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ELASTICA IN $SO(3)$

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Abstract

In a Riemannian manifold M, *elastica* are solutions of the Euler-Lagrange equation of the following second order constrained variational problem: find a unit-speed curve in M, interpolating two given points with given initial and final (unit) velocities, of minimal average squared geodesic curvature. We study elastica in Lie groups G equipped with bi-invariant Riemannian metrics, focusing, with a view to applications in engineering and computer graphics, on the group $SO(3)$ of rotations of Euclidean 3-space. For compact G, we show that elastica extend to the whole real line. For $G = SO(3)$, we solve the Euler-Lagrange equation by quadratures.

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1. Introduction

Let M be a finite-dimensional C^∞ Riemannian manifold. Let $\langle \cdot, \cdot \rangle$ denote the Riemannian metric, $\|\cdot\|$ the corresponding Riemannian norm and ∇ the corresponding Levi-Civita covariant derivative (see [3, 6] for background). Given $T > 0$ and $p_i \in M$, $v_i \in T_{p_i}M$ for $i = 0, 1$, let $\mathscr{C}_{v_0,v_1}$ be the space of C^∞ curves $x : [0, T] \to M$ satisfying $x(iT) = p_i$, $\dot{x}(iT) = v_i$ for $i = 0, 1$, where $\dot{x} = dx/dt$, and define a functional $\Phi : \mathscr{C}_{v_0,v_1} \to [0, \infty)$ by

$$\Phi(x) = \int_0^T \left\| \nabla_{d/dt} \dot{x} \right\|^2 dt.$$

The critical points of Φ, *Riemannian cubics*, are studied in [7–9, 11] and references therein. When v_0 and v_1 are unit vectors, the *elastic problem* is to minimise Φ over curves $x \in \mathscr{C}_{v_0,v_1}$ subject to the nonholonomic constraint

$$\|\dot{x}(t)\|^2 = 1, \tag{1.1}$$

for all $t \in [0, T]$. That is, the curves are required to have unit speed; in this case, $||\nabla_{d/dt}\dot{x}||^2$ is the squared geodesic curvature of x. The first order necessary conditions for $x \in \mathscr{C}_{v_0,v_1}$ to solve the elastic problem are given in the following theorem, which can be proved using the Pontryagin Maximum Principle (which can be found in [5]).

THEOREM 1.1. *A curve $x \in \mathscr{C}_{v_0,v_1}$ solving the elastic problem satisfies, for all $t \in [0, T]$, the constraint* (1.1) *and the (Euler-Lagrange) equation*

$$\nabla^3_{d/dt}\dot{x} + R\left(\nabla_{d/dt}\dot{x}, \dot{x}\right)\dot{x} + \nabla_{d/dt}(\lambda\dot{x}) = \mathbf{0}, \tag{1.2}$$

for some C^∞ function $\lambda : [0, T] \to \mathbb{R}$.

Here R is the Riemannian curvature of M, defined, with opposite sign convention to [3, 6], for C^∞ vector fields X, Y, Z on M by

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

where $[\cdot, \cdot]$ is the Lie bracket. We call any C^∞ curve $x : I \to M$ (where I is an open interval) satisfying (1.1) and (1.2) on I, for some λ, an *elastic curve*.

Elastic curves, or *elastica*, in simply-connected 2 and 3-dimensional spaces of constant curvature have been studied by Jurdjevic [4, 5]. In the present paper, we investigate elastica in the Lie group $SO(3)$ of rotations of Euclidean 3-space E^3, equipped with a bi-invariant Riemannian metric. As in previous work on Riemannian cubics [7–9, 11], we have in mind applications to engineering and computer graphics, including trajectory planning for rigid body motion (as a set, the configuration space of a rigid body is $SO(3) \times E^3$).

Since the unit 3-sphere $S^3 \subset E^4$ with standard Riemannian metric double-covers bi-invariant $SO(3)$ by a local isometry, our elastica are locally equivalent to the *symmetric elastica* in S^3 studied in [5]. Jurdjevic's construction of symmetric elastica in S^3 in terms of quadratures was obtained using optimal control theory and the Hamiltonian formalism, and extends naturally to spheres of arbitrary dimension. We use an essentially different approach to study the locally equivalent elastica in $SO(3)$. First we reduce the Euler-Lagrange equation (1.2) to a second order differential equation for an auxiliary curve $V = x^{-1}\dot{x}$ in the Lie algebra of $SO(3)$. Once V is known, the first order equation $\dot{x} = xV$ can be solved for x. This approach has previously been used to study Riemannian cubics in $SO(3)$ [7–9, 11]. While it relies in a crucial way on the geometry of $SO(3)$, it is arguably more elementary than Jurdjevic's. Our resulting global construction (by quadratures) of elastica of a bi-invariant Riemannian metric on $SO(3)$ is a useful alternative to the local construction of [5].

We begin with the following generally applicable result, which characterises elastic curves as solutions of an *unconstrained* differential equation, subject to initial conditions of a particular form. It will be used in section 3.

