Canad. Math. Bull. Vol. 59 (4), 2016 pp. 865–877
http://dx.doi.org/10.4153/CMB-2016-058-9

Moduli of Rank 2 Stable Bundles and Hecke Curves

Sarbeswar Pal

Abstract. Let X be a smooth projective curve of arbitrary genus $g > 3$ over the complex numbers. In this short note we will show that the moduli space of rank 2 stable vector bundles with determinant isomorphic to L_x, where L_x denotes the line bundle corresponding to a point $x \in X$, is isomorphic to a certain variety of lines in the moduli space of S-equivalence classes of semistable bundles of rank 2 with trivial determinant.

1 Introduction

Let X be a smooth projective curve of genus $g > 3$ over the complex numbers. Let $M_L(r)$ denote the moduli space of rank r semistable vector bundles over X with fixed determinant L. A Fano variety M is a (possibly singular) projective variety with anti-canonical divisor $-K_M$ that is ample and Cartier. It is well known that when $(r, \deg(L)) = 1$ (which we will call the rank r coprime case), the moduli space $M_L(r)$ is a smooth Fano variety and the theta divisor, which is isomorphic to the anti-canonical line bundle, generates the Picard group [16]. It is a fundamental result that for any point x in a Fano manifold M with Picard number 1, there exists a rational curve in M passing through x (see [9]). Since the 1980s, rational curves inside an algebraic variety has been an active area of study in the classification of higher dimensional varieties. This played a significant role in Mori's minimal model program. We should also mention that finding rational curves with a fixed topological class inside a variety V played an important role in the study of Gromov–Witten theory and the quantum cohomology ring of the variety V; this has been an active field of research in the last twenty years. In the rank 2 coprime case, the quantum cohomology ring of $M_L(2)$ was computed by Vincente Munoz [11, 12].

The set of rational curves with fixed topological datum inside an algebraic variety is a proper subvariety of the Hilbert scheme of curves inside V. Sambaiah Kilaru first initiated the study of rational curves of fixed degree inside the moduli space of vector bundles. In the rank 2 coprime case Kilaru gave an explicit description of the space of rational curves inside the moduli space $M_L(2)$ that are of low degree (degree 1 and 2) in terms of certain Grassmann bundles over the Picard variety of X [8, Theorems 8 and 9].

Narasimhan and Ramanan [13, 14] first introduced the Hecke modification of a vector bundle. The rational curves in $M_L(r)$ that arise from Hecke modification of

Received by the editors June 19, 2015; revised April 3, 2016.
Published electronically September 29, 2016.
AMS subject classification: **14D21**.
Keywords: Hecke curve, (0,1) stable bundle.

a vector bundle over X are called Hecke rational curves. In the rank 2 coprime case, Hwang [5,6] showed the existence of rational curves in $M_L(2)$ (also known as Hecke curves). These Hecke curves satisfy certain minimality conditions on the curves (also known as minimal rational curves) in terms of degree of the pullback of the canonical line bundle. Later Xiaotao Sun [17] showed that for the higher rank and coprime case all minimal rational curves passing through a generic point are Hecke curves and of degree $2r$.

In this paper we attempt to describe rational curves inside $M_L(2)$ for the non-coprime case. Let L_0 be a degree 0 line bundle in $\mathrm{Pic}(X)$. Drezet and Narasimhan [3] showed that the Picard group of $M_{L_0}(r)$ is isomorphic to $\mathbb{Z}$. They also showed that the moduli space is locally factorial with Gorenstein singularities, and its dualizing sheaf is isomorphic to $\mathcal{L}^{-(r,c_1(\mathcal{L}))}$, where $\mathcal{L}$ is the ample determinant bundle in $M_L(r)$. Hence $M_L(r)$ is a singular Fano variety with Picard number 1. In this paper we give an explicit description of rational curves in $M_{L_0}(2)$. We plan to generalize our result to a higher rank in a future paper.

We will briefly explain our idea here. Our approach is quite similar to that of Hwang. Let L_x be the line bundle associated with the sheaf $\mathcal{O}(x)$ for a point x in X. For simplicity we write $\mathcal{M}_0$ and $\mathcal{M}_x$ instead of $\mathcal{M}_{L_0}(2)$ and $\mathcal{M}_{L_x}(2)$, respectively. It is known that the Kummer variety K, which is the quotient of the Jacobian J by the involution $a \mapsto -a$, is embedded inside $\mathcal{M}_0$ as the locus of non-stable points that are precisely the singular loci of $\mathcal{M}_0$. First we will show that for any rational curve l in $\mathcal{M}_0$ that is not contained entirely inside the Kummer variety, there exists a universal bundle $\mathcal{E}_l$ on $X \times l$. Then we will show that this universal bundle has exactly one jumping line (see §5) at some point $t \in X$. Let $\mathcal{M}_0^t(R(0,1))$ denote the set of all rational curves l in $\mathcal{M}_0$ such that the universal bundle $\mathcal{E}_l$ with $c_1(\mathcal{E}_l)$ trivial, $c_2(\mathcal{E}_l) = 1$, and having unique jumping line at t. In Section 3, we will show that $\mathcal{M}_0^t(R(0,1))$ has a structure of smooth variety. Then we prove the following theorem.

Theorem 1.1 *$\mathcal{M}_x$ is isomorphic to $\mathcal{M}_0^x(R(0,1))$.*

Remark 1.2 If the explicit description is known for $\mathcal{M}_0$, then one can try to give an explicit description for $\mathcal{M}_x$ in terms of lines in $\mathcal{M}_0$. For example, if the genus of X is 3, then $\mathcal{M}_0$ is known to be isomorphic to a quartic hypersurface in $\mathbb{P}^7$ [15].

2 Existence of Universal Bundles on Rational Curves

In this section we will show that for a rational curve $l \in \mathcal{M}_0$ intersecting the Kummer variety in at most finitely many points, there exists a universal family of vector bundles on X parametrised by l.

