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OSCILLATION OF SEMILINEAR ELLIPTIC INEQUALITIES BY RICCATI TRANSFORMATIONS

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1. Introduction. A generalized Riccati transformation will be utilized to derive a Riccati-type inequality (3) associated with a semilinear elliptic inequality $yL(y; x) \leqq 0$ possessing a positive solution y in an exterior domain in Euclidean n-space. On the basis of (3), general sufficient conditions for the elliptic inequality to be oscillatory are developed in § 3. The matrix of coefficients of the second derivative terms in $L(y; x)$ (i.e. (A_{ij}) in (1)) is not restricted in any way beyond the usual ellipticity hypothesis (iv) below, and thereby one of the difficulties mentioned in [**9**] and inherent in the method there is resolved. Furthermore, the nonlinear term $B(x, y)$ in (1) is not required to be one-signed.

In § 4 the results are specialized to the case that $L(y; x)$ is a perturbation of a linear elliptic operator, without sign restrictions. The theorems are not deducible via comparison theorems since the coefficients are not uniformly one-signed. Several corollaries yield sharper oscillation criteria than those known previously, even in the case of linear Schrödinger operators. Examples are given of oscillatory operators by our criteria for which earlier criteria give no information.

The superlinear results in § 5 are obtained by first establishing a priori lower bounds $R(r)$ for positive solutions, and then applying our Comparison Theorem 10. In particular, the theorems of §§ 3 and 4 can be applied to yield new superlinear oscillation criteria, thereby extending earlier results of the authors [**9**] for Schrödinger inequalities to the case that A is not the identity matrix in (1). Some of these criteria are of Allegretto's type [**1, 2**], but are sharper and more specific under the intersection of his and our hypotheses. The sublinear result in § 6 improves a recent result of Kitamura and Kusano [**6**] in the 2-dimensional Schrödinger case. Some limitations and open questions are mentioned in the conclusion.

2. Preliminaries. Points in n-dimensional Euclidean space E^n will be denoted by $x = (x_1, \ldots, x_n)$, the Euclidean length of x by $|x|$, and differentiation with respect to x_i by D_i, $i = 1, \ldots, n$. Let S_a denote the

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sphere of radius a, $G(a, b)$ the annulus between spheres of radii a and b, and G_a the complement in E^n of the closed ball of radius a:

$$S_a = \{x \in E^n: |x| = a\},\ a > 0$$
$$G(a, b) = \{x \in E^n: a < |x| < b\}$$
$$G_a = G(a, \infty).$$

Let (r, θ) be hyperspherical coordinates of a point x in E^n: $r = |x|$, $\theta = \theta_1, \ldots, \theta_{n-1}$. The measure on S_a and S_1 will be denoted by s and ω, respectively; thus $ds = a^{n-1}d\omega$, $s(S_a) = a^{n-1}\omega(S_1)$. The outward unit normal ν to S_a at $x \in S_a$ has components $\nu_i = x_i/a$, $i = 1, \ldots, n$. For an exterior domain Ω in E^n, there exists a positive number a such that $G_a \subset \Omega$.

The partial differential inequality under consideration is $yL(y; x) \leqq 0$, where

$$(1) \qquad L(y; x) \equiv \sum_{i,j=1}^{n} D_i[A_{ij}(x)D_jy] + B(x, y)$$

under the assumptions listed below.

Assumptions.

(i) $B(x, t)$ is continuous in $\Omega \times E^1$;

(ii) $B(x, t) \geqq p(x)\phi(t)$ for all $x \in \Omega$ and for all $t \geqq 0$, where p is continuous in Ω, $\phi \in C^1[0, \infty)$, and $\phi(t) > 0$ if $t > 0$;

(iii) $B(x, t) \leqq -p_1(x)\phi_1(-t)$ for all $x \in \Omega$ and for all $t < 0$, where p_1 is continuous in Ω, $\phi_1 \in C^1(0, \infty)$ and $\phi_1(t) > 0$ if $t > 0$.

(*iv*) Each A_{ij} involved in (1) is a real-valued function of class $C^1(\Omega)$, and the matrix $A = (A_{ij})$ is symmetric and positive definite in Ω (ellipticity condition).

Motivated by the one-dimensional and matrix Riccati transformation used by Coles [**3**], Howard [**5**], Reid [**11**] and others, we employ a general Riccati transformation defined in terms of an arbitrary positive absolutely continuous function α in $[0, \infty)$. This transformation maps positive C^1 functions y in Ω into n-vector functions w defined by

$$(2) \qquad w(x) = -\frac{\alpha(|x|)}{\phi(y(x))}(A\nabla y)(x).$$

Matrix notation will be used throughout; in particular A^{-1} denotes the inverse of A and * denotes the transpose.

LEMMA 1. *If y is a positive-valued solution of $L(y; x) \leqq 0$ in Ω, under the preceding assumptions* (i)–(iv), *then the n-vector function w given by* (2) *satisfies the Riccati inequality*

$$(3) \qquad \operatorname{div} w(x) \geqq \alpha(r)p(x) + \frac{\phi'(y(x))}{\alpha(r)}(w^*A^{-1}w)(x) + \frac{\alpha'(r)}{\alpha(r)}w^*(x)\nu(x),$$

where $\nu(x) = x/r$ is the outward unit normal to S_r, $r = |x|$.

