T G$ action on a subshift of finite type consists of a subshift of finite type with its shift transformation, together with a group, G, of homeomorphisms of the subshift and a group automorphism T, so that the commutation relation $\sigma\circ g = Tg\circ\sigma$ is satisfied, where σ is the shift and g is any element of G.

Here, we investigate these actions when G is finite. We show that if Σ_A is any positive entropy subshift of finite type, G is any finite group and T is any automorphism of G then there is a non-trivial $\mathbb{Z}\otimes_T G$ action on Σ_A. We then classify all such actions up to 'almost topological' conjugacy.

0. *Introduction*

A continuous $\mathbb{Z}\otimes_T G$ action on a subshift of finite type (ssft), Σ_A, consists of a group, G, of homeomorphisms of Σ_A and an automorphism T of G satisfying $\sigma\circ g = Tg\circ\sigma$ where σ is the shift. The problem we consider is that of classifying these actions up to almost topological conjugacy. When G is a finite group we give necessary and sufficient conditions for this classification. This type of group action arises naturally and is crucial in classifying equal entropy factors and extensions of ssft. We will treat this topic in a subsequent paper. The parallel measure-theoretic problems have been investigated by D. Rudolph in [**6**] and [**7**]. We obtain the same conditions as Rudolph but in our context the proofs are quite different.

The overall approach to proving the main theorem is to follow the line of reasoning in [**1**] as closely as possible. There it is shown that if two irreducible ssft have the same entropy and ergodic period they are almost topologically conjugate. Knowledge of this argument is not taken for granted, it is contained in this discussion in a slightly altered form. However, a certain familiarity with ideas and constructions common to symbolic dynamics is assumed. All proofs (except in the last section) are complete and self-contained but the reader unfamiliar with this type of argument may find them somewhat sketchy.

We begin in § 1 with a little background material. The necessary notation and definitions are set up. § 2 examines groups acting on ssft. There are a number of observations made about these actions when the group is of the form $\mathbb{Z}\otimes_T G$, with G finite. We also observe that if G is any finite group, T is any automorphism and Σ_A is any ssft then there is a $\mathbb{Z}\otimes_T G$ action on Σ_A. Next we examine the case where G is a compact topological group and there is a $\mathbb{Z}\otimes_T G$ action on an ssft. This implies that G must be zero-dimensional and we show that if $T^p = \text{identity}$ for some $p \neq 0$ we may regard it as a finite group. After this we prove that if G is finite and there is a $\mathbb{Z} \otimes_T G$ action on an ssft, we can find an extension of this action where the new $\mathbb{Z} \otimes_T G$ action has a simple form that allows us to simplify many of the constructions that follow.

§ 3 proceeds along the lines of the argument in [**1**]. The steps are the same and it is shown that each of the constructions made there can be repeated here if we are sufficiently careful. We prove that if Σ_A and Σ_B are aperiodic ssft with the same entropy, and both have $\mathbb{Z} \otimes_T G$ actions then they are almost topologically conjugate.

§ 4 deals with irreducible but periodic ssft. When two ssft have $\mathbb{Z} \otimes_T G$ actions, the same entropy and ergodic period greater than one, we see that another obstruction to almost topological conjugacy may exist. The subgroup that fixes each cyclic subset, as well as certain properties of its cosets must be taken into account. We show that once this extra condition is met the two $\mathbb{Z} \otimes_T G$ actions will be almost topologically conjugate.

Finally, in the last section we observe that all these results carry over to sofic systems.

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1. *Background*

In the discussion that follows we will assume the reader is familiar with the basic ideas concerning subshifts of finite type (ssft). These include irreducibility, aperiodicity, Perron-Frobenius theory, topological entropy, ergodic period, block maps, cylinder sets, Markov partitions, and the basic operations of going to higher blocks, splitting states, and amalgamating states. All of these ideas are discussed in [**1**], and the basic operations are also examined in § 2.

For a zero-one transition matrix, A, which we will always assume to be irreducible with spectral radius greater than one, Σ_A will denote the ssft defined by A, L_A will denote the alphabet of Σ_A and λ_A (or λ) will denote A's maximal eigenvalue. For a symbol $I \in L_A$ let

$$f_A(I) = \{J \in L_A \colon \; A_{IJ} = 1\}$$

(or $f(I)$) and

$$p_A(I) = \{K \in L_A \colon A_{KI} = 1\}.$$

These are the *successor* and *predecessor* sets of a symbol. Specifically, they will be denoted by

$$f(I) = \{I_1, \ldots, I_{|f(I)|}\} \quad \text{and} \quad p(I) = \{I_1^*, \ldots, I_{|p(I)|}^*\}.$$

For a subset $E \subseteq L_A$ let

$$f(E) = \bigcup_{I \in E} f(I) \quad \text{and} \quad p(E) = \bigcup_{I \in E} p(I).$$

If $\phi : \Sigma_A \to \Sigma_B$ is a one-block map then we will use ϕ interchangeably to stand for the map from Σ_A to Σ_B or the map from L_A to L_B. If $\phi : \Sigma_A \to \Sigma_B$ is a continuous, onto, shift-commuting map then we will say that Σ_A is an *extension* of Σ_B and Σ_B is a *factor* of Σ_A or that ϕ is a *factor map.* Right or left-resolving is a property of some factor maps that we will use repeatedly. Say a map $\phi:\Sigma_A\to\Sigma_B$ is *right resolving* if it is a one-block map and $\phi:f_A(I)\to f_B(\phi(I))$ defines a set isomorphism for each $I\in L_A$. Similarly, say $\phi:\Sigma_A\to\Sigma_B$ is *left resolving* if it is a one-block map and $\phi:p_A(I)\to p_B(\phi(I))$ defines a set isomorphism for each $I\in L_A$. A right (or left) resolving map is always onto and boundedly finite-to-one (see [**1**] for a discussion).

A factor map $\phi:\Sigma_A\to\Sigma_B$ is said to be *one-to-one a.e.* if it is one-to-one when restricted to the doubly transitive points of Σ_A and Σ_B. In this case Σ_A is said to be an *almost conjugate extension of* Σ_B and Σ_B is an *almost conjugate factor* of Σ_A. This is not the definition in [1] but the two definitions are equivalent when dealing with subshifts of finite type. A discussion of some of this can be found in [**2**]. A map $\phi:\Sigma_A\to\Sigma_B$ is one-to-one a.e. if and only if a resolving block (or magic-word) exists for ϕ. A *resolving block* (or *magic-word*) for ϕ, when it is recoded to a one block map, is a cylinder set $[J_1,\ldots,J_n]$ in Σ_B whose pre-image

$$\phi^{-1}([J_1,\ldots,J_n])=\bigcup_{k=1}^{t}[I_1^k,\ldots,I_n^k]$$

is composed of blocks that all agree in some entry, $I_m^k=I$ for some m and all k. Two ssft Σ_A and Σ_B are said to be *almost topologically conjugate* when there exists a common almost conjugate extension.

2. *Group actions and skew group actions*

Let Σ_A be an ssft, G be a topological group, $T:G\to G$ an automorphism. We say G acts (continuously) on Σ_A if

(i) each $g\in G$ defines a homeomorphism $g:\Sigma_A\to\Sigma_A$, so that the map $(g,x)\to g(x)$ is also continuous;

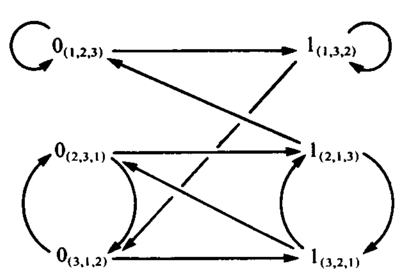
(ii) $g(h(x))=gh(x)$, i.e. group multiplication is the same as map composition;

(iii) for each $g\in G$, $g\neq \mathrm{id}$ there is an x (in each cyclic subset, if Σ_A is periodic) so that $g(x)\neq x$.

If $\sigma\circ g=g\circ\sigma$ for each $g\in G$ we say we have a $\mathbb{Z}\times G$ action on Σ_A. If $\sigma\circ g=Tg\circ\sigma$ we say we have a $\mathbb{Z}\otimes_T G$ action or *G skew action* on Σ_A.

Example 1. Let $S(2)=\{(1,2),(2,1)\}$ be the permutation group on two elements. Define a $\mathbb{Z}\times S(2)$ action on the full two-shift, $\{1,2\}^{\mathbb{Z}}$, by letting $S(2)$ act on $L_A=\{1,2\}$ in the natural way.

Example 2. Define Σ_A by



Let $S(3)=\{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}$ be the permutation group on three elements. Let $G=\{(1,2,3),(2,3,1),(3,1,2)\}$ be a subgroup of $S(3)$. Let $T:G\to G$ be defined by

$$T(i,j,k)=(1,3,2)(i,j,k)(1,3,2).$$

Define a $\mathbb{Z}\otimes_T G$ action on Σ_A by letting G act on L_A by permuting the subscripts in the natural way, $g(I_h)=I_{gh}$.

A *factor* map from a $\mathbb{Z}\otimes_T G$ action on Σ_A to a $\mathbb{Z}\otimes_T G$ action on Σ_B is a factor map between Σ_A and Σ_B that also commutes with each element of G. In this case we say that the $\mathbb{Z}\otimes_T G$ action on Σ_B is a *factor* of the one on Σ_A, and that the action on Σ_A is an *extension* of the one on Σ_B. If the map is a homeomorphism we say the actions are *topologically conjugate*. Two $\mathbb{Z}\otimes_T G$ actions are *almost topologically conjugate* if they have a common one-to-one a.e. extension. When G is a finite group we make the following observations.

Observation 1. Every $\mathbb{Z}\otimes_T G$ action is topologically conjugate to a one-block action, i.e. there is a G action on L_A so that the G part of the $\mathbb{Z}\otimes_T G$ action acts on Σ_A by

$$g(x)=(\ldots, T^{-1}g(x_{-1}), g(x_0), Tg\,(x_1),\ldots).$$

Observation 2. If we have a one-block $\mathbb{Z}\otimes_T G$ action on Σ_A and P_g is the permutation matrix indexed by L_A that represents g, then $P_gA=AP_{Tg}$ for all $g\in G$. This means $p(g(I))=T^{-1}g(p(I))$ and $f(g(I))=Tg(f(I))$ for all $I\in L_A$ and $g\in G$.

Observation 3. If we have a $\mathbb{Z}\otimes_T G$ action on Σ_A then $g(x)\neq x$ for all $g\in G$ and all doubly transitive $x\in\Sigma_A$.