THEOREM 1.2. *A C^∞ curve $x : I \to M$ is an elastic curve if and only if*

$$\nabla^3_{d/dt}\dot{x} + R\left(\nabla_{d/dt}\dot{x}, \dot{x}\right)\dot{x} + \nabla_{d/dt}\left(\left(\frac{3}{2}\left\|\nabla_{d/dt}\dot{x}\right\|^2 + \tilde{b}\right)\dot{x}\right) = \mathbf{0} \tag{1.3}$$

for some constant $\tilde{b} \in \mathbb{R}$ and all $t \in I$ and, for some $t_0 \in I$,

$$1 = \|\dot{x}(t_0)\|^2, \tag{1.4}$$

$$0 = \left\langle \nabla_{d/dt}\dot{x}\Big|_{t=t_0}, \dot{x}(t_0)\right\rangle, \tag{1.5}$$

$$0 = \left\langle \nabla^2_{d/dt}\dot{x}\Big|_{t=t_0}, \dot{x}(t_0)\right\rangle + \left\| \nabla_{d/dt}\dot{x}\Big|_{t=t_0}\right\|^2. \tag{1.6}$$

PROOF. First suppose x is an elastic curve. Then (1.1) holds on I, and the first two derivatives of $||\dot{x}||^2$ vanish on I, giving the following identities:

$$\left\langle \nabla_{d/dt}\dot{x}, \dot{x}\right\rangle = 0, \tag{1.7}$$

$$\left\langle \nabla^2_{d/dt}\dot{x}, \dot{x}\right\rangle + \left\|\nabla_{d/dt}\dot{x}\right\|^2 = 0. \tag{1.8}$$

In particular, (1.4)–(1.6) hold for any $t_0 \in I$. It remains to show that

$$\lambda = \frac{3}{2}\left\|\nabla_{d/dt}\dot{x}\right\|^2 + \tilde{b} \tag{1.9}$$

for some constant $\tilde{b}$. For this, take the inner product of (1.2) with $\dot{x}$ and use (1.1), (1.7), the fact that $\langle R(X, Y)Z, Z\rangle = 0$ for all X, Y, Z, and (1.8) to give

$$\dot{\lambda} = -\left\langle \nabla^3_{d/dt}\dot{x}, \dot{x}\right\rangle = \frac{d}{dt}\left(\frac{1}{2}\left\|\nabla_{d/dt}\dot{x}\right\|^2 - \left\langle \nabla^2_{d/dt}\dot{x}, \dot{x}\right\rangle\right) = \frac{3}{2}\frac{d}{dt}\left\|\nabla_{d/dt}\dot{x}\right\|^2.$$

Now suppose x satisfies (1.3) for some $\tilde{b}$ and (1.4)–(1.6) for some t_0. Then (1.2) holds on I with λ defined by (1.9), so we just need to show that $||\dot{x}(t)||^2 = 1$ for all $t \in I$. Write $I = (s_1, s_2)$. We show that $||\dot{x}(t)||^2 = 1$ for all $t \in [t_0, s_2)$ (the proof for $(s_1, t_0]$ is similar). Suppose not, and write $\mathscr{S} = \{t \in [t_0, s_2) : ||\dot{x}(t)||^2 = 1\}$ and $\tau = \sup(\mathscr{S})$. We show that in fact $||\dot{x}(t)||^2 = 1$ on some open interval containing τ; this contradicts $\tau = \sup(\mathscr{S})$, and it follows that $||\dot{x}(t)||^2 = 1$ on $[t_0, s_2)$. Write $x_1 = x$, $x_2 = \dot{x}$, $x_3 = \nabla_{d/dt}\dot{x}$ and $x_4 = \nabla^2_{d/dt}\dot{x}$. Then (1.3) can be written as the following system:

$$\begin{aligned} &\dot{x}_1 = x_2, \quad \nabla_{d/dt}x_2 = x_3, \quad \nabla_{d/dt}x_3 = x_4, \\ &\nabla_{d/dt}x_4 = -R(x_3, x_2)x_2 - 3\langle x_4, x_3\rangle x_2 - \left(\frac{3}{2}\|x_3\|^2 + \tilde{b}\right)x_3. \end{aligned} \tag{1.10}$$