Proof. Differentiation of the i-th component of (2) with respect to x_i gives

$$D_iw_i = -\alpha'(r)D_ir\frac{(A\nabla y)_i}{\phi(y)} - \frac{\alpha(r)}{\phi^2(y)}\left[\phi(y)\sum_j D_i(A_{ij}D_jy) - \phi'(y)\sum_j A_{ij}D_iyD_jy\right]$$

for $i = 1, \ldots, n$. Since $D_ir = x_i/r = \nu_i$, summation over i and use of (1) leads to

$$(4) \qquad \operatorname{div} w \geqq -\frac{\alpha'(r)}{\phi(y)}\sum_i (A\nabla y)_i\nu_i + \frac{\alpha(r)B(x,y)}{\phi(y)} + \frac{\alpha(r)\phi'(y)}{\phi^2(y)}(\nabla y)^*A\nabla y.$$

In view of hypothesis (ii) we obtain

$$\operatorname{div} w \geqq -\frac{\alpha'(r)}{\phi(y)}(A\nabla y)^*\nu + \alpha(r)p(x) + \frac{\phi'(y)}{\alpha(r)}(A^{-1}w)^*AA^{-1}w,$$

which is equivalent to (3).

We note that Lemma 1 is considerably simplified in the case of a linear elliptic differential equation

$$(5) \qquad Ly \equiv \sum_{i,j=1}^{n} D_i[A_{ij}(x)D_jy] + p(x)y = 0.$$

LEMMA 2. *If y is a zero-free solution of* (5) *in Ω where p is continuous in Ω and assumption* (iv) *holds, then the n-vector function $w = -\alpha y^{-1}A\nabla y$ satisfies the Riccati equation below in Ω:*

$$(6) \qquad \operatorname{div} w(x) = \alpha(r)p(x) + \frac{1}{\alpha(r)}(w^*A^{-1}w)(x) + \frac{\alpha'(r)}{\alpha(r)}w^*(x)\nu(x).$$

In fact, equality occurs in (4) on account of (5), $\phi(y) = y$, and $B(x,y) = p(x)y$; consequently (4) implies (6) if y is a positive-valued solution of (5). The same is true if y is negative-valued since $Y = -y$ is a positive-valued solution of (5) and

$$y^{-1}A\nabla y = Y^{-1}A\nabla Y.$$

3. The main theorems. The Riccati inequality (3) will be used to derive sufficient conditions for the nonexistence of positive solutions of $L(y;x) \leqq 0$, and hence oscillation criteria for elliptic differential inequalities. Our approach is a (considerable) amplification of the method used by Coles [**3**] in the one-dimensional case of (1). Various specializations of Theorems 1 and 2 give new oscillation criteria for a class of nonlinear inequalities $yL(y;x) \leqq 0$ and also sharpen known linear oscillation criteria.

Let $\lambda(x)$ denote the largest (necessarily positive) eigenvalue of the matrix $(A_{ij}(x))$ and let f be any piecewise C^1 function in $(0, \infty)$ satisfying $f(r) \geqq \max_{|x|=r} \lambda(x)$.

THEOREM 3. *The elliptic inequality $L(y; x) \leqq 0$ has no eventually positive solution y in an exterior domain Ω of E^n if there exists a positive absolutely continuous function α in $[0, \infty)$ and positive numbers a and k such that the following conditions are satisfied:*

(7) $\phi'(t) \geqq k$ *for all* $t > 0$;

$$(8) \quad \lim_{r\to\infty} \left\{ \int_{G(a,r)} \left[\alpha(|x|)p(x) - \frac{[\alpha'(|x|)]^2 f(|x|)}{4k\alpha(|x|)} \right] dx + \tfrac{1}{2}\omega(S_1)k^{-1}r^{n-1}f(r)\alpha'(r) \right\} = +\infty;$$

and

$$(9) \quad \int_a^\infty \frac{r^{1-n}dr}{\alpha(r)f(r)} = +\infty,$$

where $\omega(S_1)$ denotes the area of the unit sphere in E^n.

Proof. Suppose to the contrary that y is a positive solution of $L(y; x) \leqq 0$ in $\Omega \cap G_b$ for some $b \geqq 0$. Since Ω is an exterior domain, we can assume that b is large enough so that $G_b \subset \Omega$, $\Omega \cap G_b = G_b$; then the Riccati inequality (3) holds in G_b. Since $\lambda^{-1}(x)$ is the smallest (necessarily positive) eigenvalue of $A^{-1}(x)$,

$$(10) \quad (w^*A^{-1}w)(x) \geqq \lambda^{-1}(x)|w(x)|^2 \geqq [f(r)]^{-1}|w(x)|^2.$$

Since $\phi'(y(x)) \geqq k$ by hypothesis (7), inequalities (3) and (10) imply that

$$(11) \quad \operatorname{div} w(x) \geqq \alpha(r)p(x) + \frac{k|w(x)|^2}{\alpha(r)f(r)} + \frac{\alpha'(r)(w^*\nu)(x)}{\alpha(r)}.$$

Define

$$(12) \quad W(x) = w(x) + (2k)^{-1}f(r)\alpha'(r)\nu(x),\ x \in G_a.$$

An easy calculation using (12) transforms (11) into

$$(13) \quad \operatorname{div} W \geqq \operatorname{div}\frac{f\alpha'\nu}{2k} + \alpha p - \frac{f(\alpha')^2}{4k\alpha} + \frac{k}{\alpha f}W^*W,$$

where the dependence on $x \in G_b$ has been suppressed in the notation.