Proofs. (1) Let $\mathcal{P}$ be the partition on Σ_A of one-block time zero cylinder sets. Define a new partition, $\mathcal{P}'=\bigvee_{g\in G} g(\mathcal{P})$. This new partition is finite, invariant under G, and Markov with respect to σ. Let A' be the transition matrix for this new partition. Then $\Sigma_{A'}$ is a new ssft, topologically conjugate to Σ_A, with a one-block $\mathbb{Z}\otimes_T G$ action.

(2) These three equations follow directly from the fact that $\mathbb{Z}\otimes_T G$ is a one-block action, using $\sigma\circ g=Tg\circ\sigma$.

(3) Since T is an automorphism of a finite group there is a p such that $T^p=\mathrm{id}$. Then the $\mathbb{Z}\otimes_T G$ action on Σ_A gives a $\mathbb{Z}\times G$ action on each irreducible component of Σ_{A^p}. Consider each one of these separately. Observe that if x is doubly transitive for σ on Σ_A, then x is doubly transitive on its component for σ^p. So we are reduced to the case of a one-block $\mathbb{Z}\times G$ action on an irreducible ssft. For each $g\in G$ there is an $I\in L_A$ such that $g(I)\neq I$; since every doubly transitive point, x, contains I, $g(x)\neq x$. □

At this point we will digress to examine how a skew group action affects the standard higher block, state splitting, and amalgamation operations.

Higher blocks. Consider a $\mathbb{Z}\otimes_T G$ action on an ssft Σ_A. Define a natural conjugate action on the two-block system of Σ_A by:

$$g([I,J])=[g(I),Tg(J)],\qquad \forall g\in G.$$

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This gives a one-block $\mathbb{Z}\otimes_T G$ action on the two block system because if $[I, J]\to[J, K]$,

$$g([I, J])=[g(I), Tg(J)]\to[Tg(J), T^2g(K)]=Tg[J, K].$$

The construction extends to the k-block system in the natural way.

State splitting. Fix an $I\in L_A$ and let $f(I)=E_1\cup E_2$ be a partition of the follower set of I into two sets. When $I\notin f(I)$ split I into two symbols I^1, I^2 with

$$p(I^1)=p(I^2)=p(I),\quad f(I^1)=E_1\quad\text{and}\quad f(I^2)=E_2.$$

When $I\in f(I)$ suppose $I\in E_1$, split I into two symbols I^1, I^2 with

$$p(I^1)=p(I^2)=(p(I)-\{I\})\cup\{I^1\},\quad f(I^1)=(E_1-\{I\})\cup\{I^1, I^2\}\quad\text{and}\quad f(I^2)=E_2.$$

This defines a new ssft conjugate to the original. If we also want to get a conjugate $\mathbb{Z}\otimes_T G$ action, then we must split more states. For each $g\in G$, split each state $g(I)$ according to the partition $f(g(I))=Tg(E_1)\cup Tg(E_2)$. This splits each $g(I)$ into $(g(I))^1$ and $(g(I))^2$ as before. For $h\in G$ define $h((g(I))^1)=(hg(I))^1$ and $h((g(I))^2)=(hg(I))^2$. This defines a new one-block $\mathbb{Z}\otimes_T G$ action. The construction may be carried out using the predecessors instead of the followers.

Amalgamations. Let $I, J\in L_A$ with $I\neq J$, $p(I)=p(J)$ and $f(I)\cap f(J)=\varnothing$. Then the amalgamation $I\sim J$, defines an ssft conjugate to Σ_A; namely, throw out the states I and J, and add the state $\{I, J\}$ with transitions

$$p(\{I, J\})=p(I)=p(J),\qquad f(\{I, J\})=f(I)\cup f(J),$$

if I or J is not in $f(I)\cup f(J)$. If I or J is in $f(I)\cup f(J)$ we may without loss of generality assume $\{I, J\}\cap(f(I)\cup f(J))=\{I\}$. This means $I\in p(I)=p(J)$. Now the amalgamation $I\sim J$ defines an ssft conjugate to Σ_A with

$$p(\{I, J\})=(p(I)-\{I\})\cup\{\{I, J\}\},$$
$$f(\{I, J\})=(f(I)-\{I\})\cup f(J)\cup\{\{I, J\}\}.$$

In order to get an amalgamation that respects the $\mathbb{Z}\otimes_T G$ action, we assume $J\notin G(I)=\{g(I)\colon g\in G\}$ and then amalgamate the other pairs $g(I)\sim g(J)$, $g\in G$. This all fits together since

$$p(g(I))=T^{-1}g(p(I))=T^{-1}g(p(J))=p(g(J)),$$

and

$$f(g(I))\cap f(g(J))=Tg(f(I))\cap Tg(f(J))=\varnothing.$$

A G-action is defined by

$$h(\{g(I), g(J)\})=\{hg(I), hg(J)\}.$$

This gives a $\mathbb{Z}\otimes_T G$ action on the new ssft that is naturally conjugate to the original. This procedure may also be carried out if the roles of predecessors and successors are exchanged.

Next, we observe that $\mathbb{Z}\otimes_T G$ actions, for finite G, are plentiful.

THEOREM 1. *If G is any finite group, $T\colon G\to G$ is any automorphism, Σ_A is any irreducible ssft then there is a $\mathbb{Z}\otimes_T G$ action on Σ_A.*

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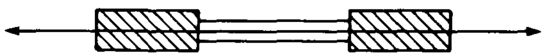
Proof. This is essentially the proof of the corresponding statement in [3]. Let G, T, and Σ_A be as in the hypothesis. First we will produce a $\mathbb{Z}\times G$ action on Σ_A. Since every finite group is isomorphic to a subgroup of the permutation group $S(n)$ for sufficiently large n, it suffices to produce a $\mathbb{Z}\times S(n)$ action. Choose a simple cycle, $[I_1, I_2, \ldots, I_p, I_1]$, in the graph of A. By using the elementary conjugacies (of course without the group action) just discussed, if necessary, we can assume there is a $J\in f(I_1)$, $J\neq I_2$ and a $K\in p(I_1)$, $K\neq I_p$. Choose an integer N large enough so that $(A^{N-1})_{JK}>n$. Choose n of these N-blocks and number them. $S(n)$ can act on these blocks by permutation. Define maps on Σ_A by letting $g\in S(n)$ act on x by leaving



all blocks in x alone except when one of the numbered blocks occurs between two blocks of $[I_1, \ldots, I_p, I_1]$ repeated cyclically of length at least $2N$. When such a configuration occurs, the numbered block is permuted by g. The cyclic blocks of length at least $2N$ are markers needed to insure that the numbered blocks being permuted don't interfere with one another. This gives a $\mathbb{Z}\times G$ action for any finite G. Now we easily produce the $\mathbb{Z}\otimes_T G$ action. Permute the blocks by g if the numbered block begins at time zero, $x_0=J$. If the numbered block begins at time n, $x_n=J$, permute it by $T^n g$. □

Next we examine the case where the group acting is a compact topological group. We conclude that in many cases, because of the topology of Σ_A, there are really no new actions.

Observation 4. If G is a compact topological group and we have a $\mathbb{Z}\otimes_T G$ action on Σ_A, then G is zero-dimensional.

Proof. First notice that if $x\in\Sigma_A$ is doubly transitive then $g(x)\neq x$ for all $g\neq$ identity. Fix a doubly transitive $x\in\Sigma_A$, so $G(x)\subseteq\Sigma_A$ is a closed subset of Σ_A. Define a map from $G(x)$ to G by sending $y\in G(x)$ to the unique element $g\in G$ where $g(x)=y$. This is a homeomorphism which also defines a group operation on $G(x)$ by

$$y\circ z=g(x)\circ h(x)=gh(x)=w.$$

This makes the map a group isomorphism, $G(x)\cong G$. □

Example 3. Let Σ_A be the full four-shift with alphabet $L_A=\{(0,0), (0,1), (1,0), (1,1)\}$. Let $G=\{0,1\}^{\mathbb{Z}}$ with coordinate wise addition modulo two the group operation. Let T be the shift. Define a $\mathbb{Z}\otimes_T G$ action on Σ_A by

$$g(x)=\cdots(x^1_{-1}, x^2_{-1}+g_{-1})(x^1_0, x^2_0+g_0)(x^1_1, x^2_1+g_1)\cdots$$

where

$$x=\cdots(x^1_{-1}, x^2_{-1})(x^1_0, x^2_0)(x^1_1, x^2_1)\cdots$$
$$g=\cdots g_{-1}g_0g_1\cdots$$

and the additions are modulo two.

We would like to thank D. Lind for his help with the preceding observation and example.

THEOREM 2. *Assume G is a compact topological group with an automorphism T, T^p = identity for some p, and we have a $\mathbb{Z}\otimes_T G$ action on Σ_A. Then there is a finite group G' with an automorphism T', and a group homomorphism $\rho: G \to G'$ giving rise to a $\mathbb{Z}\otimes_{T'} G'$ action, such that $\rho \circ T = T' \circ \rho$ and $g = \rho(g)$ as a map of Σ_A, for each $g \in G$.*

Proof. Assume G has a one-block action on Σ_A. Define an equivalence relation on G by saying $g \sim_0 h$ if and only if $g(I) = h(I)$ for all $I \in L_A$. The partition $\mathcal{P}_0$ of G defined by this equivalence relation is finite and open-closed whenever G acts continuously on Σ_A. Define a second equivalence relation on G by saying $g \sim h$ if and only if $g(x) = h(x)$ for all $x \in \Sigma_A$. Both equivalence relations are preserved by the group operation. This means the quotient topological group defined by second relation $G/\sim = G'$ acts continuously on Σ_A. The quotient map $\rho: G \to G'$ makes $g = \rho(g)$ as homeomorphisms of Σ_A and also induces an automorphism T' of G'. The partition of G defined by the second equivalence relation is $\mathcal{P} = \bigvee_{i=-\infty}^{\infty} T^i(\mathcal{P}_0)$. Any time this is finite G' is finite. We have assumed T^P is the identity for some p, so in this case it is finite. □

This raises the question of what are necessary and sufficient conditions on G and T to have a $\mathbb{Z}\otimes_T G$ action on a given Σ_A.