Let X be a smooth, projective curve over the complex numbers with genus > 3. Let E be a vector bundle of rank r on X. Recall that a parabolic structure of length $p (\leq r)$ at a point $x \in X$ is a filtration $E_x = F^1E_x \supsetneqq F^2E_x \supsetneqq \cdots \supsetneqq F^pE_x$, where E_x denotes the fibre of E at x and weight α_i is attached to F^iE_x for each i with $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_p < 1$, $i = 1, \ldots, p$. Set $k_i = \dim F^iE_x - \dim F^{i+1}E_x$. Then the parabolic degree of E is defined as $\mathrm{pardeg}\, E = \deg E + \sum k_i\alpha_i$. We write $\mathrm{par}\,\mu(E) = \mathrm{pardeg}\, E/\mathrm{rank}\, E$.

If W is a subbundle of E, it acquires, in an obvious way, a quasi parabolic structure by taking the induced distinct flags. To make it a parabolic subbundle, attach weights as follows. Given i_0 with $F^{i_0}W \subset F^jE$ for some j, let j_0 be such that $F^{i_0}W \subset F^{j_0}E$ and $F^{i_0}W \not\subseteq F^{j_0+1}E$. Then the weight of $F^{j_0}E$ is defined to be the weight of $F^{i_0}W$. Define E to be parabolic stable (resp., semistable) if for every proper parabolic subbundle $W$of E, one has par $\mu(W) <$ par $\mu(E)$ (resp., $\leq$).

Let E be a vector bundle of rank 2 over X with trivial determinant. Suppose we are given a parabolic structure at x defined by a 1-dimensional subspace

$$F^2E_x \subset F^1E_x = E_x$$

with small weights such that

- parabolic semistable is the same as parabolic stable
- parabolic stable implies that the underlying bundle is semistable.

Let T be the torsion $\mathcal{O}_X$ module given by $T_x = E_x/F^2E_x$ and $T_y = 0$ if $y \neq x$.

Consider the canonical surjective homomorphism $E \to T$, and let W be the kernel of this map. Then W is locally free of rank 2 and its determinant is isomorphic to L_x^{-1}. In other words, each parabolic structure at x on E gives a rank 2 vector bundle as above.

Let $\mathcal{H}$ be the moduli space of rank 2 parabolic stable bundles over X with trivial determinant. Let $\mathcal{M}_0$ denote the moduli space of rank 2 semistable vector bundles over X with trivial determinant over X, and let $\mathcal{M}_x$ denote the moduli space of rank 2 stable vector bundles over X with determinant isomorphic to L_x, where $x \in X$ is a fixed point. Then we have the correspondence

$$\begin{tikzcd} & \mathcal{H} \arrow[dl, "\psi"'] \arrow[dr, "f"] & \\ \mathcal{M}_x \simeq \mathcal{M}_{-x} & & \mathcal{M}_0 \end{tikzcd}$$

where the map f sends a parabolic bundle E to the underlying bundle E, and $\psi(E) = W$, with W as above. Here we have used the canonical isomorphism of $\mathcal{M}_x$ and $\mathcal{M}_{-x}$ via tensoring by $\mathcal{O}(-x)$. Denote $f^{-1}(\mathcal{M}_0^s)$ by $\mathcal{H}^s \subset \mathcal{H}$, where $\mathcal{M}_0^s$ denotes the stable locus. Then we have the following proposition.

Proposition 2.1 *The map $\psi: \mathcal{H} \to \mathcal{M}_x, E \to W$ is a $\mathbb{P}^1$-bundle, locally trivial in the Zariski topology. The morphism f is such that $f: \mathcal{H}^s \to \mathcal{M}_0^s$ is a $\mathbb{P}^1$-bundle.*

Proof See [1, Proposition 3.1]. ∎

Remark 2.2 There exist universal bundles on $X \times \mathcal{H}$ [10,16].

Proposition 2.3 *Set $S = X \times \mathbb{P}^1$. Then any $\mathbb{P}^n$-bundle on S lifts to a vector bundle on S.*

Proof The obstruction of lifting a projective bundle to a vector bundle is an element of the algebraic Brauer group. Since $X \times \mathbb{P}^1$ is a smooth projective variety, this is isomorphic to the analytic Brauer group, which is a torsion subgroup of $H^1(\mathcal{O}_{an}^*)$.

https://doi.org/10.4153/CMB-2016-058-9 Published online by Cambridge University Press

From the exact sequence of analytic sheaves

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_{\mathrm{an}} \longrightarrow \mathcal{O}^*_{\mathrm{an}} \longrightarrow 1,$$

we get

$$H^1(S, \mathcal{O}^*_{\mathrm{an}}) \longrightarrow H^2(S, \mathbb{Z}) \longrightarrow H^2(S, \mathcal{O}_{\mathrm{an}}) \longrightarrow H^2(S, \mathcal{O}^*_{\mathrm{an}}) \longrightarrow H^3(S, \mathbb{Z}) \longrightarrow 0.$$

Since $H^2(S, \mathcal{O}_{\mathrm{an}})$ and $H^3(S, \mathcal{O}_{\mathrm{an}})$ are zero, $H^2(S, \mathcal{O}^*_{an}) \simeq H^3(S, \mathbb{Z})$. On the other hand, $H^3(S, \mathbb{Z}) \simeq H^1(X, \mathbb{Z})$, which is torsion free. Therefore, the obstruction to lifting a $\mathbb{P}^n$ bundle to a vector bundle is zero. Hence the proposition follows. ∎

Proposition 2.4 *Let l be a rational curve in $\mathcal{M}_0$ intersecting the Kummer variety in at most finitely many points, (in other words, l contains at most finitely many non-stable bundles). Then there exists a universal family of vector bundles which determine the line l.*

Proof Case (1): $l \subset \mathcal{M}_0^s$. It is known that there is a natural universal projective bundle on $X \times \mathcal{M}_0^s$. By restricting it to $X \times l$, we get a universal projective bundle on $X \times l$. Hence by Proposition 2.3 and by the argument given in [17, Lemma 2.1], there is a universal vector bundle on X parametrised by l. We will denote this universal vector bundle on $X \times l$ by $\mathcal{E}_l$.