Integrating (13) over $G(b, r)$ and using the divergence theorem we obtain

$$(14) \quad \int_{G(b,r)} \operatorname{div} W(x)dx = \int_{S_r} W^*\nu ds - \int_{S_b} W^*\nu ds \geqq \int_{S_r} \frac{f\alpha'}{2k} ds - \int_{S_b} \frac{f\alpha'}{2k} ds + \int_{G(b,r)} g(x)dx + R(r)$$

where

$$(15) \quad g(x) = \alpha(r)p(x) - \frac{[\alpha'(r)]^2 f(r)}{4k\alpha(r)}, \quad r = |x|$$

$$(16) \quad R(r) = \int_b^r \int_{S_r} \frac{kW^*(x)W(x)}{\alpha(r)f(r)} ds\, dr,$$

and consequently

$$\frac{dR}{dr} = \frac{k}{\alpha(r)f(r)} \int_{S_r} W^*(x)W(x)ds.$$

An application of the Cauchy-Schwarz inequality gives

$$(17) \quad \frac{dR}{dr} \geqq \frac{kr^{1-n}}{\omega(S_1)\alpha(r)f(r)} \left| \int_{S_r} W^*(x)\nu(x)ds \right|^2.$$

In view of hypothesis (8), it is seen from (14) that there exists a number $r_1 \geqq b$ such that

$$(18) \quad \int_{S_r} W^*(x)\nu(x)dS > R(r) \geqq 0$$

for $r \geqq r_1$, and consequently (17) implies that

$$(19) \quad \frac{dR}{dr} \geqq \frac{kr^{1-n}R^2(r)}{\omega(S_1)\alpha(r)f(r)}.$$

It follows from (16) and the Cauchy-Schwarz inequality again that

$$\left[\int_{S_r} W^*(x)\nu(x)ds\right]^2 \leqq r^{n-1}\omega(S_1)\int_{S_r} W^*(x)W(x)ds,$$

$$(20) \quad R(r) \geqq \int_b^r \frac{kr^{1-n}}{\omega(S_1)\alpha(r)f(r)} \left[\int_{S_r} W^*(x)\nu(x)ds\right]^2 dr$$

and therefore $R(r) > 0$ from (18) whenever $r \geqq r_1$. Then integration of (19) over (r_1, r) gives

$$\frac{1}{R(r_1)} > \int_{r_1}^r \frac{R'(r)dr}{R^2(r)} \geqq \frac{k}{\omega(S_1)} \int_{r_1}^r \frac{r^{1-n}dr}{\alpha(r)f(r)}.$$

This proves that the right side is a bounded function of r, contradicting hypothesis (9).

A function $f:\Omega \to E^1$ is called (weakly) *oscillatory* in Ω if, and only if, $f(x)$ has a zero in $\Omega \cap G_b$ for all $b \geqq 0$. The inequality $yL(y;x) \leqq 0$ is called *oscillatory* in Ω whenever every solution y of the inequality is oscillatory in Ω. Define

$$(21) \quad \hat{p}(x) = \min\{p(x), p_1(x)\}, \quad x \in \Omega.$$

THEOREM 4. *The inequality $yL(y; x) \leqq 0$ is oscillatory in an exterior domain Ω of E^n if there exists a positive absolutely continuous function α in $[0, \infty)$ and positive numbers a and k such that* (7), (9), *and the following conditions are fulfilled:*

(22) $\phi_1'(t) \geqq k$ *for all* $t > 0$;

(23) $$\lim_{r\to\infty}\left\{\int_{G(a,r)}\left[\alpha(|x|)\hat{p}(x) - \frac{[\alpha'(|x|)]^2 f(|x|)}{4k\alpha(|x|)}\right]dx + \tfrac{1}{2}\omega(S_1)k^{-1}r^{n-1}f(r)\alpha'(r)\right\} = +\infty.$$

Proof. Since $p(x) \geqq \hat{p}(x)$ for all $x \in \Omega$, (23) implies (8), and hence no solution y of the differential inequality can remain positive in $\Omega \cap G_b$ for $b \geqq 0$ by Theorem 3. If y were a negative-valued solution of $L(y; x) \geqq 0$ throughout $\Omega \cap G_b$ for some $b \geqq 0$, then by (1) and assumption (iii), $z = -y$ would be a positive-valued solution of

$$\sum_{i,j} D_i[A_{ij}(x)D_j z] \leqq B(x, -z) \leqq -p_1(x)\phi_1(z).$$

This has the same form as the inequality in Lemma 1 and Theorem 3 with p and ϕ replaced by p_1 and ϕ_1, respectively, and therefore Theorem 3 is contradicted in view of the hypotheses (22) and (23).