Suppose we have a $\mathbb{Z}\otimes_T G$ action on Σ_A, for G finite. If each $g \in G$ is a one-block map, and for $g \neq$ identity, $g(I) \neq I$ all $I \in L_A$, say we have a *special* $\mathbb{Z}\otimes_T G$ action on Σ_A. The following theorem allows us to deal only with special actions in the rest of this paper. This will greatly simplify the main constructions.

THEOREM 3. *If we have a $\mathbb{Z}\otimes_T G$ action on Σ_A, for finite G and irreducible Σ_A, then there is a $\Sigma_{\bar{A}}$ with a special $\mathbb{Z}\otimes_T G$ action that is an almost conjugate extension of the $\mathbb{Z}\otimes_T G$ action on Σ_A. The factor map can be made to be either left or right resolving.*

Proof. We will first examine the case where we have a $\mathbb{Z}\times G$ action (see example 4). The construction we make is simpler when there is no skewing function. Assume that we have recoded so that we are dealing with a one-block G action, and notice that if $g(x) \neq x$ for all $g \neq$ identity and $x \in \Sigma_A$ then we need only go to a higher block presentation to arrive at the desired result. Suppose this is not the case. We shall show that by going to a sufficiently high block system we can choose a K in the alphabet so that:

(i) $g(K) \neq K$ for each $g \in G$, $g \neq$ identity;

(ii) the ssft Σ_F obtained by deleting the set $G(K)$ from the alphabet of Σ_A is irreducible and has the same ergodic period as Σ_A.

To see that this is possible, first notice that it is possible to choose a block B in Σ_A so that $g(B) \neq B$ for each $g \in G$, $g \neq$ identity. Fix this block and let I be the first symbol of B and J the last. Make sure B is long enough so that there is a block B' in Σ_A that has the same length as B, begins with I, ends with J, and is not in $G(B)$. Next find two blocks M and N so that the block MBN is in Σ_A and so that MBN cannot overlap itself in such a way that the B's overlap. This is choosing markers just as in the proof that all $\mathbb{Z}\otimes_T G$ actions exist for G finite. The construction there will produce such an M and N.

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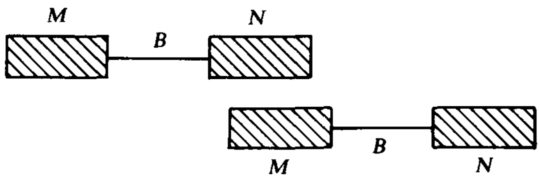

Do this for all blocks in $G(B)$ so that no block $g(MBN)$ can overlap MBN so much that $g(B)$ and B overlap. Let Σ_F be the ssft obtained by deleting $G(MBN)$ from the blocks of Σ_A. Σ_F is irreducible. To prove this we will produce a doubly transitive point for Σ_F. Take any doubly transitive point $x \in \Sigma_A$. Let y be the point in Σ_F obtained by replacing each occurrence of $g(MBN)$ in x by the block $g(M)g(B')g(N)$ for each $g \in G$. This is a doubly transitive point for Σ_F. To see that Σ_F can be made to have the same period we need only to choose x and y periodic in Σ_A with gcd {period x, period y} = period of Σ_A and then choose B so that x, $y \in \Sigma_F$. Now go to a sufficiently high block presentation so that $K = MBN$ is in the alphabet.

Once K is chosen and Σ_F is produced the rest is easy. The idea is to make an extension of Σ_A so that Σ_F has exactly $|G|$ pre-images and the symbol K has one. Assume now we have $L_F \subseteq L_A$, $G(K) = L_A - L_F$, and G acts as one-block maps on L_A. Define $\Sigma_{\bar{A}}$ by

$$L_{\bar{A}} = (L_A - L_F) \cup (L_F \times G)$$

and say

$$\begin{aligned} I &\to J && \text{if } I \to J \text{ in } \Sigma_A, \\ I &\to (J, g) && \text{if } I \to J \text{ in } \Sigma_A, \\ (I, g) &\to (J, g) && \text{if } I \to J \text{ in } \Sigma_A, \\ (I, g) &\to J && \text{if } J = g(K) \text{ and } I \to J \text{ in } \Sigma_A. \end{aligned}$$

This is irreducible because Σ_A and Σ_F are. Define a $\mathbb{Z} \times G$ action on $\Sigma_{\bar{A}}$ by:

$$h(I) \text{ is defined as on } L_A, \quad h(I, g) = (h(I), hg).$$

Notice that this is a special action since $(I, g) \neq (h(I), hg)$ unless h = identity. The obvious map $\Sigma_{\bar{A}} \to \Sigma_A$ is left resolving, one-to-one a.e., $|G|$-to-one at most, and commutes with the $\mathbb{Z} \times G$ actions. Notice that every element of $G(K)$ is a magic word and that the map is exactly $|G|$-to-one on Σ_F. The map could have been made right resolving instead of left by reversing the roles of the successors and predecessors in the construction of $\Sigma_{\bar{A}}$.

The construction for a $\mathbb{Z} \otimes_T G$ action uses the same idea but is slightly more involved. The idea is instead of covering Σ_F by $|G|$ copies of itself we cover it with $|G|$ copies of a periodic ssft. The standard p periodic cover (or extension) of an ssft type is defined as follows. Let $L = L_F \times \{0, \ldots, p-1\}$. Let q be the period of Σ_F and $L_F^0, \ldots, L_F^{q-1}$ be the decomposition of L_F into its cyclic subsets. Then say $(I, k) \to (J, l)$ if and only if $I \to J$ in Σ_F and either (i) $l = k + 1 \leq q - 1$, or (ii) $k = q - 1$ and $l = 0$. This new ssft is irreducible and has period p times the period of Σ_F. The obvious map from it onto Σ_F is exactly p-to-one. The covers of Σ_F in the new

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construction won't be copies of Σ_F now, but will be standard p periodic covers of Σ_F, where p is the period of T.

To begin the construction assume we have a $\mathbb{Z}\otimes_T G$ action on an irreducible ssft Σ_A. Choose Σ_F as before and let p be the period of T. Define $\Sigma_{\bar{A}}$ by

$$L_{\bar{A}} = (L_A - L_F) \cup (L_F \times G \times \{0, \dots, p-1\}),$$

and say

$$\begin{aligned} I &\to J && \text{if } I \to J \text{ in } \Sigma_A, \\ I &\to (J, g, n) && \text{if } I \to J \text{ in } \Sigma_A, \\ (I, g, n) &\to (J, g, n+1) \bmod p, && \text{if } I \to J \text{ in } \Sigma_A, \\ (I, g, 0) &\to J && \text{if } J = g(K) \text{ and } I \to J \text{ in } \Sigma_A. \end{aligned}$$

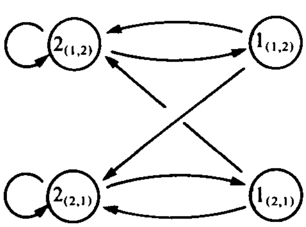
This is irreducible because Σ_A and Σ_F are. Define a $\mathbb{Z}\otimes_T G$ action on this by:

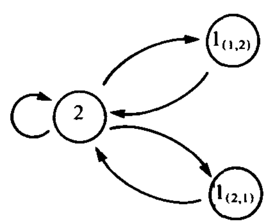
$$h(I) \text{ is defined as on } L_A, \quad h(I, g, n) = (h(I), (T^{p-n+1}h)g, n).$$

Notice that for a fixed g the symbols $\{(I, g, n)\}$, $I \in L_F$, $0 \le n < p$ and all transitions among them define an irreducible ssft $\Sigma_F^g \subseteq \Sigma_{\bar{A}}$. It is a standard p periodic cover of Σ_F. Let $\Sigma_F^{(g,n)}$ be the cyclic subset of Σ_F^g consisting of all points with (I, g, n) for $I \in L_F$ in the time zero coordinate, then $\sigma(\Sigma_F^{(g,n)}) = \Sigma_F^{(g,n+1)} \bmod p$ and $\Sigma_F^g = \Sigma_F^{(g,0)} \cup \dots \cup \Sigma_F^{(g,p-1)}$. Let $L_F^{(g,n)} = \{(I, g, n)\colon I \in L_F\}$, and observe that only $L_F^{(g,0)}$ has successors outside of L_F^g. We have a well-defined $\mathbb{Z}\otimes_T G$ action on $\Sigma_{\bar{A}}$ because of this and because if $(I, g, n) \to (J, g, n+1)$ in Σ_F^g then $h(I, g, n) \to Th(J, g, n+1)$ in Σ_F^k where $k = (T^{p-n+1}h)g$. The obvious map $\Sigma_{\bar{A}} \to \Sigma_A$ is left resolving, one-to-one a.e., $p|G|$-to-one at most and commutes with the $\mathbb{Z}\otimes_T G$ actions. Again, every element of $G(K)$ is a magic word, but now the inverse image of Σ_F is $|G|$ copies of the standard p periodic cover of Σ_F. □

From this point on we will assume that all actions are special.

Example 4. Define $\Sigma_{\bar{A}}$ by the first of the two graphs that follow, and Σ_A by the second. Then the symbol K in the previous proof can be taken to be $1_{(1,2)}$ in L_A, and $\Sigma_F \subseteq \Sigma_A$ will be the single fixed point of all 2's. Using these, the construction in the proof will produce $\Sigma_{\bar{A}}$.





3. *Classification of $\mathbb{Z}\otimes_T G$ action; the aperiodic case*
In this section we will prove the following:

THEOREM 4. *If Σ_A and Σ_B are aperiodic ssft, G is a finite group, T is an automorphism of G, and Σ_A and Σ_B both have $\mathbb{Z}\otimes_T G$ actions, then these actions are almost topologically conjugate if and only if the two ssft have the same topological entropy.*

The proof is a slight generalization of the proof of the corresponding thorem in [**1**]. There are five parts: (i) getting a suitable positive integer matrix F satisfying $AF = FB$, (ii) defining the tableau, (iii) filling in the tableau to get magic words when Σ_A and Σ_B have fixed points, (iv) showing that every aperiodic ssft is almost topologically conjugate to an ssft with a fixed point, (v) building a 'tower' to see that (iii) and (iv) together imply the theorem. In each section we will first explain what is needed for the $\mathbb{Z}$ action alone and then show what changes are needed to deal with the $\mathbb{Z}\otimes_T G$ action. The difference is that any symbolic manipulation must be pushed around by the group action.

Part (i).

LEMMA (Furstenberg). *Let A and B be irreducible transition matrices.* $\lambda_A = \lambda_B$ *if and only if there exists a positive integral matrix F such that* $AF = FB$.