Case (2): $l \nsubseteq \mathcal{M}_0^s$. Let l be a line in $\mathcal{M}_0$ intersecting the Kummer variety at finitely many points. Then $l^s = l \cap \mathcal{M}_0^s$ is an open subset of l. Now by Proposition 2.1 since $f\colon \mathcal{H}^s \to \mathcal{M}^s$ is a $\mathbb{P}^1$-bundle, $f|_{f^{-1}(l^s)} : f^{-1}(l^s) \to l^s$ has a non-zero section $\sigma\colon l^s \to f^{-1}(l^s) \subset \mathcal{H}$. Since $\mathcal{H}$ is projective, we can complete l^s to a line l' in $\mathcal{H}$ mapping isomorphically by f to l. By Remark 2.2, there exists a universal bundle on $X \times \mathcal{H}$, the restriction of which on l' gives a universal bundle on l. Again by a similar argument given in [17, Lemma 2.1], we get a universal vector bundle on X which determines the line l. ∎

3 The Variety $\mathcal{M}_0(R(0,1))$

In this section we will define the variety $\mathcal{M}_0(R(0,1))$ consisting of rational curves in $\mathcal{M}_0$ intersecting the singular locus in at most finitely many points; the universal family of vector bundles parametrizing the curve has trivial first Chern class and second Chern class = 1. We will prove that $\mathcal{M}_0(R(0,1))$ is smooth.

Let Z and Y be projective varieties over a field k. $\mathrm{Hom}(Z, Y)$ is the functor

$$\mathrm{Hom}(Z, Y)(T) = \{T\text{-morphisms} : Z \times T \longrightarrow Y \times T\}.$$

Then we have the following theorem.

Theorem 3.1 *Let Z and Y be projective varieties over a field k. Then* $\mathrm{Hom}(Z, Y)$ *is represented by an open subscheme* $\mathrm{Hom}(Z, Y) \subset \mathrm{Hilb}(Z \times Y)$*, where* $\mathrm{Hilb}(Z \times Y)$ *is the Hilbert scheme of graphs of morphisms $Z \to Y$.*

Proof See [4, Theorem 5.23]. ∎

Let $Z = \mathbb{P}^1$ and $Y = \mathcal{M}_0$. Let L be the ample generator of $\mathrm{Pic}(\mathcal{M}_0)$ [3].

https://doi.org/10.4153/CMB-2016-058-9 Published online by Cambridge University Press

Let $\mathrm{Hom}_1(\mathbb{P}^1, \mathcal{M}_0) \subset \mathrm{Hom}(\mathbb{P}^1, \mathcal{M}_0)$ be the subscheme that parametrizes morphisms $f: \mathbb{P}^1 \to \mathcal{M}_0$ of degree 1 with respect to L and whose image intersects the smooth locus of $\mathcal{M}_0$. We call the points of $\mathrm{Hom}_1(\mathbb{P}^1, \mathcal{M}_0)$ lines in $\mathcal{M}_0$ and denote them by l. By Proposition 2.4 for such a line l, the universal bundle $\mathcal{E}_l$ that determines the embedding, exists.

We set $\mathcal{M}_0(R(0,1)) = \{l \in \mathrm{Hom}_1(\mathbb{P}^1, \mathcal{M}_0) \mid c_1(\mathcal{E}_l) = 0, c_2(\mathcal{E}_l) = 1\}$. Since the singular locus (semistable locus) is of codimension at least 2 in $\mathcal{M}_0$, a general line in $\mathcal{M}_0$ consists of stable bundles only. Let $\mathcal{E}_l$ be the universal bundle corresponding to a generic rational curve l of degree 1 in the stable locus $\mathcal{M}_0^s$. Since genus of $X > 3$, we have that l is a Hecke curve [17] and thus $c_1(\mathcal{E}_l) = 0$ and $c_2(\mathcal{E}_l) = 1$. Thus $\mathcal{M}_0(R(0,1))$ is open in $\mathrm{Hom}_1(\mathbb{P}^1, \mathcal{M}_0)$. Therefore to see that $\mathcal{M}_0(R(0,1))$ is smooth at l, it is enough to show l is a smooth point of $\mathrm{Hom}_1(\mathbb{P}^1, \mathcal{M}_0)$. But it is known that l is a smooth point of $\mathrm{Hom}_1(\mathbb{P}^1, \mathcal{M}_0)$ if $\mathrm{Ext}^1(\Omega_{\mathcal{M}_0}|_l, \mathcal{O}_l) = 0$ [7, Theorem 2.16]. On the other hand, $\Omega_{\mathcal{M}_0}|_l$ can be identified with the sheaf $p_{2*}(\mathrm{ad}(\mathcal{E}_l) \otimes p_1^* K)$, where p_1, p_2 denote the projections of $X \times l$ to the first and second factors, respectively, and $\mathrm{ad}(\mathcal{E})$ denotes the bundle of trace free endomorphisms.

Lemma 3.2 $\mathrm{Ext}^1(p_{2*}(\mathrm{ad}(\mathcal{E}_l) \otimes p_1^* K), \mathcal{O}_l) = 0.$

Proof By Serre duality,

$$\mathrm{Ext}^1(p_{2*}(\mathrm{ad}(\mathcal{E}_l) \otimes p_1^* K), \mathcal{O}_l) \cong (H^1(l, (p_{2*}(\mathrm{ad}(\mathcal{E}_l) \otimes p_1^* K)^*))^*.$$

On the other hand, the sheaf $(p_{2*}(\mathrm{ad}(\mathcal{E}_l) \otimes p_1^* K))^*$ can be canonically identified with $R^1(p_2)_* \mathrm{ad}(\mathcal{E}_l)$. Now there is a spectral sequence $E_2^{p,q} = H^p(l, R^q p_{2*} \mathrm{ad}(\mathcal{E}))$ converging to $H^{p+q}(X \times l, \mathrm{ad}(\mathcal{E}))$. Since $E_2^{p,q} = 0$ for $p \neq 0, 1$, we have a short exact sequence

$$0 \to E_2^{0,n} \to H^n(X \times l, \mathrm{ad}(\mathcal{E})) \to E_2^{1,n-1} \to 0.$$

Thus we have $H^1(l, R^1 p_{2*} \mathrm{ad}(\mathcal{E})) \cong H^2(X \times l, \mathrm{ad}(\mathcal{E}))$. Now consider the sheaf $R^i(p_1)_* \mathrm{ad}(\mathcal{E})$. Since $\mathcal{E}_x \cong \mathcal{O} \oplus \mathcal{O}$ or $\mathcal{O}(1) \oplus \mathcal{O}(-1)$, where $\mathcal{E}_x = \mathcal{E}|_{\{x\} \times l}$, $R^i(p_1)_* \mathrm{ad}(\mathcal{E}) = 0$ for all $i \geq 1$. Thus by the spectral sequence for the projection p_1, we have $H^2(X \times l, \mathrm{ad}(\mathcal{E})) \cong H^2(X, (p_1)_* \mathrm{ad}(\mathcal{E})) = 0$. ■

Thus the variety $\mathcal{M}_0(R(0,1))$ is smooth.