4. Linear and perturbed linear equations. The results of § 3 will now be specialized to the case that $B(x, y)$ in (1) has the form

(24) $$B(x, y) = q_1(x)y + \sum_{j=2}^{J} q_j(x)\psi_j(y), \quad x \in \Omega$$

under the assumptions listed below.

Assumptions.

(v) Each q_j is a continuous real-valued function in Ω; $j = 1, 2, \ldots, J$;

(vi) Each ψ_j is an odd C^1 function in $(-\infty, \infty)$ with $\psi_j(t) > 0$ and $\psi_j'(t) \geqq 0$ for $t > 0$.

It is not required that any of the functions q_j be everywhere positive in Ω. Define

(25) $$p(x) = \min[q_1(x), q_2(x), \ldots, q_J(x)]$$

(26) $$\phi(y) = y + \sum_{j=2}^{J} \psi_j(y).$$

Then $B(x, y) \geqq p(x)\phi(y)$ for all $y > 0$ and for all $x \in \Omega$, and

$$B(x, y) \leqq -p(x)\phi(-y) \text{ for all } y < 0 \text{ and } x \in \Omega.$$

Consequently the basic assumptions (ii) and (iii) of § 2 are satisfied with $p_1 = p$ and $\phi_1 = \phi$.

If each $q_j(x) > 0$ throughout Ω $(j = 2, \ldots, J)$ then a solution y of $yL(y; x) \leqq 0$ satisfies

$$(27) \quad Q_1(x) = q_1(x) + \sum_{j=2}^{J} q_j(x) y^{-1}(x) \psi_j(y(x)) > q_1(x)$$

for all $x \in \Omega$ with $y(x) \neq 0$, and oscillation criteria for $L(y; x) = 0$ follow from known nodal oscillation criteria for linear (only) differential equations

$$(28) \quad L(y; x) \equiv \sum_{i,j=1}^{n} D_i[A_{ij}(x) D_j y] + q_1(x) y = 0.$$

The latter criteria, sufficient for the existence of a nontrivial solution of (28) with a nodal domain in $\Omega \cap G_a$ for all $a > 0$, have been known for some time [**4**, **8**]. If a one-signed solution of $yL(y; x) \leqq 0$ existed under such nodal oscillation criteria, then $y(x)$ would satisfy a linear inequality of type (28) with q_1 replaced by Q_1. In view of (27), a standard linear comparison theorem would yield a contradiction. However, this argument cannot be accomplished in any straightforward way when the functions q_j in (24) change sign in Ω.

Since assumptions (ii) and (iii) are satisfied under the structure (24), and (7), (22) hold with $k = 1$ by (26), conditions (9), (23) of Theorem 4 (or (8), (9) of Theorem 3) with $k = 1$ are oscillation criteria for the perturbed linear inequality $yL(y; x) \leqq 0$. Specializations of these results to the cases $\alpha(r) = \log r$ $(r > 1)$ if $n = 2$ and $\alpha(r) = r^{2-n}$ $(r > 0)$ if $n \geqq 3$ are as follows:

THEOREM 5. *The perturbed linear inequality* $yL(y; x) \leqq 0$ *given by* (1), (24) *is oscillatory in an exterior domain* Ω *of* E^2 *under assumptions* (iv), (v), *and* (vi) *if the largest eigenvalue* $\lambda(x)$ *of* $A(x)$ *satisfies*

$$(29) \quad \max_{|x|=r} \lambda(x) \leqq C[\log(\log r)]^{\delta}, \ r > e$$

for some numbers $C > 0$ *and* δ, $0 < \delta \leqq 1$, *and*

$$(30) \quad \int_a^\infty \left[r \log r p_M(r) - \frac{f(r)}{4r \log r} \right] dr = +\infty$$

for some $a > 0$, *where* $p_M(r)$ *denotes the spherical mean*

$$(31) \quad p_M(r) = \frac{1}{\omega(S_1)} \int_{S_1} p(x) d\omega.$$

THEOREM 6. *The perturbed linear inequality* $yL(y; x) \leqq 0$ *given by* (1), (24) *is oscillatory in an exterior domain of* E^n $(n \geqq 3)$ *under assumptions* (iv), (v), *and* (vi) *if*

$$(32) \quad \max_{|x|=r} \lambda(x) \leqq C(\log r)^{\delta}, \ r > 1$$

for some numbers $C > 0$ *and* δ, $0 < \delta \leqq 1$, *and*

$$(33)\quad \int_a^\infty \left[rp_M(r) - \frac{(n-2)^2 f(r)}{r} \right] dr = +\infty$$

for some $a > 0$, *where* $p_M(r)$ *is given by* (31).