Proof. (Note: this is not the proof in [**1**], it is the original one given by Furstenberg and can be found in [**5**] – the reason we use this proof here is that it allows us to construct an F with some additional properties.) Suppose $\lambda_A = \lambda_B$. Choose a strictly positive right eigenvector, $\imath$, for A, and a strictly positive left eigenvector, ℓ, for B. Let $N = \imath\ell$ (matrix multiplication). It is a positive real matrix satisfying $AN = NB$. For $\varepsilon > 0$ choose a $t \in \mathbb{R}^+$ such that $tN = F + U$ where F is a strictly positive integer matrix and U has all entries less than ε in absolute value. Then $AF - FB$ is an integer matrix. But

$$|AF - FB| \leq |A(F - tN)| + |(tN - F)B| \leq (|A| + |B|)|U|.$$

By choosing ε sufficiently small we get that $AF - FB = 0$. □

In [**1**] any choice of F will do; in this case we must be more careful. The extra requirement is spelled out in the following lemma.

LEMMA. *Let* Σ_A, Σ_B *be irreducible ssft. Suppose* $\lambda_A = \lambda_B$ *and they have* $\mathbb{Z}\otimes_T G$ *actions. Then there is a positive integral F such that* $AF = FB$ *and* $F_{I,J} = F_{gI,hJ}$ *for all* $g, h \in G$.

Proof. Recall the construction of F in the previous lemma. $F \approx t\imath\ell$ where $t \in \mathbb{R}^+$, $A\imath = \lambda\imath$, and $\ell B = \lambda\ell$. The extra condition is satisfied because a $\mathbb{Z}\otimes_T G$ action means $P_g A = AP_{Tg}$ and $Q_g B = BQ_{Tg}$ for the appropriate families of permutation matrices $\{P_h\}, \{Q_h\}$ as defined in observation 2. This means $P_{T^{-n}g}A^n = A^n P_g$; since $T^p = \mathrm{id}$ for some $p > 0$, $P_g A^p = A^p P_g$ and $\lambda^p P_g \imath = P_g A^p \imath = A^p P_g \imath$, so $P_g \imath = \imath$, for all $g \in G$. The same is true for B, $\ell Q_g = \ell$ and since $F \approx t\imath\ell$ we have the desired result. □

Part (ii). Here we define the tableau and then show that it can be filled in so that we have a common finite-to-one extension for the special $\mathbb{Z}\otimes_T G$ actions. The tableau is defined just as it was in [**1**]. Take A and B to be irreducible transition matrices with a common maximal eigenvalue. Suppose they have alphabets $L_A = \{1, \ldots, |L_A|\}$ and $L_B = \{1, \ldots, |L_B|\}$ and that we have chosen a positive integral matrix F satisfying $AF = FB$. Define a new symbol set

$$L_C = \{(I, J, K): I \in L_A, J \in L_B, 1 \leq K \leq F_{IJ}\}.$$

We call the following list of partially assigned transitions the *tableau associated with* A, B *and* F.

For each $I \in L_A$ we get a page in the tableau, and for each $J \in L_B$ we get a paragraph on each page. Pick $I \in L_A$, its page is:

$$\begin{array}{ccccc}
(I, 1, 1) & \rightarrow & (\cdot, 1_1, \cdot) & \cdots & (\cdot, 1_{|f(1)|}, \cdot) \\
\vdots & & & & \\
(I, 1, F_{I1}) & \rightarrow & (\cdot, 1_1, \cdot) & \cdots & (\cdot, 1_{|f(1)|}, \cdot) \\
\vdots & & & & \\
(I, J, 1) & \rightarrow & (\cdot, J_1, \cdot) & \cdots & (\cdot, J_{|f(J)|}, \cdot) \\
\vdots & & & & \\
(I, J, F_{IJ}) & \rightarrow & (\cdot, J_1, \cdot) & \cdots & (\cdot, J_{|f(J)|}, \cdot) \\
\vdots & & & & \\
(I, |L_B|, 1) & \rightarrow & (\cdot, |L_B|_1, \cdot) & \cdots & (\cdot, |L_B|_{|f(|L_B|)|}, \cdot) \\
\vdots & & & & \\
(I, |L_B|, F_{I|L_B|}) & \rightarrow & (\cdot, |L_B|_1, \cdot) & \cdots & (\cdot, |L_B|_{|f(|L_B|)|}, \cdot).
\end{array}$$

The paragraph corresponding to a $J \in L_B$ is composed of the rows where J appears as the middle symbol of the left most entry.

If we see in the tableau $(I, J, K) \rightarrow (\cdot, J', \cdot)$ we have by definition that $J \rightarrow J'$ is a B-transition. We want to fill in the first component on the right with an I' for which $I \rightarrow I'$ is an A-transition and the third component with a K', $1 \leq K' \leq F_{I'J'}$ such that the resulting (I', J', K') is used only once in the page for I. We also require that each (I', J', K') that can appear in the page for I does appear. That this can be done is a consequence of the matrix equation $AF = FB$.

Generally, there are many choices for filling in the tableau. Once the page is filled in for each $I \in L_A$ we have defined a tansition matrix C for the symbols L_C. We define one-block mappings, π and $\bar{\pi}$ from Σ_C to Σ_A and Σ_B by $\pi(I, J, K) = I$ and $\bar{\pi}(I, J, K) = J$. Because of the construction of the tableau π is left resolving and $\bar{\pi}$ is right resolving. Σ_C may not be irreducible, but if there are resolving blocks for both π and $\bar{\pi}$ it will be. This follows from the fact that one of the maps is left resolving and the other right.

Now suppose Σ_A and Σ_B have $\mathbb{Z} \otimes_T G$ actions. We want to show that a tableau can be produced and filled in so that the resulting Σ_C has a $\mathbb{Z} \otimes_T G$ action that is a continuous extension of the $\mathbb{Z} \otimes_T G$ actions on Σ_A and Σ_B. First choose F as described in the second lemma of part one. Here we will use the condition that $F_{IJ} = F_{gIhJ}$ all $g, h \in G$. Write down the tableau as before. Notice that if we examine the page for I, it looks just like the ones for gI, each $g \in G$. That is, for each row in I's page

$$(I, J, K) \rightarrow (\cdot, J_1, \cdot)(\cdot, J_2, \cdot) \cdots (\cdot, J_{|f(J)|}, \cdot)$$

there is a corresponding one in $g(I)$'s page

$$(g(I), g(J), K) \rightarrow (\cdot, Tg(J_1), \cdot)(\cdot, Tg(J_2), \cdot) \cdots (\cdot, Tg(J_{|f(J)|}), \cdot).$$

The method for filling in the tableau to produce the desired G action on Σ_C is obvious. L_A is partitioned into disjoint G orbits. Choose one representative from each. Fill in the page for each of these representatives. Then use G to push these to the rest of the tableau. So if we have $(I, J, K) \to (I', J', K')$ we fill in $(g(I), g(J), K) \to (Tg(I), Tg(J), K')$. Since $F_{IJ} = F_{gIhJ}$ everything fits. This produces a Σ_C with the desired $\mathbb{Z} \otimes_T G$ action. The maps π and $\bar{\pi}$ take this action to the ones on Σ_A and Σ_B.

Part (iii). This is the crucial step. Here we assume that Σ_A and Σ_B have fixed points and show that the tableau can be filled in so that the resulting maps are one-to-one a.e. This is done by carefully filling in the tableau so that resolving blocks are produced for both maps. This ensures that Σ_C is irreducible and that the maps are one-to-one a.e. As before, we will present the argument worrying only about the shift, then show how it can be modified to take into account the entire $\mathbb{Z} \otimes_T G$ actions.

Suppose A and B are irreducible transition matrices with fixed states. The symbol sets are of the form $L_A = \{1, \ldots, |L_A|\}$, $L_B = \{1, \ldots, |L_B|\}$. In both cases assume 1 is the fixed state. In the directed graph defined by A we 'grow' a tree rooted on 1 of simple A-paths which can only join but not cross each other, leading from any $I \in L_A$, $I \neq 1$, forward to 1. For each I denote the tree transition by $I \to I_1$ and the non-tree ones by $I \to I_2, \ldots, I_{|f(I)|}$. Let $I_1 = 1$ when $I = 1$. Likewise in the directed graph determined by B grow a tree rooted on 1 of simple non-crossing B-paths from each $J \in L_B$ backwards to 1. These tree transitions are denoted by $J_1^* \to J$, with $J_1^* = J$ for $J = 1$, and the non-tree ones by $J_2^*, \ldots, J_{|p(J)|}^* \to J$. Next we will sequentially fill in the tableau in five steps. When making an assignment in this procedure, say, $(I, J, K) \to (I', J', K')$ two conditions must prevail in order to fill in the tableau legally:

(i) no $(\cdot, J', \cdot)$ has previously been assigned as a successor to (I, J, K);

(ii) (I', J', K') has not been previously assigned as a successor to any $(I, \cdot, \cdot)$.

As the following steps are presented we leave it to the reader to check that these conditions are met. I and J will be used as generic symbols of L_A and L_B, respectively.

Step 1. $I \neq 1$, $J = 1$, set

$$
\begin{array}{ccc}
(I, 1, 1) & \to & (I_1, 1, 1) \\
\vdots & & \vdots \\
(I, 1, F_{I_1 1}) & \to & (I_1, 1, F_{I_1 1}) \\
(I, 1, F_{I_1 1} + 1) & \to & (I_2, 1, 1) \\
\vdots & & \vdots \\
(I, 1, F_{I_1 1} + F_{I_2 1}) & \to & (I_2, 1, F_{I_2 1}) \\
\vdots & & \vdots \\
(I, 1, F_{I1}) & \to & (I_R, 1, N)
\end{array}
$$

where $1 \leq R \leq |f(I)|$ and N is whatever positive integer is needed.