4 $(0,1)$-Stable Bundles and Hecke Curves

In this section we will recall the notion of (k,l)-stability [14] and define the Hecke curve associated with a vector bundle $E \in \mathcal{M}_x$. We say this association is a Hecke correspondence. Then we will show that the Hecke correspondence defines an injective morphism from $\Phi: \mathcal{M}_x \to \mathcal{M}_0(R(0,1))$.

If E is a vector bundle $(\neq 0)$ on X and $k \in \mathbb{Z}$, we denote by $\mu_k(E)$ the rational number $(\deg E + k)/rkE$.

Definition 4.1 A vector bundle E on X is said to be (k,l)-stable (resp., (k,l)-semistable) if, for every proper subbundle F of E, we have $\mu_k(F) < \mu_{-l}(E/F)$ (resp., $\mu_k(F) \leq \mu_{-l}(E/F)$).

Remark 4.2 Note that usual Mumford stability is equivalent to $(0,0)$-stability.

Lemma 4.3 *Let $x \in X$ and $0 \to E' \to E \to \mathcal{O}_x \to 0$ be an exact sequence of sheaves with E', E locally free. If E is (k,l)-stable, then E' is $(k, l-1)$-stable. In particular, if E is $(0,1)$-stable, then E' is stable. Similar statements are valid when stable is replaced by semistable.*

Proof See [14, Lemma 5.5]. ∎

Lemma 4.4 (i) *Let E be a $(0,1)$-stable bundle of rank n and E' a stable vector bundle of rank n whose determinant is isomorphic to $\det E \otimes L_x^{-1}$. If $f: E' \to E$ is a non-zero homomorphism, we have an exact sequence*

$$0 \longrightarrow E' \longrightarrow E \longrightarrow \mathcal{O}_x \longrightarrow 0.$$

(ii) *Moreover,* $\dim H^0(X, \mathrm{Hom}(E', E)) \le 1$.

Proof See [14, Lemma 5.6]. ∎

Let E be a vector bundle of rank 2 on X. Then any non-zero element of E_x^* (the fibre of E^* at $x \in X$) can be thought of as a surjective homomorphism of the sheaf E onto $\mathcal{O}_x$, the structure sheaf of x. The kernel depends only on the one-dimensional subspace of E_x^* generated by the given element.

We will indicate how one can construct a family of vector bundles on X parametrised by $\mathbb{P}(E_x^*)$. The construction is the same as that in [13], except that here we will take one vector bundle instead of a family of vector bundles.

The construction goes as follows. Let p_i, p_2 denote the ith-projection map from $\mathbb{P}(E_x^*) \times X$ onto the first and second factors, respectively. Consider the natural surjective homomorphism $\beta_E: p_2^* E \to p_2^*(\mathcal{O}_x) \otimes p_1^* \tau$, where τ is the tautological hyperplane bundle on $\mathbb{P}(E_x^*)$. Clearly the kernel of the homomorphism β_E, denoted by $H(E)$, is locally free. Then $H(E)$ is the bundle we are looking for. We denote by $K(E)$ the dual of $H(E)$.

Remark 4.5 The construction also works for a family of stable vector bundles. See [13].

If $E \in \mathcal{M}_x$, then $K(E)$ is a family of semistable vector bundles on X with trivial determinant. More precisely, we have a morphism $\mathbb{P}(E_x^*) \to \mathcal{M}_0$. In fact, this map is the same as the map $f|_{\psi^{-1}(E)}: \psi^{-1}(E) \to \mathcal{M}_0$ (Proposition 2.1). We claim that the map is non-constant.

If E is a $(0,1)$ stable bundle in $\mathcal{M}_x$, then by [14, Lemma 5.9], the morphism $\mathbb{P}(E_x^*) \to \mathcal{M}_0$ defines a rational curve in $\mathcal{M}_0$.

If the above map is constant for some E, then the restriction of the pullback of any ample line bundle on $\mathcal{M}_0$ via f to $\psi^{-1}(E)$ is trivial. On the other hand, $(0,1)$ stable bundles form a dense open subset of $\mathcal{M}_x$ and restriction of the pullback of an ample line bundle on $\mathcal{M}_0$ at fibres over $(0,1)$ stable bundles is not trivial (Remark 4.6), a contradiction. Thus the map $\mathbb{P}(E_x^*) \to \mathcal{M}_0$ defines a rational curve in $\mathcal{M}_0$. A rational curve on $\mathcal{M}_0$ constructed in this way is called a Hecke curve.

Remark 4.6 The Hecke curves are of degree 1 rational curves in $\mathcal{M}_0$ with respect to the ample generator [5].

The following lemma is well known to the experts but for the lack of proper references and for the sake of completeness we include a proof.

Lemma 4.7 *If E is a rank 2 stable vector bundle with determinant isomorphic to L_x, then E contains finitely many line subbundles of degree zero.*

Proof Let ξ be a degree zero line subbundle of E.

Case (1): $H^0(X, \xi^{-2} \otimes \mathcal{O}(x)) = 0$. Consider the base-point-free line bundle $K \otimes \xi^{-2} \otimes \mathcal{O}(x)$, where K is the canonical line bundle over X. This gives rise to a morphism

$$\pi: X \to \mathbb{P}((H^0(X, K \otimes \xi^{-2} \otimes \mathcal{O}(x)))^*) \simeq \mathbb{P}(H^1(X, \xi^2 \otimes \mathcal{O}(-x)))$$

given by mapping, each $y \in X$ to the point in $\mathbb{P}(H^1(X, \xi^2 \otimes \mathcal{O}(-x))$ corresponding to the kernel of the surjective map

$$H^1(X, \xi^2 \otimes \mathcal{O}(-x)) \longrightarrow H^1(X, \xi^2 \otimes \mathcal{O}(-x) \otimes \mathcal{O}(y)).$$

Clearly, $\pi^{-1}(e)$is finite, where e is the point in $\mathbb{P}(H^1(X, \xi^2 \otimes \mathcal{O}(-x)))$ corresponding to the bundle E. Therefore, the line bundles $\xi, \xi^{-1} \otimes \mathcal{O}(x) \otimes \mathcal{O}(-x_i)$ such that $x_i \in \pi^{-1}(e)$ are contained in E.