Proofs. For $n = 2$, we set $k = 1$ and $\alpha(r) = \log r$ $(r > 1)$ in (23) and note that (9) is satisfied for $f(r) = C[\log(\log r)]^\delta$, $C > 0$, $0 < \delta \leqq 1$:

$$\int^\infty \frac{r^{1-n}dr}{\alpha(r)f(r)} = \frac{1}{C}\int^\infty \frac{dr}{r\log r[\log(\log r)]^\delta} = +\infty.$$

The first term on the left side of (23) reduces to

$$\lim_{r\to\infty} \int_a^r \int_{S_1} \left[p(x)\log r - \frac{f(r)}{4r^2 \log r} \right] r d\omega\, dr,$$

which diverges to $+\infty$ by (30) and (31). The conclusion of Theorem 5 then follows from Theorem 4.

For $n \geqq 3$ we set $k = 1$ and $\alpha(r) = r^{2-n}$ in (23) and again see that (9) is satisfied in view of (32), and (23) is satisfied because of (33). Theorem 6 then also follows as a special case of Theorem 4.

The case that $A(x)$ is bounded above is included in Theorems 5 and 6, and the case of a constant matrix A leads to the following simplification.

COROLLARY 7. *For a constant matrix* A, *the inequality* $yL(y; x) \leqq 0$ *given by* (1), (24) *is oscillatory in an exterior domain* Ω *of* E^n *under assumptions* (iv), (v), *and* (vi) *if*

$$\int_a^\infty \left[r\log r p_M(r) - \frac{\lambda}{4r\log r} \right] dr = +\infty, \quad n = 2$$

$$\int_a^\infty \left[rp_M(r) - \frac{(n-2)^2\lambda}{r} \right] dr = +\infty, \quad n \geqq 3$$

for some $a > 0$, *where* λ *denotes the largest eigenvalue of* A.

If the growth condition (29) or (32) fails, oscillation criteria are still available by different choices of $\alpha(r)$ in Theorem 4. The following is obtained for $\alpha(r) = 1$ identically.

THEOREM 8. *The perturbed linear inequality* $yL(y; x) \leqq 0$ *is oscillatory in an exterior domain* Ω *of* E^n *under assumptions* (iv), (v), *and* (vi) *if*

$$(34)\quad \int_\Omega p(x)dx = +\infty$$

and

$$\int_a^\infty \frac{r^{1-n}dr}{f(r)} = +\infty, \quad \textit{for some } a > 0.$$

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Theorems 4, 5, 6 and 8 give new oscillation criteria even in the case of a linear equation (28) or inequality $yL(y; x) \leqq 0$. The first reason for this is that $p(x) = q_1(x)$ in (28) is allowed to change sign in Ω, and the second reason is that (30) and (33) are sharper than the usual criteria, e.g. those in [**9**]. For example, (30) is satisfied if (28) is the Schrödinger equation $\triangle y + p(x)y = 0$ and if either

$$\liminf_{r\to\infty} r^2 p_M(r)(\log r)^2 > \tfrac{1}{4}$$

or

$$p_M(r) \sim \frac{1}{4r^2 \log^2 r} + \frac{C}{r^2 \log^2 r[\log (\log r)]^\delta}$$

as $r \to \infty$ for some $C > 0$, $0 < \delta \leqq 1$. The standard criterion [**9**]

$$\int_a^\infty r(\log r)^\rho p_M(r)dr = +\infty, \quad \rho < 1$$

(and also the criterion (34)) are stronger than (30) and are not always fulfilled in the preceding examples. Similar remarks can be made attesting to the sharpness of (33) in dimensions $n \geqq 3$.

5. Superlinear inequalities. In this section results similar to those in § 4 will be obtained for $yL(y; x) \leqq 0$ under the superlinear hypotheses below.

Superlinear Assumptions.

(vii) The functions p and p_1 in assumptions (ii) and (iii) are identical and everywhere nonnegative in Ω;

(viii) The functions ϕ and ϕ_1 are identical and $\psi(t) = \phi(t)/t$ is nondecreasing for all $t > 0$.

Theorem 4 cannot be applied since ϕ does not satisfy condition (7):

$$\phi'(t) \geqq k > 0 \text{ for all } t > 0.$$

For example, (7) fails for the superlinear prototype $\phi(t) = t^\gamma$, $\gamma > 1$. Instead Lemma 9 below will be employed, replacing (7) by an a priori lower bound $Y(a)R(|x|)$ on any positive solution $y(x)$ of $L(y; x) \leqq 0$ in Ω, where $Y(a)$ and $R(r)$ are defined by (38) and (37) below, respectively. The superlinear oscillation criteria can then be deduced from the linear results in § 4. Alternatively, we could have employed the Riccati inequality (3) directly to obtain analogues of Theorems 3 and 4 when (7) is replaced by the a priori bound in Lemma 9.