Step 2. For $I = 1$, $J = 1$, (assume $F_{11} > 1$) set

$$
\begin{array}{ccc}
(1, 1, 1) & \to & (1, 1, 1) \\
(1, 1, 2) & \to & (1_2, 1, 1) \\
(1, 1, 3) & \to & (1, 1, 2) \\
\vdots & & \vdots \\
(1, 1, F_{11}) & \to & (1, 1, F_{11} - 1)
\end{array}
$$

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The idea of step 1 is to proceed from $(I, 1, K)$ by forward transitions in C in which, if possible, the first coordinate moves along tree transitions while the third stays constant as 1 is being repeated successively in the second. If this is not possible, the first coordinate takes a non-tree transition and the third is reduced. Eventually either the third coordinate is reduced to 1 or we arrive at $(1, 1, K)$. In step 2, if $1 < K \leq F_{11}$, we spiral down from $(1, 1, K)$ to $(1, 1, 2)$ and then out to $(1_2, 1, 1)$. Then every $(I, 1, 1)$ leads to $(1, 1, 1)$ along tree transitions as 1 is repeated in the second coordinate. Then no matter how the rest of the tableau is filled in the block $B = [1, 1, \ldots, 1]$ with 1 repeated $|L_C|$ times, maybe even less, is resolving for $\bar{\pi}$ - i.e. all C-admissible $|L_C|$-blocks in $\bar{\pi}^{-1}B$ are of the form $[(I, 1, K), \ldots, (1, 1, 1)]$.

Step 3. For $I = 1$, $J \neq 1$, set

$$\begin{array}{ccc}
(1, J_1^*, 1) & \rightarrow & (1, J, 1) \\
\vdots & & \vdots \\
(1, J_1^*, F_{1J_1^*}) & \rightarrow & (1, J, F_{1J_1^*}) \\
(1, J_2^*, 1) & \rightarrow & (1, J, F_{1J_1^*}+1) \\
\vdots & & \\
(1, J_2^*, F_{1J_2^*}) & \rightarrow & (1, J, F_{1J_1^*}+F_{1J_2^*}) \\
\vdots & & \vdots \\
(1, J_R^*, N) & \rightarrow & (1, J, F_{1J})
\end{array}$$

where $1 \leq R \leq |p(J)|$ and N is whatever positive integer is needed.

Step 4. Assuming $F_{11} > 1$ set

$$(1, 1_2^*, 1) \rightarrow (1, 1, F_{11}).$$

Step 5. Fill in the rest of the tableau arbitrarily but according to the rules of part 2.

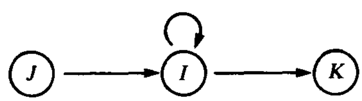
Now moving from $(1, J, K)$ by backward transitions in C the second coordinate according to step 3 moves back along tree transitions when possible while keeping the third coordinate fixed as 1 is repeated in the first. On the other hand if this is not possible the second coordinate takes a non-tree transition and the third is reduced. Eventually either the third coordinate is reduced to 1 or we arrive at $(1, 1, K)$. If $1 < K \leq F_{11}$, by steps 2 and 4 we spiral back up from $(1, 1, K)$ to $(1, 1, F_{11})$ and then back to $(1, 1_2^*, 1)$. Then $(1, J, 1)$ leads back to $(1, 1, 1)$ along tree transitions as 1 is repeated in the first coordinate. Then no matter how the rest of the tableau is filled in by step 5, the block $B = [1, \ldots, 1]$ of $|L_C|$ consecutive 1's is resolving for π - i.e. all C-admissible $|L_C|$-blocks in $\pi^{-1}(B)$ are of the form $[(1, 1, 1), \ldots, (1, J, K)]$. In the case $F_{11} = 1$ replace F by $2F$.

At this point we have shown that if A and B are irreducible transition matrices with positive trace and the same largest eigenvalues then Σ_A and Σ_B are almost conjugate factors of Σ_C.

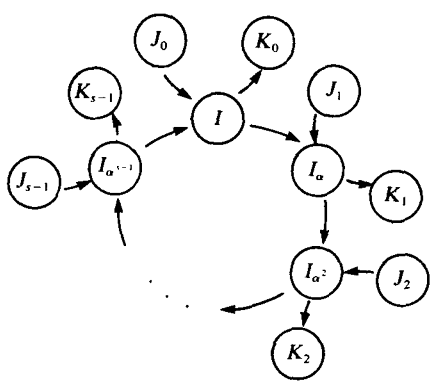
Let $I = g(1)$; it need not be a fixed symbol. If $Tg \neq g$ then $g(1) \neq Tg(1)$ since we are dealing with a special action. For each $I \in G(1)$ denote by $I \to I_\alpha$ (or $I^*_\alpha \to I$) the special transition where $I = g(1)$ and $I_\alpha = Tg(1)$ (or $I^*_\alpha = T^{-1}g(1)$). For a $g \in G$ let s be the least integer so that $T^s g = g$. Taking $I_{\alpha^s} = T^s g(1)$ we have the loop $I \to I_\alpha \to I_{\alpha^2} \to \cdots \to I_{\alpha^{s-1}} \to I$. To complete this part of the argument we must choose our trees carefully, taking into account these symbols $I \in G(1)$. The extra condition on our trees is spelled out in the following lemma.

LEMMA. *Suppose A is an aperiodic transition matrix with non-zero trace, G is a finite group, and $\mathbb{Z} \otimes_T G$ acts on Σ_A. Then there is a $\Sigma_{\bar{A}}$, with a conjugate $\mathbb{Z} \otimes_T G$ action, where we can choose a tree in the graph of $\bar{A}$, based on a fixed symbol,* 1, *so that for $J \in G(1)$, $J \neq 1$; $J \neq I_1$ for any $I \in L_{\bar{A}}$.*

Proof. Begin with the graph of A where 1 is the fixed symbol, by using the elementary conjugacies we can easily make sure that 1 has exactly one predecessor and one successor other than itself, and that neither is in the G-orbit of 1.

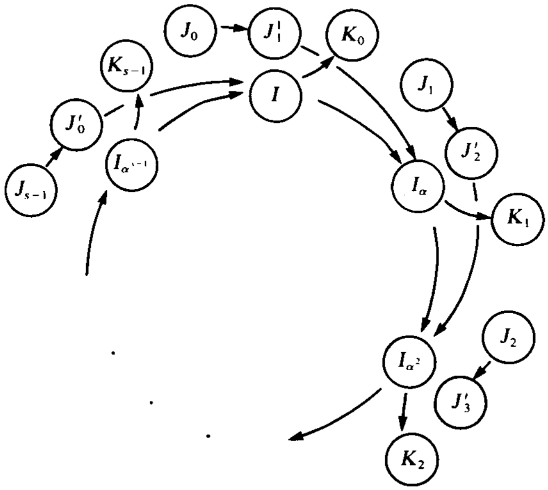


This means that if we look at any α loop in $G(1)$, it looks like:



where J_t and K_t are not in $G(1)$ for all t.

First we will show that we can make sure there are no tree transitions into the α loop from the outside. Assume $J_0 \to I$ is a tree transition, and that $I \to I_\alpha$, $I_\alpha \to I_{\alpha^2}, \ldots, I_{\alpha^{n-1}} \to I_{\alpha^n}$, $I_{\alpha^n} \to K_n$ are also tree transitions for some n, $1 \leq n \leq s-1$. We will show by induction that we can get rid of this. Split each element in $G(1)$ by predecessors, so in particular I_{α^t} gets split into $[I_{\alpha^{t-1}}, I_{\alpha^t}]$ which we will again call I_α and $[J_t, I_{\alpha^t}]$ which we will now call J^1_{t+1}. This results in a new conjugate aperiodic ssft, that inherits a tree and a $\mathbb{Z} \otimes_T G$ action. The α loop now looks like:



Notice what we have accomplished. Before splitting we had a tree transition enter the loop proceed along the loop for n steps and then exit. Now the transition enters, proceeds for $n-1$ steps and exits. By repeating this construction a sufficient number of times we may assume no tree transition enters an α loop from the outside.

It may still be that there are tree transitions that agree with α transitions. Notice that because of the previous construction we can throw out all of $G(1)$, except 1, and the remaining graph will still contain a tree leading from each vertex, forward to 1. This means that in the whole graph if $I \to I_\alpha$ is a tree transition, we can change I's tree transition to $I \to J$ where J is I's successor outside of $G(1)$, and we will still have a tree. Do this for all $I \in G(1)$, $I \neq 1$ and we have the desired result. □

We are now in a position to complete this part of the proof. Repeat the procedure of step 1, but this time only for those I's not in the orbit of 1. Call these original transitions. Next fill in the transitions forced by G. If $(I, J, K) \to (I', J', K')$ is an original transition then $(gI, gJ, K) \to (Tg(I'), Tg(J'), K')$ is a forced transition for each $g \in G$. We must see that no contradictions have arisen. First, notice that the pages for $I \in G(1) \subseteq L_A$, are still blank. Next examine the page of an $I \notin G(1)$. Notice that only those paragraphs corresponding to a $J \in G(1) \subseteq L_B$ have any rows with any transitions filled in. In fact, each row in such a paragraph has exactly one transition filled in because there is a unique $h \in G$ such that $hJ = 1$. At this point there are no contradictions.

Step 2. This is now for $I \in G(1) \subseteq L_A$; $J = 1$, (note that $F_{I1} = F_{11}$) set

$$\begin{array}{ccc}
(I, 1, 1) & \to & (I_1, 1, 1) \\
(I, 1, 2) & \to & \begin{cases} (I_\alpha, 1, 1) & \text{if } I \neq 1 \\ (I_2, 1, 1) & \text{if } I = 1 \end{cases} \\
(I, 1, 3) & \to & (I_\alpha, 1, 2) \\
\vdots & & \vdots \\
(I, 1, F_{11}) & \to & (I_\alpha, 1, F_{11} - 1).
\end{array}$$

These are original transitions and the ones forced by G are filled in as in step 1. We must again check for contradictions. First notice that the pages affected in step 1 and those affected here are disjoint. No contradictions can arise from conflict between the two steps. Step 2 is internally consistent by the same reasoning as applied in step 1.

At this point we have constructed a resolving block for $\bar{\pi}$. The reasoning is identical to that after step 2 in the original argument except that $(I, 1, K)$ for $I \in G(1)$, $K \geq 2$ proceeds by forward transitions down to $(I_{\alpha^{K-2}}, 1, 2)$ as 1 is repeated in the second coordinate. Then if $I \neq 1$, and another 1 is repeated in the second coordinate, $(I_{\alpha^{K-1}}, 1, 1)$ results. If $I = 1$, another transition results in $(1_2, 1, 1)$. In either case we are led to $(1, 1, 1)$ as 1 is repeated in the second coordinate.