Conversely, if η is a degree zero line sub-bundle of E, then the exact sequence

$$0 \longrightarrow \xi \longrightarrow E \longrightarrow \xi^{-1} \otimes \mathcal{O}(x) \longrightarrow 0$$

induces a morphism $\eta \to \xi^{-1} \otimes \mathcal{O}(x)$. If this map is zero, then the map $\eta \to E$ factors through $\eta \to \xi$. Since both line bundles are of degree zero, this is an isomorphism. If the map is not zero, then we have $\eta \otimes \mathcal{O}(y) \simeq \xi^{-1} \otimes \mathcal{O}(x)$ for some $y \in X$, *i.e.*, $\eta \simeq \xi^{-1} \otimes \mathcal{O}(x) \otimes \mathcal{O}(-y)$.

Case (2): $H^0(X, \xi^{-2} \otimes \mathcal{O}(x) \neq 0$. In other words, $\xi^{-2} \otimes \mathcal{O}(x) = \mathcal{O}(y)$ for some $y \in X$. In that case, the line bundle $K \otimes \xi^{-2} \otimes \mathcal{O}(x)$ is not base point free. Its base locus is $\{y\}$ and therefore tensoring it by the line bundle $\mathcal{O}(-y)$, we get the base point free canonical line bundle K. Using the isomorphism of $H^0(X, K)$ and $H^0(X, K \otimes \xi^{-2} \otimes \mathcal{O}(x))$, we again have a morphism

$$\pi: X \to \mathbb{P}(H^0(X, K)^*) \simeq \mathbb{P}(H^0(X, K \otimes \xi^{-2} \otimes \mathcal{O}(x))^*)) \simeq \mathbb{P}(H^1(X, \xi^2 \otimes \mathcal{O}(-x))),$$

given by mapping each point $z \in X$ to the point in $\mathbb{P}(H^1(X, \xi^2 \otimes \mathcal{O}(-x)))$ corresponding to the kernel of the map $H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}(z))$. By the same argument as earlier, E contains only finitely many line subbundles of degree zero. ■

Proposition 4.8 *The Hecke curves intersect the Kummer variety at finitely many points.*

Proof Let C be a Hecke curve corresponding to a stable bundle E. If C contains infinitely many points of the Kummer variety, then E contains infinitely many line bundles of degree zero, a contradiction to Lemma 4.7. ■

For each family of stable bundles we have a morphism of the parameter space into $\mathcal{M}_0$. Since these morphisms are clearly functorial, we get a morphism Φ from $\mathcal{M}_x$ which sends E to the scheme of rational curves in $\mathcal{M}_0$ given by the associated Hecke curve.

Proposition 4.9 *If E and E' are two nonisomorphic stable vector bundles in $\mathcal{M}_x$, then the respective associated Hecke curves are distinct.*

Proof This is known if E and E' are $(0,1)$ stable bundles [14]. For the sake of completeness I am including a proof.

Case (1): E and E' are $(0,1)$ stable bundles. If not, let F and F' be two points of intersection. Then we have the following exact sequences

$$0 \longrightarrow F \xrightarrow{f_{\lambda_1}} E \xrightarrow{\lambda_1} \mathcal{O}_x \longrightarrow 0, \tag{4.1}$$

$$0 \longrightarrow F \xrightarrow{f_{\lambda_1'}} E' \xrightarrow{\lambda_1'} \mathcal{O}_x \longrightarrow 0, \tag{4.2}$$

$$0 \longrightarrow F' \xrightarrow{f_{\lambda_2}} E \xrightarrow{\lambda_2} \mathcal{O}_x \longrightarrow 0, \tag{4.3}$$

$$0 \longrightarrow F' \xrightarrow{f_{\lambda_2'}} E' \xrightarrow{\lambda_2'} \mathcal{O}_x \longrightarrow 0. \tag{4.4}$$

Now since $F \simeq F^*$ and $F' \simeq F'^*$, where F^* denotes its dual, we have two non-zero homomorphisms from E^* to E' of maximal rank, one from the dual sequences of (4.1) and (4.2) and another from the dual sequences of (4.3) and (4.4). Since $\lambda_1 \neq \lambda_2$ and $\lambda_1' \neq \lambda_2'$, and f_{λ_i} and f'_{λ_i} are of maximal rank, these two homomorphisms cannot be equal, which is a contradiction to Lemma 4.3.

Case (2): E and E' are not $(0,1)$ stable bundles (in this case the associated Hecke curves contain at least one S-equivalence class of semistable bundles). Then the bundles E and E' can be written as extensions of the forms,

$$0 \longrightarrow \xi \longrightarrow E \longrightarrow \xi^* \otimes \mathcal{O}(x) \longrightarrow 0, \qquad 0 \longrightarrow \xi \longrightarrow E' \longrightarrow \xi^* \otimes \mathcal{O}(x) \longrightarrow 0,$$

respectively, where ξ is a line bundle of degree zero and ξ^* denotes its dual. Since the generic points of Hecke curves are stable bundles, there exists a stable bundle, say F, in the Hecke curve associated with E. Thus we have the following diagram.

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & F \arrow[d] & & \\
0 \arrow[r] & \xi \arrow[r] & E \arrow[r] \arrow[d] & \xi^* \otimes \mathcal{O}(x) \arrow[r] & 0 \\
 & & \mathcal{O}_x \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

https://doi.org/10.4153/CMB-2016-058-9 Published online by Cambridge University Press

This gives a nonzero morphism $f\colon F \to \xi^* \otimes \mathcal{O}(x)$.

Claim: $\dim \operatorname{Hom}(F, \xi^* \otimes \mathcal{O}(x)) \le 1$. Consider the vector bundle $F^* \otimes \xi^* \otimes \mathcal{O}(x)$. Since F is stable, $F^* \otimes \xi^* \otimes \mathcal{O}(x)$ is stable and hence the maximal degree of a line subbundle is 0. Let s be the integer such that the maximal degree of a line subbundle of $F^* \otimes \xi^* \otimes \mathcal{O}(x)$ is $\frac{d-s}{2}$, where d is the degree of $F^* \otimes \xi^* \otimes \mathcal{O}(x)$. Then $s = d$. Then the claim follows from [2, Corollary 1].