The following notation will be used:

$$(35)\quad L_0(y;x) = \sum_{i,j=1}^{n} D_i[A_{ij}(x)D_jy]$$

$$(36)\quad \rho(r) = \sup_{|x|=r} |L_0(|x|;x)|\left[\sum_{i,j} A_{ij}(x)D_i|x|D_j|x|\right]^{-1}$$

$$S(r;b) = \int_r^b \exp\left[\int_a^t -\rho(s)ds\right]dt, \quad a \leqq r \leqq b$$

$$R(r;b) = S(r;b)/S(a;b)$$

$$(37)\quad R(r) = \lim_{b\to\infty} R(r;b)$$

$$(38)\quad Y(a) = \inf_{|x|=a} y(x).$$

LEMMA 9. *Every positive solution y of $L_0(y;x) \leqq 0$ for $|x| \geqq a$ satisfies the inequality $y(x) \geqq Y(a)R(|x|)$ for $|x| \geqq a$.*

Proof. The function u defined in $[a, b]$ by $u(r) = Y(a)R(r;b)$ is the unique solution of the ordinary boundary problem

$$(39)\quad \begin{cases} u''(r) - \rho(r)|u'(r)| = 0, \; a \leqq r \leqq b \\ u(a) = Y(a), \; u(b) = 0, \end{cases}$$

as is well-known and easily checked by direct substitution. We define $v(x) = u(|x|) = u(r)$ and compute $L_0(v;x)$ using (35), (36), and (39):

$$\begin{aligned} L_0(v;x) &= \sum_{i,j} (A_{ij}D_irD_jr)u''(r) + \sum_{i,j} D_i(A_{ij}D_jr)u'(r) \\ &= \sum_{i,j} (A_{ij}D_irD_jr)u''(r) + L_0(|x|;x) \\ &\geqq \left[\sum_{i,j} A_{ij}D_irD_jr\right][u''(r) - \rho(r)|u'(r)|] = 0. \end{aligned}$$

Then $v(x)$ is a solution of the boundary problem

$$\begin{cases} L_0(v;x) \geqq 0 \text{ in } a \leqq r \leqq b \\ \quad v(x) = Y(a) \text{ on } |x| = a;\; v(x) = 0 \text{ on } |x| = b. \end{cases}$$

However $y(x)$ satisfies

$$\begin{cases} L_0(y;x) \leqq 0 \text{ in } a \leqq r \leqq b \\ \quad y(x) \geqq Y(a) \text{ on } |x| = a;\; y(x) > 0 \text{ on } |x| = b, \end{cases}$$

and it follows from the Hopf maximum principle [**10**] applied to the annulus $G(a, b)$ that $y(x) \geqq v(x) = u(|x|)$ throughout $a \leqq |x| \leqq b$, i.e.,

$$y(x) \geqq Y(a)R(r;b).$$

Since this is valid for arbitrary $b > a$, the proof of Lemma 9 is complete.

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THEOREM 10. *Under assumptions* (i)–(iv), (vii), *and* (viii), *the superlinear inequality* $yL(y;x) \leqq 0$ *is oscillatory in an exterior domain* Ω *of* E^n *if the linear inequality*

$$(40) \quad L_0(y;x) + p(x)\psi(\epsilon R(|x|))y \leqq 0$$

has no eventually positive solution in Ω *for any positive number* ϵ.

Proof. If $y(x)$ is a positive solution of $yL(y;x) \leqq 0$ throughout G_b for some $b > 0$ (chosen large enough so that $G_b \subset \Omega$ without loss of generality), then

$$\begin{aligned} 0 &\geqq L_0(y(x);x) + B(x, y(x)) \\ &\geqq L_0(y(x);x) + p(x)\psi(y(x))y(x) \\ &\geqq L_0(y(x);x) + p(x)\psi(Y(a)R(|x|))y(x) \end{aligned}$$

on account of Lemma 9 and the nondecreasing hypothesis (viii) on ψ. This means that $y(x)$ satisfies (40) with $\epsilon = Y(a) > 0$, and so $y(x)$ cannot be eventually positive.

If $y(x)$ were a negative solution of $L(y;x) \geqq 0$ throughout G_b, then $z(x) = -y(x)$ would be a positive solution of

$$\begin{aligned} 0 &\leqq L_0(-z(x);x) + B(x, -z(x)) \\ &\leqq -L_0(z(x);x) - p(x)\psi(z(x))z(x) \end{aligned}$$

and consequently

$$0 \geqq L_0(z(x);x) + p(x)\psi(Z(a)R(|x|))z(x),$$

which is impossible by the first part of the proof.

COROLLARY 11. *Under the same assumptions, the superlinear inequality* $yL(y;x) \leqq 0$ *is oscillatory in an exterior domain* E^n *if there exists a positive absolutely continuous function* α *in* $[0, \infty)$ *and a positive number* a *such that*

$$(41) \quad \lim_{r\to\infty} \left\{ \int_{G(a,r)} \left[\alpha(|x|)p(x)\psi(\epsilon R(|x|)) - \frac{[\alpha'(|x|)]^2 f(|x|)}{4\alpha(|x|)} \right] dx + \tfrac{1}{2}\omega(S_1)r^{n-1}f(r)\alpha'(r) \right\} = +\infty$$

and

$$(42) \quad \int_a^\infty \frac{r^{1-n}dr}{\alpha(r)f(r)} = +\infty$$

for all $\epsilon > 0$.