Step 3. For $I = 1, J \notin G(1) \subseteq L_B$ the transitions are filled in just as in the original argument. Recall that the only lines where any transitions are filled in are in the paragraphs with $J \in G(1)$ on pages with $I \notin G(1)$ from step 1 and in the paragraphs with $J \in G(1)$ on pages with $I \in G(1)$ from step 2. In step 3 we have filled in transitions in rows in paragraphs for arbitrary J, on pages with $I \in G(1)$. We must see that no contradictions arise from step 2. Recall that in the transitions from step 2, $(I, J, K) \to (I', J', K')$, all have $J' \in G(1)$. The transitions from step 3 never have $J' \notin G(1)$. There are no contradictions in step 3 arising from either step 1 or step 2. Step 3 is internally consistent by the same reasoning as applied in steps 1 and 2.

Step 4. For $J \in G(1)$, $(F_{1J} = F_{11})$, set

$$\begin{array}{ll} (1, J_1^*, 1) \to (1, J, F_{11}) & \text{if } J \neq 1, \\ (1, J_2^*, 1) & \text{if } J = 1. \end{array}$$

Since the transitions arising here all have $I \in G(1)$ the only possible external contradictions are with either step 2 or step 3. When $J \neq 1$, $J_1^* \notin G(1)$, so here the only possible contradiction is with step 3. It may be that there is a $J' \notin G(1)$ with $(J')_n^* = J_1^*$ so that in step 3 we have $(1, (J')_n^*, 1) = (1, J_1^*, 1) \to (1, J', K)$ and in step 4 we have $(1, J_1^*, 1) \to (1, J, F_{11})$ but since $J' \notin G(1)$ and $J \in G(1)$ there won't be a contradiction. When $J = 1$ we have

$$(1, 1_2^*, 1) \to (1, 1, F_{11}).$$

Since $1_2^* \notin G(1)$ the same reasoning applies here. Finally, we check to see that there are no internal contradictions in step 4. This is the same as steps 1-3.

Now we have constructed a resolving block for π. It works just as it did in the original argument with one exception. If we are at $(1, J, K)$ for $J \in G(1)$ and move backward as 1 is repeated in the first coordinate, J takes successive J_α^* transitions, K is increased until it reaches F_{11}. Then we go to a $(1, J', 1)$ and then by tree transitions to $(1, 1, 1)$, never seeing another $J \in G(1)$ as a centre coordinate.

Step 5. Fill in the rest of the tableau arbitrarily but according to the rules of part 2 to get a $\mathbb{Z} \otimes_T G$ action on Σ_C that commutes with the maps π and $\bar{\pi}$.

Part (iv). Here we will show that any aperiodic ssft with a $\mathbb{Z} \otimes_T G$ action is almost-topologically conjugate to a $\mathbb{Z} \otimes_T G$ action on an ssft with a fixed point under σ. This does not follow the proof of the corresponding statement in [1]. Instead it follows proofs of that statement done independently by W. Krieger [4] and M. Keane. The idea is fairly straightforward, we begin with A and find a conjugate representation whose graph contains a simple cycle all of whose vertices have a common predecessor and disjoint successors. This is done by splitting states and making amalgamations. This ssft can then be covered, with an almost-conjugate extension, by making two copies of this cycle, each with the same successors but dividing the predecessors. Each vertex in one copy will have only the single common predecessor outside of the cycle. The corresponding vertex in the other copy will have all the other predecessors. The map obtained by identifying the two cycles is one-to-one a.e. and right resolving. Finally, this extension has an almost conjugate factor obtained by collapsing the copy of the cycle, where each vertex has the common predecessor, to a single fixed vertex. This is a one-to-one a.e. left resolving map whose image is the desired ssft with a fixed point. Our proof here follows this outline. The problem is to make sure that all these operations can be made consistent with the $\mathbb{Z}\otimes_T G$ action. We begin with the following lemma.

LEMMA. *Given a $\mathbb{Z}\otimes_T G$ action there is a conjugate $\mathbb{Z}\otimes_T G$ action for which the graph of the ssft contains a simple p-cycle*

$$J^0 \to J^1 \to J^2 \to \cdots \to J^{p-1} \to J^0,$$

and there exists an integer N, a state J_0 and allowable paths $\gamma_R = [J_0^R, J_1^R, \ldots, J_N^R]$, $0 \le R < p$ such that

(1) *each $J_0^R = J_0$, $J_N^R = J^R$, and*

(2) *$\{J_i^R\}$, $0 \le R < p$, $0 \le i \le N$ are all in distinct G-orbits.*

Note. After proving this, we will then show that in fact N can be chosen to be one.

Proof. First, we show that there is a σ-periodic point, x, such that $g(x) \notin \sigma\text{-orb}(x)$ for all $g \neq \text{id}$. In fact, any periodic point with least period p, for p prime and greater than the order of G has this property. Let k be the period of T; then $\sigma^k \circ g = g \circ \sigma^k$ for all $g \in G$. For a point, x, with prime period p, $p > k$, $\sigma\text{-orb}(x) = \sigma^k\text{-orb}(x)$. Suppose for some $g \neq$ identity, $g(x) \in \sigma\text{-orb}(x)$. Then $g(x) \in \sigma^k\text{-orb}(x)$ so $g(x) = \sigma^{mk}(x)$ for some $0 \le m < p$. Let j be the order of g. We have $x = g^j(x) = \sigma^{jmk}(x)$, so p divides jmk, since $j \neq 0$ and is less than p, p divides mk, but this means $g(x) = x$. This contradicts the fact that the $\mathbb{Z}\otimes_T G$ action is special. For x of prime period greater than the order of G, and $g \neq$ identity, $g(x) \notin \sigma\text{-orb}(x)$. Choose such an x, then by going to a sufficiently high block system we can assume that the σ-orbit of x is represented by a simple p-cycle,

$$I^0 \to I^1 \to \cdots \to I^{p-1} \to I^0,$$

in the graph of A. Also that for $g \neq$ identity $g(\{I^R\}_{0 \le R < p}) \cap \{I^R\}_{0 \le R < p} = \emptyset$. Since A is aperiodic, we can find an $n > 0$, an $I_0 \in f(I^0)$, and paths $\gamma_R = [I_0^R, I_1^R, \ldots, I_n^R]$, $0 \le R < p$, where:

$I_0^R = I_0$, $I_n^R = I^R$ for all R;
$\{I_j^R\}_{0 \le R < p, 1 \le j \le n}$ are distinct;
$\{I_j^R\}_{0 \le R < p, 0 \le j < n}$ all miss the G-orbit of $\{I^R\}_{0 \le R < p}$.

It may be necessary to go to an even higher block presentation to do this. Let $[K_0, \ldots, K_{n-1}] = [I^{p-n+1}, \ldots, I^{p-1}, I^0]$ (where $I^{p-t} \equiv I^{p-t(\bmod p)}$ if $p-t$ is negative). For $0 \le R < p$, and $0 \le t \le 2n$ let

$$J_t^R = \begin{cases} [K_t, \ldots, K_{n-1}, I_0^R, \ldots, I_t^R] & 0 \le t < n \\ [I_{t-n}^R, \ldots, I_n^R, I^{R+1}, \ldots, I^{R+t-n}] & n \le t \le 2n \end{cases}$$

(where $I^s \equiv I^{s(\bmod p)}$ for $s \ge p$). This defines symbols, J_t^R, in the $(n+1)$-block presentation. The periodic point x is represented by the cycle

$$J^0 \to J^1 \to \cdots \to J^{p-1} \to J^0,$$

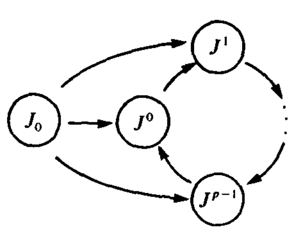
where $J^R = [I^R, I^{R+1}, \ldots, I^{R+n}]$. Now let $N = 2n$ and $\gamma_R = [J_0^R, \ldots, J_N^R]$ and we will see that the γ_R satisfy conditions (1) and (2) of the lemma. Condition (1) is clearly satisfied. To see that (2) is satisfied, observe that since each $J_t^R, 0 \le t < n$, begins with an I^k; these have disjoint G-orbits by the conditions imposed on $\{I^0, \ldots, I^{p-1}\}$. By the same reasoning each J_t^R, $n \le t \le 2n$, ends with an I^k; so these have disjoint G-orbits. Since each $J_t^R, 0 \le t < n$ ends with an I_s^R, $s < n$, and each J_t^R, $n \le t \le 2n$ ends with an I^k the condition that $\{I_j^R\}_{0 \le R < p, 0 \le j \le n}$ all miss the G-orbit of $\{I^0, \ldots, I^{p-1}\}$ means that the G-orbits of the J_t^R with $t \ge n$ and the J_t^R with $t < n$ are disjoint. This means condition (2) is satisfied. □

Next, by induction, we show that N can be chosen to be one. The rough idea is that we would like to amalgamate the $\{J_1^R\}$, $0 \le R < p-1$. But as it stands these states don't satisfy the hypotheses for amalgamation. We need some preliminary splitting. Let

$$E_1 = \{J_2^0\}, \qquad E_2 = f(J_1^0) - \{J_2^0\}.$$

This is a partition of $f(J_1^0)$ and this describes a state splitting as described earlier. Recall in the discussion of state splitting the way this is pushed around by the G action. Condition (2) of the previous lemma makes sure that this doesn't affect any of the other J_i^R. This replaces J_1^0 by a new (split state) J_1^0 and keeps all other conclusions of the lemma in effect. Do this for each R. The result is that each new J_1^R has exactly one successor, namely J_2^R, and the conclusions of the lemma still hold. Now split by predecessors so that each new J_1^R has exactly one predecessor J_0^R and one successor J_2^R. We can now amalgamate the $\{J_1^R\}$. The new conjugate ssft still has a special $\mathbb{Z} \otimes_T G$ action, and the lemma is still satisfied but the length of the paths has been reduced to $N-1$. This means that up to topological conjugacy we may assume that we have special $\mathbb{Z} \otimes_T G$ action and that somewhere in the graph we see the picture:

(1)



(2) and the $\{J_0, J^0, \ldots, J^{p-1}\}$ are in disjoint G-orbits.

(3) By splitting states we can easily guarantee that the $\{J^R\}$, $0 \le R < p$ have disjoint follower sets.

Next we build an almost conjugate extension, Σ_B, in which somewhere in the graph of Σ_B we have the above picture and also

(4) $$p(J^R) = \{J_0, J^{R-1}\}.$$

To produce Σ_B let

$$L_B = L_A \cup \{K_g^R\}, \qquad 0 \le R < p, g \in G.$$

Eliminate the transitions

$$T^{-1}g(J_0) \to g(J^R), \qquad g \in G, 0 \le R < p,$$

from those allowed by A and add new transitions

(i) each $T^{-1}g(J_0) \to K_g^R$;

(ii) for $L \in f(J^R)$, $L \ne J^{R+1}$, let $K_g^R \to Tg(L)$;

(iii) $K_g^R \to K_{Tg}^{R+1}$.