If F also lies in the Hecke curve associated with E', then the map f can be lifted to a map $F \to E'$ in the exact sequence

$$0 \longrightarrow \xi \longrightarrow E' \longrightarrow \xi^* \otimes \mathcal{O}(x) \longrightarrow 0,$$

which means that the corresponding extension class $\delta(E') \in H^1(X, \xi^2 \otimes \mathcal{O}(-x))$ is in the kernel of the natural map $H^1(X, \xi^2 \otimes \mathcal{O}(-x)) \to H^1(X, F^* \otimes \xi)$. On the other hand, we have the exact sequence

$$0 \longrightarrow \xi \otimes \mathcal{O}(-x) \longrightarrow F \longrightarrow \xi^* \otimes \mathcal{O}(x) \longrightarrow 0$$

which gives the following exact sequence

$$0 \longrightarrow \xi^2 \otimes \mathcal{O}(-x) \longrightarrow F^* \otimes \xi \longrightarrow \mathcal{O}(x) \longrightarrow 0.$$

Since F is stable, then $H^0(X, F^* \otimes \xi) = 0$. Thus from the long exact sequence of cohomology for the above exact sequence we have that the kernel of the map

$$H^1(X, \xi^2 \otimes \mathcal{O}(-x)) \longrightarrow H^1(X, F^* \otimes \xi)$$

is one-dimensional. Therefore the extension class $\delta(E) = \lambda\delta(E')$ for some scalar λ. Hence E is isomorphic to E', a contradiction. ∎

Proposition 4.10 *Let E be a point in $\mathcal{M}_x$ and $H(E)$ be the family of semistable vector bundles parametrised by the associated Hecke curve. Then $c_1(H(E))$ is trivial and $c_2(H(E)) = 1$.*

Proof Let p_1, p_2 denote the projections from $\mathbb{P}(E_x^*) \times X$ onto the first and second factors, respectively. Let τ be the tautological hyperplane line bundle on $\mathbb{P}(E_x^*)$. The natural homomorphism $p_2^*E \to p_1^*\tau$ maps the subsheaf $H(E)$ of p_2^*E into $p_1^*\tau \otimes p_2^*L_x^{-1}$. Now we have the following commutative diagram on $\mathbb{P}(E_x^*) \times X$.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & p_1^*\tau \otimes p_2^*L_x^{-1} \arrow[r] \arrow[d] & p_1^*\tau \otimes p_2^*L_x^{-1} \arrow[d] & \\
0 \arrow[r] & H(E) \arrow[r] \arrow[d] & p_1^*\tau \otimes p_2^*L_x^{-1} \oplus p_2^*E \arrow[r] \arrow[d] & p_1^*\tau \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H(E) \arrow[r] & p_2^*E \arrow[r] \arrow[d] & p_2^*\mathcal{O}_x \otimes p_1^*\tau \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

From the definition of $H(E)$, it is clear that $H(E)|_{\mathbb{P}(E_x^*)\times y}$ is trivial for $y \neq x$. On the other hand, we restrict the above diagram to $\mathbb{P}(E_x^*) \times x$ and note that the map $\tau \otimes L_x^{-1}|_x \to \tau$ is zero. Now, using the canonical exact sequence

$$0 \longrightarrow \Omega^1_{\mathbb{P}(E_x^*)} \otimes \tau \longrightarrow \mathcal{O} \otimes E_x^* \longrightarrow \tau \longrightarrow 0,$$

we have $H(E)|_{\mathbb{P}(E_x^*)\times x} \simeq \tau \otimes L_x^{-1}|_x \oplus \Omega^1_{\mathbb{P}(E_x^*)} \otimes \tau \simeq \tau \oplus \tau^{-1}$. The same is true for the vector bundle $K(E)$. Hence $c_2(H(E)) = 1$.

Since the restriction of $H(E)$ on each fibre over $\mathbb{P}(E_x^*)$ is of trivial determinant, using the above splitting type of $H(E)$ on fibres over the first projection, we can conclude that $c_1(H(E))$ is trivial. ∎

Thus by Remark 4.5 and Proposition 4.10 we have a morphism

$$\Phi\colon \mathcal{M}_x \to \mathcal{M}_0(R(0,1)), \tag{4.5}$$

which is injective by Proposition 4.9.

5 Jumping Line at a Point

Let $S = X \times \mathbb{P}^1$ be a ruled surface and $p_1\colon S \to X$ be the projection. Then for any torsion free sheaf $\mathcal{E}$ on the ruled surface S, its restriction to a generic fibre $p_1^{-1}(t) = S_t$ has the form $\mathcal{E}|_{S_t} = \bigoplus_{i=1}^n \mathcal{O}_{S_t}(\alpha_i)^{\oplus r_i}, \alpha_1 > \alpha_2 > \cdots > \alpha_n$. The $\alpha = (\alpha_1^{\oplus r_1}, \dots, \alpha_n^{\oplus r_n})$ is called the *generic splitting type* of $\mathcal{E}$. Any such $\mathcal{E}$ admits a relative Harder–Narasimhan filtration $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = \mathcal{E}$ of which the quotient sheaves $\mathcal{F}_i = \mathcal{E}_i/\mathcal{E}_{i-1}$ are torsion free with generic splitting type $(\alpha_i^{\oplus r_i})$, respectively. Then it is easy to see that

$$2c_2(\mathcal{E}) = 2\sum_{i=1}^n c_2(\mathcal{F}_i) + 2\sum_{i=1}^n c_1(\mathcal{E}_{i-1})c_1(\mathcal{F}_i) = 2\sum_{i=1}^n c_2(\mathcal{F}_i) + c_1(\mathcal{E})^2 - \sum_{i=1}^n c_1(\mathcal{F}_i)^2. \tag{5.1}$$

Lemma 5.1 *Any torsion free sheaf $\mathcal{E}$ of rank r on a ruled surface with generic splitting type $(0^{\oplus r})$ must have $c_2(\mathcal{E}) \geq 0$.*

Proof See [17, Lemma 2.1]. ∎

Definition 5.2 A rank r vector bundle $\mathcal{E}$ on a ruled surface $X \times \mathbb{P}^1$ with generic splitting type $0^{\oplus r}$ is said to have a *jumping line* $S_t = p_1^{-1}(t)$ at $t \in X$ if

$$\mathcal{E}|_{S_t} = \bigoplus_{i=1}^n \mathcal{O}_{S_t}(\alpha_i)^{\oplus r_i}, \alpha_1 > \cdots > \alpha_n$$

with the type $(\alpha_1^{\oplus r_1}, \dots, \alpha_n^{\oplus r_n})$ different from $(0^{\oplus r})$.