Proof. The linear elliptic inequality (40) is of the type considered in Theorem 3 in the special case $\phi(t) = t$, so (7) and (8) hold for $k = 1$, with $p(x)$ replaced by $p(x)\psi(\epsilon R(|x|))$.

COROLLARY 12. *Under the same assumptions and* (29) *in addition, the*

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superlinear inequality $yL(y; x) \leqq 0$ *is oscillatory in an exterior domain* Ω *of* E^2 *if there exists a positive number* a *such that*

$$(43)\quad \int_a^\infty \left[r \log r \psi(\epsilon R(r)) p_M(r) - \frac{f(r)}{4r \log r} \right] dr = +\infty$$

for all $\epsilon > 0$, *where* $p_M(r)$ *denotes the spherical mean* (31).

COROLLARY 13. *Under the same assumptions and* (32) *in addition,* $yL(y; x) \leqq 0$ *is oscillatory in an exterior domain of* E^n $(n \geqq 3)$ *if there exists a positive number* a *such that*

$$(44)\quad \int_a^\infty \left[r\psi(\epsilon R(r)) p_M(r) - \frac{(n-2)^2 f(r)}{r} \right] dr = +\infty$$

for all $\epsilon > 0$.

Proofs. For $n = 2$ we choose $\alpha(r) = \log r$ $(r > 1)$ in Corollary 11 and note that (42) is satisfied for

$$f(r) = C[\log(\log r)]^\delta,\ C > 0,\ 0 < \delta \leqq 1.$$

Furthermore (41) is satisfied by assumption (43) as in the proof of Theorem 5. For $n \geqq 3$ we choose $\alpha(r) = r^{2-n}$ in (41) and (42) and argue similarly.

The next corollary is a specialization to the Emden-Fowler prototype

$$(45)\quad L(y; x) = \Delta y + p(x) y^\gamma = 0,\ \gamma > 1$$

where γ is a quotient of odd integers.

COROLLARY 14. *Under the same assumptions,* (45) *is oscillatory in an exterior domain in* E^n *if there exists a positive number* a *such that*

$$(46)\quad \int_a^\infty \left[\epsilon r \log r p_M(r) - \frac{1}{4r \log r} \right] dr = +\infty \quad (n = 2)$$

$$(47)\quad \int_a^\infty \left[\epsilon r^\sigma p_M(r) - \frac{(n-2)^2}{r} \right] dr = +\infty \quad (n \geqq 3),$$

for all $\epsilon > 0$, *where* $\sigma = (n - 1) - \gamma(n - 2)$.

Proof. In the case $L_0(y; x) = \Delta y$, one checks from (37) that $R(r) = (a/r)^{n-2}$, $n \geqq 2$. For $f(r) = 1$ and $\psi(\epsilon_1 R(r)) = \epsilon_1^{\gamma-1} = \epsilon$, (43) specializes to (46). For $f(r) = 1$ and

$$\psi(\epsilon_1 R(r)) = \left[\epsilon_1 \left(\frac{a}{r} \right)^{n-2} \right]^{\gamma-1} = \epsilon r^{(2-n)(\gamma-1)},$$

where $\epsilon = (\epsilon_1 a^{n-2})^{\gamma-1}$, and consequently

$$r\psi(\epsilon_1 R(r)) = \epsilon r^\sigma,\ \sigma = n - 1 - \gamma(n-2),$$

(44) specializes to (47).

The criteria (46) and (47) are very similar to those obtained by the authors [**9**] using the totally different method of spherical means and ordinary differential inequalities.

6. A sublinear oscillation criterion. The Riccati method will be illustrated in the case that (1) reduces to the Schrödinger operator

$$(48) \quad L(y;x) = \Delta y + B(x,y), \quad \Delta = \sum_{i=1}^{2} D_i D_i, \quad x \in E^2.$$

The assumptions are (i)–(iii) and the

Sublinear Assumption.

(ix) The functions ϕ and ϕ_1 in (ii), (iii) are identical, $\phi'(t) > 0$ for all $t > 0$, and $\Phi(t) < \infty$ for all $t > 0$, where

$$(49) \quad \Phi(t) = \int_0^t \frac{ds}{\phi(s)}.$$

If y is a positive solution of $yL(y;x) \leqq 0$, then $\Delta y \leqq -p(x)\phi(y)$ by (48) and assumption (ii), and hence (49) gives

$$(50) \quad \Delta\Phi(y) = -\frac{\phi'(y)}{\phi^2(y)}|\nabla y|^2 + \frac{\Delta y}{\phi(y)} \leqq -p(x).$$

The oscillation theorem below is stated in terms of the spherical mean

$$(51) \quad \hat{p}_M(r) = \frac{1}{2\pi}\int_0^{2\pi} \hat{p}(r,\theta)d\theta, \quad r > 0$$

where $\hat{p}(r,\theta) = \hat{p}(x)$ is given by (21).