Now observe that the cycle

$$K_e^0 \to K_e^1 \to \cdots \to K_e^{p-1} \to K_e^0 \qquad (e \text{ is the group identity element}),$$

satisfies conditions (1), (2), (3), and (4) with J_0 unchanged but each J^R is replaced by K_e^R. The G action is extended to Σ_B by defining

$$g(K_h^R) = K_{gh}^R \qquad \text{for } 0 \le R < p;\ g, h \in G.$$

This gives a special $\mathbb{Z} \otimes_T G$ action on Σ_B. Define a right resolving, one-to-one a.e. factor map, π, from $\Sigma_B \to \Sigma_A$ by

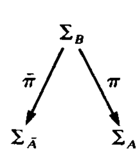
$$\pi(I) = I, \qquad \text{for } I \in L_A,$$
$$\pi(K_g^R) = g(J^R) \qquad \text{for all } R \text{ and } g.$$

This takes the $\mathbb{Z} \otimes_T G$ action on Σ_B to the one on Σ_A.

Finally, we create, $\Sigma_{\bar{A}}$, an almost conjugate factor of Σ_B. To do this let:

$$L_{\bar{A}} = L_A \cup \{K_g\} \qquad g \in G$$
$$T^{-1}g(J_0) \to K_g \qquad \text{all } g \in G$$
$$K_g \to K_{Tg}$$
$$K_g \to L \qquad \text{when } K_g^R \to L \text{ any } R$$

This is the image of Σ_B under the left resolving map $\bar{\pi}(I) = I$ for $I \in L_A$, $\bar{\pi}(K_g^R) = K_g$. A G action is defined on $\Sigma_{\bar{A}}$ by $g(K_h) = K_{gh}$. The map, $\bar{\pi}$, takes the special $\mathbb{Z} \otimes_T G$ action on Σ_B to the one on $\Sigma_{\bar{A}}$. Notice that $K_e \to K_e$ so $\Sigma_{\bar{A}}$ has a fixed point. We have the picture



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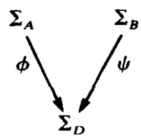
where π and $\bar{\pi}$ are one-to-one a.e. factor maps that conjugate the appropriate special $\mathbb{Z}\otimes_T G$ actions. $\Sigma_{\bar{A}}$ has a fixed point, and $\bar{\pi}$ is left resolving while π is right resolving. We could have made $\bar{\pi}$ right resolving and π left resolving by switching the roles of predecessors and successors. This proves the statement at the beginning of this part.

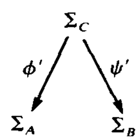
Part (v). Here we observe that the standard tower construction works for $\mathbb{Z}\otimes_T G$ actions. Suppose we are given:



where Σ_A, Σ_B, and Σ_D have special G actions, ϕ and ψ are factor maps that also commute with these actions. When we do the standard constructions:

$$L_C = \{(I, J): I \in L_A, J \in L_B \text{ and } \phi(I) = \psi(J)\},$$

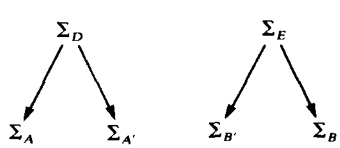
$(I, J) \to (I', J')$ if and only if $I \to I'$ in A and $J \to J'$ in Σ_B, we get Σ_C a cover for Σ_A and Σ_B.



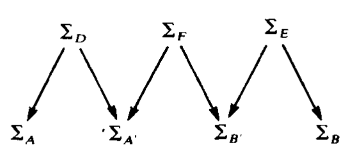
Recall that ϕ' retains all the characteristics of ψ and ψ' all those of ϕ. Σ_C has a $\mathbb{Z}\otimes_T G$ action defined on it by $(g(I), g(J))$. This is well-defined and commutes with the maps because

$$g \circ \phi'(I, J) = g(I) = \phi'(g(I), g(J)).$$

This is all we need. We now put all five parts of this section together for a proof of the theorem stated at the beginning. Given Σ_A and Σ_B aperiodic with $\mathbb{Z}\otimes_T G$ actions and $\lambda_A = \lambda_B$, begin (using part (iv)) by finding $\Sigma_{A'}$, Σ_D, $\Sigma_{B'}$, Σ_E such that



$\Sigma_{A'}$ and $\Sigma_{B'}$ have fixed points, while all four maps are one-to-one a.e. Next use parts (i), (ii), (iii) to get Σ_F such that



with both maps one-to-one a.e. Finally, use part (v) three times to get:

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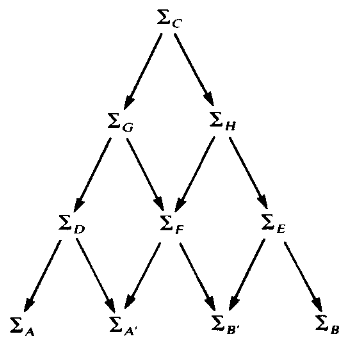


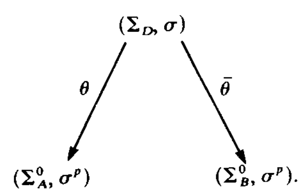
The resulting map $\Sigma_C \to \Sigma_A$ is left resolving, one-to-one a.e. and the one $\Sigma_C \to \Sigma_B$ is right resolving one-to-one a.e. This completes the proof of the theorem. □

4. *The periodic case*

Here we deal with the case where A and B are irreducible but periodic. First, we will give the proof when there is no G acting [**1**]:

THEOREM 5. *Let Σ_A and Σ_B be irreducible ssft. They are almost topologically conjugate if and only if they have the same entropy and ergodic period.*

Proof. We have the proof when A and B are aperiodic. Suppose Σ_A and Σ_B are irreducible with the same entropy and period. Let $\Sigma_A^0, \ldots, \Sigma_A^{p-1}$ and $\Sigma_B^0, \ldots, \Sigma_B^{p-1}$ be the cyclic subsets of Σ_A and Σ_B, under σ. This means (Σ_A^n, σ^p) and (Σ_B^n, σ^p), $0 \le n < p$, are aperiodic ssft with p times the entropy of Σ_A or Σ_B. By the previous section there is an aperiodic Σ_D and one-to-one a.e. factor maps θ and $\bar{\theta}$ such that



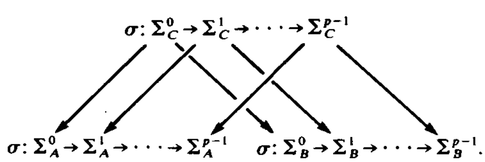
Define Σ_C an irreducible ssft of period p by:

$$L_C = L_D \times \{0, \ldots, p-1\}$$

and

$$(I, k) \to (J, l) \qquad \text{if } I = J \text{ and } l = k+1 \le p-1$$
$$\text{or } I \to J \text{ in } \Sigma_D \text{ and } l = 0,\ k = p-1.$$

Notice $(\Sigma_C^0, \sigma^p) = (\Sigma_D, \sigma)$ so θ and $\bar{\theta}$ can be considered as maps from Σ_C^0 to Σ_A^0 and Σ_B^0. Now define maps from Σ_C to Σ_A and Σ_B by:



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$\pi: \Sigma_C \to \Sigma_A$ is defined on each cyclic subset $\pi_n: \Sigma_C^n \to \Sigma_A^n$ by $\pi_n = \sigma^n \circ \theta \circ \sigma^{-n}$ and $\bar{\pi}$ similarly. □

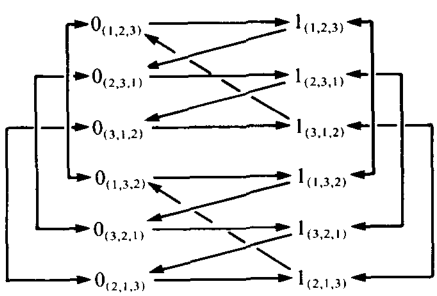
Now suppose Σ_A is an irreducible ssft of period $p > 1$ and it has a $\mathbb{Z} \otimes_T G$ action. Let $\Sigma_A^0, \Sigma_A^1, \ldots, \Sigma_A^{p-1}$ be the cyclic decomposition of Σ_A $(\sigma(\Sigma_A^t) = \Sigma_A^{t+1 \bmod p})$. For each n and t, $0 \le n, t \le p-1$ define

$$H_t^n = \{g \in G: \exists x \in \Sigma_A^t \text{ with } g(x) \in \Sigma_A^{t+n \bmod p}\}.$$

Then we have:

LEMMA. *If the H_t^n, $0 \le t, n < p$ are defined as above,*
(i) $g(\Sigma_A^t) = \Sigma_A^{t+n \bmod p}$, *for* $g \in H_t^n$, $0 \le n < p$;
(ii) H_t^0 *is a subgroup,* $0 \le t < p$;
(iii) $H_t^1, \ldots, H_t^{p-1}$ *are the left cosets for* H_t^0, $0 \le t < p$;
(iv) $H_t^n = T^t(H_0^n)$;
(v) *if* $h \in H_t^n$, $T^{-n} g \in H_t^m$ *then* $gh \in H_t^{n+m}$.

Example 5. Define Σ_A by



It has period 2 with $L_A^0 = \{0_{(1,2,3)}, 0_{(2,3,1)}, 0_{(3,1,2)}, 1_{(1,3,2)}, 1_{(3,2,1)}, 1_{(2,1,3)}\}$ and $L_A^1 = \{0_{(1,3,2)}, 0_{(3,2,1)}, 0_{(2,1,3)}, 1_{(1,2,3)}, 1_{(2,3,1)}, 1_{(3,1,2)}\}$. Obtain a $\mathbb{Z} \times S(3)$ action by letting $S(3)$ permute the subscripts of the symbols in the natural manner. Then $H_0^0 = H_1^0 = \{(1,2,3), (2,3,1), (3,1,2)\}$ and $H_0^1 = H_1^1 = \{(1,3,2), (3,2,1), (2,1,3)\}$.