Remark 5.3 Note that for any E in $\mathcal{M}_x$, the associated Hecke curve has a unique jumping line at x.

Let $\mathcal{E}$ be a rank r vector bundle on a ruled surface $S = X \times \mathbb{P}^1$ with generic splitting type $0^{\oplus r}$ and let S_t be a jumping line. Then we can perform the elementary transformation on $\mathcal{E}$ along S_t, by taking $\mathcal{F}$ to be the kernel of the surjective homomorphism $\varphi\colon \mathcal{E} \to \mathcal{E}|_{S_t} \to \mathcal{O}_{S_t}(\alpha_n)^{\oplus r_n}$. Then clearly we have the following exact sequence:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{S_t}(\alpha_n)^{\oplus r_n} \longrightarrow 0. \tag{5.2}$$

https://doi.org/10.4153/CMB-2016-058-9 Published online by Cambridge University Press

Lemma 5.4 $c_1(\mathcal{F}) = c_1(\mathcal{E}) - r_n S_t$ *and* $c_2(\mathcal{F}) = c_2(\mathcal{E}) + r_n \alpha_n$.

Proof By the exact sequence (5.2), the computation is straightforward. ∎

Lemma 5.5 *If* $c_2(\mathcal{E}) = 1$ *and* $\mathcal{E}$ *has generic splitting type* $(0^{\oplus r})$*, then* $\mathcal{E}$ *has exactly one jumping line* S_t *and the elementary transformation* $\mathcal{F}$ *along* S_t *is isomorphic to* $p_1^* V$ *for a vector bundle* V *over* X.

Proof See [17, Lemma 2.4]. ∎

Proposition 5.6 *Let* l *be a line in* $\mathcal{M}_0$ *such that the universal bundle* $\mathcal{E}_l$ *(which exists by the previous section) has* $c_1(\mathcal{E}_l) = 0$ *and* $c_2(\mathcal{E}_l) = 1$*. Then* $\mathcal{E}_l$ *has generic splitting type* $(0^{\oplus 2})$.

Proof First we claim that $0 = \mathcal{E}_0 \subset \mathcal{E}_1 = \mathcal{E}_l$ is the relative Harder–Narasimhan filtration for $\mathcal{E}_l$. If not, let $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 = \mathcal{E}_l$ be the relative Harder–Narasimhan filtration, where $\mathcal{F}_1 = \mathcal{E}_1$ and $\mathcal{F}_2 = \mathcal{E}_l/\mathcal{E}_1$ are torsion free. Since $\mathcal{F}_i, i = 1,2$ are torsion free of rank 1, $c_2(\mathcal{F}_i) = 0$. On the other hand, $0 = c_2(\mathcal{F}_i) = l(\mathcal{F}_i^{\vee\vee}/\mathcal{F}_i)$. Thus $\mathcal{F}_i, i = 1,2$ are locally free. Let (α_1) be the generic splitting type of $\mathcal{F}_i$. Then $\mathcal{F}_i \otimes p_2^*(\mathcal{O}(-\alpha_i))$ has the generic splitting type (0). So by [17, Lemma 2.2] $\mathcal{F}_i = p_1^* V_i \otimes p_2^*(\mathcal{O}_{\mathbb{P}^1}(\alpha_i))$, where the V_is are line bundles on X of degree, say, d_i. Thus we have $c_1(F_i) = p_1^*\mathcal{O}_X(d_i) + p_2^*\mathcal{O}_{\mathbb{P}^1}(\alpha_i)$, where d_i is the degree of $\mathcal{F}_i$ on the generic fiber of p_2 and α_i is the degree of $\mathcal{F}_i$ on the generic fiber of p_i. Then $c_1(F_i)^2 = 2d_i\alpha_i$. Since the degree of $\mathcal{E}$ on the generic fiber of p_i is zero, from (5.1), we have $c_2(\mathcal{E}) = -2d_1\alpha_1$. Since d_1 and α_1 are integers and $c_2(\mathcal{E}) = 1$, we get a contradiction. Then the proposition follows from the fact that $c_1(\mathcal{E}_l) = 0$. ∎

Therefore, from Proposition 5.6 and Lemma 5.5 it follows that for any such l, $\mathcal{E}_l$ has exactly one jumping line S_t at t for some $t \in X$. Also from Lemma 5.1 and Lemma 5.4, it is clear that $\mathcal{E}_l|_{S_t} = \mathcal{O}_{S_t}(1) \oplus \mathcal{O}_{S_t}(-1)$. We say that a point l in $\mathcal{M}_0(R(0,1))$ has a jumping line at x if $\mathcal{E}_l$ has jumping line at x.

Set $\mathcal{M}_0^x(R(0,1)) := \{l \in \mathcal{M}_0(R(0,1)) \mid l \text{ has unique jumping line at } x\}$ Thus by Remark 5.3, the morphism Φ in (4.5) factors through $\mathcal{M}_0^x(R(0,1))$. We also denote this morphism by Φ.