THEOREM 15. *Under assumptions* (i)–(iii) *and* (ix), *the sublinear inequality* $yL(y;x) \leqq 0$ *given by* (48) *is oscillatory in an exterior domain of* E^2 *if*

$$(52) \quad \lim_{r\to\infty} \int_{r_0}^{r} \rho\hat{p}_M(\rho)d\rho = +\infty$$

for some $r_0 > 0$.

Proof. Suppose to the contrary that $y(x)$ is a positive solution of $L(y;x) \leqq 0$ in G_a for some $a > 0$. Define

$$(53) \quad m_y(r) = \int_{S_r} \nabla\Phi(y(x))^* \nu ds = \int_0^{2\pi} \frac{\partial\Phi(y(r,\theta))}{\partial r} r d\theta$$

for $a < r < \infty$. Let $w(x)$ be defined by (2) with $\alpha(r) = 1$ identically.

Then (3) can be integrated over $G(a, r)$ to give

$$(54) \quad \int_{G(a,r)} \operatorname{div} w(x)dx \geqq \int_{G(a,r)} p(x)dx.$$

Let ρ, θ denote polar coordinates of x, $a \leqq \rho = |x| \leqq r$. Since

$$(w^*\nu)(x) = -\partial\Phi(y(\rho, \theta))/\partial\rho$$

by (2) and (49), application of the divergence theorem to (54) gives

$$m_y(r) - m_y(a) \leqq -2\pi \int_a^r \rho \hat{p}_M(\rho)d\rho.$$

By hypothesis (52), there exists a number $b \geqq a$ such that $m_y(r) \leqq K_a < 0$ for all $r \geqq b$, and consequently

$$\int_0^{2\pi} [\Phi(y(r, \theta)) - \Phi(y(b, \theta))]d\theta = \int_0^{2\pi} \int_b^r \frac{\partial\Phi(y(\rho, \theta))}{\partial\rho} d\rho d\theta$$
$$= \int_b^r m_y(\rho) \frac{d\rho}{\rho} \leqq K_a \log \frac{r}{b}$$

whenever $r \geqq b$ on account of (53). Then

$$\int_0^{2\pi} \Phi(y(r, \theta))d\theta$$

would become negative for sufficiently large r, contrary to the hypothesis that $y(x) > 0$ throughout G_a.

If $y(x) < 0$ and $L(y; x) \geqq 0$ throughout G_a for some $a > 0$, then $z(x) = -y(x)$ would be a positive solution of

$$0 \leqq -\triangle z + B(x, -z) \leqq -\triangle z - p_1(x)\phi_1(z)$$

by assumption (iii), or equivalently $\triangle z \leqq -\hat{p}(x)\phi(z)$ by (ix). Since this is exactly the inequality leading to (3), a contradiction is obtained as in the first part of the proof.

We remark that none of the functions $B(x, t)$, $p(x)$, and $p_1(x)$ are required to be everywhere positive for Theorem 15. A similar result was obtained recently by Kitamura and Kusano [**6**] in the case of a Schrödinger equation $\triangle y + p(x)\phi(y) = 0$.

7. Conclusion. The Riccati partial differential inequality (3) has led to new oscillation criteria for several types of nonlinear elliptic operators, especially perturbed linear and superlinear operators. Specialization to linear problems actually sharpened and extended earlier results. Several open questions stated in [**9**] have been resolved: (1) The matrix $A(x)$ in (1) is not required to have constant entries, as in [**9**], etc; (2) The usual sign restrictions on $B(x, t)$, $p(x)$, etc. are relaxed; and (3) Sublinear and perturbed linear problems are treated. Also, unbounded domains Ω which are not exterior domains could easily be treated formally, as in [**7**], by simply adjoining the boundary condition $\nabla y^*\nu \geqq 0$ on $\partial\Omega \cap G_a$ for some $a > 0$, e.g. the Robin condition $\nabla y^*\nu = \lambda y$ for a nonnegative boundary function λ. This approach, however, shows the existence of an oscillatory solution of $yL(y; x) \leqq 0$ only if a solution satisfying the above boundary condition is known to exist. A more realistic approach offered recently by Allegretto [**2**] proves that all solutions oscillate in a class of unbounded domains under suitable conditions, but sharp and explicit oscillation criteria of our type here are still lacking.

The sharpness of our criteria can be seen by considering the known one-dimensional results: Either the radial ordinary differential equation associated with L can be examined, or the authors' differential inequality approach [**9**] can be compared. It turns out to be true, in fact, that (46), (47) are very close to necessary and sufficient conditions for oscillation of superlinear operators. These results will appear elsewhere. [Indiana Univ. Math. J. *28* (1979), 993–1003]

Some open questions are:

(1) As already mentioned, are there sharp and/or explicit criteria (similar to linear ones) guaranteeing the existence of an oscillatory solution in a nonexterior unbounded domain?

(2) In view of the sharpness of the superlinear criteria, is it possible to obtain analogous sublinear criteria? (The result derived in § 6 and in [**6**] would seem to be not sharp in view of the well known necessary and sufficient condition for oscillation in 1 dimension.)

(3) How can one obtain sublinear criteria when A is not the identity matrix?

Added in proof. Lemma 9 is closely related to a result of Allegretto [**1**, p 935].

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