Proof. (i) Showing this is true for $t = 0$ is sufficient. Take $g \in H_0^n$, so there is an $I \in L_A^0$ such that $g(I) \in L_A^n$. If $J \in L_A^0$, and $T^N = $ identity, then there is an x with $x_0 = J$ and $x_{lNp} = I$ for some l. Then

$$\sigma^{-lNp} \circ g(x) = T^{-lNp} g \circ \sigma^{-lNp}(x) = g \circ \sigma^{-lNp}(x) \in \Sigma_A^n$$

since $(\sigma^{-lNp}(x))_0 = I$; but then $g(x) \in \Sigma_A^n$ and $g(J) \in L_A^n$. So $g(\Sigma_A^0) \subseteq \Sigma_A^n$, using the ergodicity of σ^p on Σ_A^n we have $g(\Sigma_A^0) = \Sigma_A^n$.

(ii) We have $g(\Sigma_A^t) = \Sigma_A^t$ for any $g \in H_t^0$, so clearly H_t^0 is a subgroup.

(iii) Let $g \in H_t^n$ then $g^{-1} \in H_{t+n}^{p-n}$ and $g(H_t^0) \subseteq H_t^n$. If $h \in H_t^n$ then $g^{-1} h \in H_t^0$ so that $h \in g(H_t^0)$ and $H_t^n = g(H_t^0)$.

(iv) Let $T^t g \in H_t^n$, if $x \in \Sigma_A^t$ then $T^t g(x) = T^t g \circ \sigma^t \circ \sigma^{-t}(x) = \sigma^t \circ g \circ \sigma^{-t}(x) \in \Sigma_A^{t+n}$. This means $g \circ \sigma^{-t}(x) \in \Sigma_A^n$ but $\sigma^{-t}(x) \in \Sigma_A^0$ so $g \in H_0^n$.

(v) This follows from the above observation. □

COROLLARY. *Suppose we have a* $\mathbb{Z}\times G$ *action on* Σ_A, *then we have the following strengthening of the above lemma*:

(ii′) H^0 *is a normal subgroup*;

(iii′) $H_t^1, \ldots, H_t^{p-1}$ *are left and right cosets for* H^0;

(iv′) $H_0^n = H_t^n$ *for all n and t*; *so we can ignore the subscripts*;

(v′) *if* $g \in H^n$ *and* $h \in H^m$ *then* $gh \in H^{n+m}$;

(vi′) *if* $|g| =$ *order of g, and* $g \in H^n$ *then* $n|g| = 0$ *mod p*.

Proof. (ii′) This follows from (iv′).

(iii′) This is because H^0 is normal.

(iv′) This follows from (iv) of the lemma.

(v′) This follows from (v) of the lemma.

(vi′) If $|g| =$ order of g, and $g \in H^n$ then by (iv′) $g^{|g|} \in H^{n|g|}$, but $g^{|g|} = \mathrm{id}$ so $n|g| = 0 \bmod p$. □

These H_t^n are essential in classifying $\mathbb{Z}\otimes_T G$ actions of irreducible but periodic ssft. The corollary tells us that when we have a $\mathbb{Z}\times G$ action we can ignore the subscript. If we have a $\mathbb{Z}\otimes_T G$ action, (iv) of the lemma says that the subscripts can be almost, but not entirely ignored. Using these we are led to the main theorem.

THEOREM 6. *If there are* $\mathbb{Z}\otimes_T G$ *actions, with G finite, on irreducible ssft* Σ_A *and* Σ_B *then the two actions are almost topologically conjugate if and only if they have the same topological entropy, the same ergodic period and* $H^0(A) = H^0(B), \ldots, H^{p-1}(A) = H^{p-1}(B)$, (*this means for some numbering of the cyclic subsets* $H_0^n(A) = H_0^n(B)$, $0 \le n < p$).

Proof. The necessity of these conditions is clear. We turn to proving the sufficiency. Assume we have fixed Σ_A^0 and Σ_B^0 so that $H^n(A) = H^n(B)$, for $0 \le n < p$. Also assume that we have recoded, and if necessary found almost topological extensions so that the $\mathbb{Z}\otimes_T G$ actions are special. The proof breaks down into two cases. The first case is when $H^1 \ne \varnothing$, and the second is when $H^1 = \varnothing$. We will deal with them in order.

The first case has $H^1 \ne \varnothing$ (see example 6). Choose an element $g \in H^1$. If we let $g_n = (T^{n-1}g)\cdots(Tg)g$, we have that $H_0^n = g_n H_0^0$, $0 \le n < p$. Now rename the symbols in L_A to get $L_A^0 \times \{0, \ldots, p-1\}$ by setting $I \equiv (I, 0)$ and $g_n(I) \equiv (I, n)$. This means for any $k \in G$, $k = g_n h$ for some n and $h \in H_0^0$ and it acts on L_A by $k(I, 0) = (h(I), n)$ on L_A^0, and on L_A^t by $k(I, t) = (h'(I), m)$ where $kg_n = g_m h'$. We are just using g to push the symbols in L_A^0 around through all the symbols in L_A, and keeping track of the action of G. Now we have

$$A = \begin{bmatrix} 0 & A' & & & \\ & 0 & A' & & \\ & & \ddots & & \\ & & & & A' \\ A' & & & & 0 \end{bmatrix}$$

with zeros everywhere else, assuming we have put the symbols in the same order, by the first coordinate, in each cyclic set. A' is aperiodic; as it stands it defines transitions from $(I, n) \to (J, n+1) \bmod p$. Define a new ssft $\Sigma_{A'}$ by $L_{A'} = L_A^0$ and $(I, 0) \to (J, 0)$ when $(I, n) \to (J, n+1)$. Define a special $\mathbb{Z} \otimes_{T'} H^0$ action on $\Sigma_{A'}$ by letting $h \in H^0$ act on $(I, 0)$ as it did in Σ_A. Define T': $H^0 \to H^0$ by saying

$$T'h = g^{-1} \circ Th \circ g.$$

Repeat this process on Σ_B, using the same element g, to get an aperiodic $\Sigma_{B'}$ with a $\mathbb{Z} \otimes_{T'} H^0$ action on it. Apply the results of the previous section to get Σ_D with a $\mathbb{Z} \otimes_{T'} H^0$ action that is an almost conjugate extension of $\Sigma_{A'}$ and $\Sigma_{B'}$. Now define Σ_C by

$$L_C = L_D \times \{0, \ldots, p-1\},$$

and

$$(I, n) \to (J, n+1) \qquad \text{if } I \to J \text{ in } \Sigma_D.$$

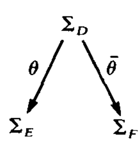
Define a $\mathbb{Z} \otimes_T G$ action on this by saying, for $k \in G$, $k = g_n h$, $k(I, 0) = (H(I), n)$ on L_C^0, and $k(I, t) = (h'(I), m)$ where $kg_t = g_m h'$ in general. We define $\pi : \Sigma_C \to \Sigma_A$ by $\pi(I, n) = (\theta(I), n)$ and observe that this takes the $\mathbb{Z} \otimes_T G$ action on Σ_C to the one on Σ_A. Then define $\bar{\pi}$ in the same way.

The final case is when $H^1 = \emptyset$. To treat this we let $p' = \gcd\{r: H^r \neq \emptyset\}$. Observe that

$$\{np' (\bmod p): n = 1, 2, \ldots\} = \{r: H^r \neq \emptyset\},$$

and so $p = lp'$ for some l. Consider $\sigma^{p'}$ on Σ_A; it will not be irreducible. There will be p' irreducible components, each of period l. Consider the subshift of finite type, Σ_E, that is made up of the orbit of Σ_A^0 under $\sigma^{p'}$, with $\sigma^{p'}$ acting on it. This has a natural $\mathbb{Z} \otimes_S G$ action on it where $S = T^{p'}$ and G is the original group acting in the original manner. Define Σ_F in the same manner beginning with Σ_B. This also has a natural $\mathbb{Z} \otimes_S G$ action on it.

Consider Σ_E and Σ_F with their $\mathbb{Z} \otimes_S G$ actions. This is the case we just dealt with; $H^1(E) = H^{p'}(A) \neq \emptyset$. Build Σ_D with a $\mathbb{Z} \otimes_S G$ action that is an almost conjugate extension of the $\mathbb{Z} \otimes_S G$ actions on Σ_E and Σ_F.



Now define a new ssft Σ_C of period p by

$$L_C = L_D^0 \cup L_D^1 \cup \cdots \cup L_D^{l-1} \times \{0, \ldots, p'-1\}$$

and

$$(I, n) \to (I, n-1) \qquad \text{for } 0 \le n \le p'-2$$

$$(I, p'-1) \to (J, 0) \qquad \text{where } I \to J \text{ in } \Sigma_D.$$

This has a $\mathbb{Z} \otimes_T G$ action defined on it by $g(I, n) = (g(I), n)$. Finally define maps $\pi : \Sigma_C \to \Sigma_A$ and $\bar{\pi} : \Sigma_C \to \Sigma_B$ as at the beginning of this section. They are defined on each cyclic subset $\pi_n : \Sigma_C^n \to \Sigma_A^n$ by $\pi_n = \sigma^m \circ \theta \circ \sigma^{-m}$ where m is n after reduction mod p'. These maps take the $\mathbb{Z} \otimes_T G$ action on Σ_C to the ones on Σ_A and Σ_B. □

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The condition that $H^0(A) = H^0(B), \ldots, H^{p-1}(A) = H^{p-1}(B)$ is just a different statement of the same condition obtained by D. Rudolph [7] in the measure-theoretic case.

Example 6. Let Σ_A with its $\mathbb{Z} \times S(3)$ action be as in example 5. Choose $g = (1, 3, 2) \in H^1$. Then the $\Sigma_{A'}$ with its $\mathbb{Z} \otimes_{T'} H^0$ action that results from the construction in the previous proof is the one described in example 2.

5. *Sofic systems*

The following theorems can be proved using the ideas contained in the previous sections and a few facts about sofic systems. We include them for completeness, but leave out the proofs.

THEOREM 7. *If S is any positive entropy sofic system, G is any finite group, and T is any automorphism of G then there is a* $\mathbb{Z} \otimes_T G$ *action on S.*

THEOREM 8. *If S is an irreducible sofic system with a* $\mathbb{Z} \otimes_T G$ *action then there is a ssft equipped with a* $\mathbb{Z} \otimes_T G$ *action that is an almost conjugate extension of the one on S.*

COROLLARY. *If S and S′ are two irreducible sofic systems that have* $\mathbb{Z} \otimes_T G$ *actions, then these actions are almost topologically conjugate if and only if they have the same topological entropy, the same ergodic period, and* $H^0(S) = H^0(S'), \ldots, H^{p-1}(S) = H^{p-1}(S')$.

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