6 Surjectivity of Φ

Let l be a point in $\mathcal{M}_0^x(R(0,1))$. Then by Lemma 5.5, we have the following exact sequence on $S = X \times l$:

$$0 \longrightarrow p_1^*(V) \longrightarrow \mathcal{E}_l \longrightarrow \mathcal{O}_{S_x}(-1) \longrightarrow 0, \tag{6.1}$$

for some vector bundle V on X. For any $p \in l$, let $\mathcal{E}_{(l,p)}$ denote $\mathcal{E}_l|_{X\times\{p\}}$. Restricting the exact sequence (6.1) to $p_2^{-1}(p) = X \times \{p\}$, we get

$$0 \longrightarrow V \longrightarrow \mathcal{E}_{(l,p)} \xrightarrow{\lambda_p} \mathcal{O}_{S_x}(-1)_p \longrightarrow 0. \tag{6.2}$$

Since $\mathcal{E}_{(l,p)}$ is a rank 2 bundle with trivial determinant, $\mathcal{E}_{(l,p)} = \mathcal{E}_{(l,p)}{}^{*}$. Taking the dual of (6.2), we get

$$0 \longrightarrow \mathcal{E}_{(l,p)} \longrightarrow V^{*} \xrightarrow{\lambda_p} \mathcal{O}_{S_x}(-1)_p \longrightarrow 0.$$

Since the line l passes through a stable point, there is a $p_0 \in l$ such that $\mathcal{E}_{(l,p_0)}$ is stable and therefore V is stable. Thus V^* is also stable. Then it is clear that the Hecke curve defined by V^* is l, which proves the surjectivity of Φ. Since the variety $\mathcal{M}_0^x(R(0,1))$ is smooth, we have the following theorem.

Theorem 6.1 *$\mathcal{M}_x$ is isomorphic to $\mathcal{M}_0^x(R(0,1))$.*

Remark 6.2 The moduli of $(0,1)$ stable bundles, which is a non-empty open subset of $\mathcal{M}_x$, is isomorphic to the variety of minimal degree rational curves (lines) with respect to $-K_{\mathcal{M}_0^s}$ in $\mathcal{M}_0^s$.

In fact, let l be a rational curve in $\mathcal{M}_0^s$. We define the degree of l with respect to the anti-canonical ample line bundle $-K_{\mathcal{M}_0^s}$ by the number $-K_{\mathcal{M}_0^s}.l$. Let $\mathcal{E}_l$ be the vector bundle on $X \times l$ that induces the embedding $l \subset \mathcal{M}_0^s$. Then using the same line of argument as in [17], it can be shown that the degree of l is minimal if and only if $c_1(\mathcal{E}_l) = 0$ and $c_2(\mathcal{E}_l) = 1$. On the other hand, it is clear from Lemma 4.3 and Remark 4.6 that this line associated with a $(0,1)$ stable bundle is contained in $\mathcal{M}_0^s$.

Acknowledgement I wish to express my thanks to Professors S. Ramanan, Peter Newstead, V Balaji, and Dr. Arijit Dey for important discussions and for their helpful suggestions and comments. I also thank the reviewer for useful comments.

References

[1] V. Balaji, *Intermediate Jacobian of some moduli space of vector bundles on curves*. Amer. J. Math. **112**(1990), no. 4, 611–629. http://dx.doi.org/10.2307/2374872

[2] J. Cilleruelo and I. Sols, *The Severi bound on sections of rank two semistable bundles on a Riemann surface*. Ann. of Math. (2) **154**(2001), no. 3, 739–758. http://dx.doi.org/10.2307/3062146

[3] J.-M. Drezet and M. S. Narasimhan, *Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques*. Invent. Math. **97**(1989), no. 1, 53–94. http://dx.doi.org/10.1007/BF01850655

[4] B. Fantechi, L. Göttsche, L. Illusie, S. Kleiman, N. Nitsure, and A. Vistoli, *Fundamental algebraic geometry. Grothendieck's FGA explained*. Mathematical Surveys and Monographs, 123. American Mathematical Society, Providence, RI, 2005.

[5] J.-M. Hwang, *Tangent vectors to Hecke curves on the moduli space of rank 2 bundles over an algebraic curve*. Duke Math. J. **101**(2000), no. 1, 179–187. http://dx.doi.org/10.1215/S0012-7094-00-10117-2

[6] ———, *Geometry of minimal rational curves on Fano manifolds*. In: School on vanishing theorems and effective results in algebraic geometry (Trieste, 2000). ICTP Lect. Notes, 6, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2001, pp. 335–393, .

[7] J. Kollár, *Rational curves on algebraic varieties*. Ergebnisse der Mathematik und ihrer Grenzgebiete, 32. Springer-Verlag, Berlin, 1996.

[8] S. Kilaru, *Rational curves on moduli spaces of vector bundles*. Proc. Indian Acad. Sci. Math. Sci. **108**(1998), no. 3, 217–226. http://dx.doi.org/10.1007/BF02844479

[9] S. Mori, *Projective manifolds with ample tangent bundles*. Ann. of Math. (2) **110**(1979), no. 3, 593–606. http://dx.doi.org/10.2307/1971241

[10] D. Mumford and P. E. Newstead, *Periods of a moduli space of vector bundles on curves*. Amer. J. Math. **90**(1968), 1201–1208. http://dx.doi.org/10.2307/2373296

[11] V. Muñoz, *Quantum cohomology of the moduli space of stable bundles over a Riemann surface*. Duke Math. J. **98**(1999), no. 3, 525–540. http://dx.doi.org/10.1215/S0012-7094-99-09816-2

[12] ———, *Another proof for the presentation of the quantum cohomology of the moduli of bundles over a Riemann surface*. Bull. London Math. Soc. **34**(2002), no. 4, 411–414. http://dx.doi.org/10.1112/S0024609301008906

[13] M. S. Narasimhan and S. Ramanan, *Deformations of the moduli space of vector bundles over an algebraic curve*. Ann. of Math. **101**(1975), 391–417. http://dx.doi.org/10.2307/1970933

[14] ———, *Geometry of Heche Cycles–I*. In: C. P. Ramanujam–a tribute. Springer Verlag, Berlin, 1978, pp. 291–345.

[15] ———, *2θ-linear systems on abelian varieties*. In: Vector bundles on algebraic varieties (Bombay, 1984), Tata Inst. Fund. Res. Stud. Math., 11. Tata Inst. Fund. Res., Bombay, 1987 pp. 415–427.

[16] C. S. Seshadri, *Fibrés vectoriels sur les courbes algébriques*. Astérisque, 96, Société Mathématique de France, Paris, 1982.

[17] S. Xiaotao, *Minimal rational curves on moduli spaces of stable bundles*. Math. Ann. **331**(2005), no. 4, 925–937. http://dx.doi.org/10.1007/s00208-004-0614-2

Indian Institute of Science Education and Research Thiruvananthapuram, Thiruvananthapuram-695016, India
e-mail: spal@iisertvm.ac.in