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Take a coordinate chart $U \subseteq M$ containing $x(\tau)$ and let $\tilde{U} \subseteq \mathbb{R}^m$ be the image of U under the chart diffeomorphism (here m is the dimension of M). Let J be the maximal open sub-interval of I containing τ and with $x(J) \subseteq U$. We show that $\|\dot{x}(t)\|^2 = 1$ for all $t \in J$. Let Γ be the C^∞ map that takes each $\tilde{u} \in \tilde{U}$ to the Christoffel transformation $\Gamma_{\tilde{u}} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ (we avoid coordinate notation for the sake of presentation). Then, in $\tilde{U}$, for any vector field Ω defined along the curve x, for all $t \in I$,

$$\tilde{\nabla}_{d/dt}\tilde{\Omega}\Big|_t = \dot{\tilde{\Omega}}(t) + \Gamma_{\tilde{x}(t)}\left(\tilde{\Omega}(t), \dot{\tilde{x}}(t)\right), \tag{1.11}$$

where $\tilde{\nabla}$, $\tilde{\Omega}$, $\tilde{x}$ are the images of ∇, Ω, x under the chart diffeomorphism. Write $\tilde{N} := \tilde{U} \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m$ and define $\tilde{X} : \tilde{N} \to \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m$ to be the mapping

$$\tilde{X} : (y_1, y_2, y_3, y_4) \mapsto \begin{bmatrix} y_2 \\ y_3 - \Gamma_{y_1}(y_2, y_2) \\ y_4 - \Gamma_{y_1}(y_3, y_2) \\ -\tilde{R}_{y_1}(y_3, y_2)y_2 - 3\langle y_4, y_3\rangle_{y_1} y_2 - \left(\frac{3}{2}\|y_3\|_{y_1}^2 + \tilde{b}\right) y_3 - \Gamma_{y_1}(y_4, y_2) \end{bmatrix}^T$$

where $\tilde{R}$ is the image of R under the chart diffeomorphism and the inner product and norm are now computed in the chart. Then, by (1.10)–(1.11), the differential equation (1.3) is described in $\tilde{U}$ by the vector field corresponding to $\tilde{X}$. That is, denoting the image of x_i under the chart diffeomorphism by $\tilde{x}_i$ and writing $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4)$, we have $\dot{\tilde{x}} = \tilde{X}(\tilde{x})$. Since $\|\dot{x}(t)\|^2 = 1$ on $[t_0, \tau)$, (1.7) and (1.8) hold on $[t_0, \tau)$. By the smoothness of x, these equalities also hold at $t = \tau$. That is, $\tilde{x}(\tau)$ lies in the submanifold of $\tilde{N}$ given by

$$\tilde{N}_* = \left\{(y_1, y_2, y_3, y_4) \in \tilde{N} : \|y_2\|_{y_1}^2 = 1,\ \langle y_3, y_2\rangle_{y_1} = 0,\ \langle y_4, y_2\rangle_{y_1} + \|y_3\|_{y_1}^2 = 0\right\}.$$

We now show that $\tilde{X}$ is tangent to $\tilde{N}_*$. It then follows that the image of the integral curve $\tilde{x} : J \to \tilde{N}$ of $\tilde{X}$ lies in $\tilde{N}_*$; in particular, $\|\dot{x}(t)\|^2 = 1$ for all $t \in J$, as claimed. Differentiating the constraints that define $\tilde{N}_*$ and applying (1.11), we find that the tangent space to $\tilde{N}_*$ at $y := (y_1, y_2, y_3, y_4) \in \tilde{N}_*$ is the set of all $(z_1, z_2, z_3, z_4) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m$ satisfying

$$0 = \langle z_2 + \Gamma_{y_1}(y_2, z_1), y_2\rangle_{y_1}, \quad 0 = \langle z_3 + \Gamma_{y_1}(y_3, z_1), y_2\rangle_{y_1} + \langle y_3, z_2 + \Gamma_{y_1}(y_2, z_1)\rangle_{y_1},$$
$$0 = \langle z_4 + \Gamma_{y_1}(y_4, z_1), y_2\rangle_{y_1} + \langle y_4, z_2 + \Gamma_{y_1}(y_2, z_1)\rangle_{y_1} + 2\langle z_3 + \Gamma_{y_1}(y_3, z_1), y_3\rangle_{y_1}.$$

A short calculation shows that $\tilde{X}(y)$ lies in this space, as required. □

The rest of the paper is organised as follows. In section 2 we take M to be an arbitrary Lie group G with $\langle\cdot,\cdot\rangle$ bi-invariant. In this setting, solutions of (1.3) with (1.1) (elastic curves) satisfy a system consisting of a first order equation relating x to an auxiliary curve V in the unit sphere of the Lie algebra of G, and a second order equation for V. In section 3 we show that if G is compact, (1.3) subject to (1.4)–(1.6) is solvable on the whole real line. We then take $G = SO(3)$ and construct the elastic curves $x : \mathbb{R} \to SO(3)$ in terms of quadratures: V is found in sections 4 and 5, and x in section 6.

2. Elastica in Lie Groups

From now on, suppose the manifold M is a Lie group G and assume the Riemannian metric $\langle \cdot, \cdot \rangle$ of G is bi-invariant, that is, invariant with respect to both left and right multiplications (note that any Lie group admits a left-invariant metric, and any compact Lie group admits a bi-invariant metric). Let e be the identity of G, $\mathcal{G} = T_eG$ the Lie algebra and $[\cdot, \cdot]$ the Lie bracket. Let $\|\cdot\| : \mathcal{G} \to [0, \infty)$ be the norm corresponding to the restriction of $\langle \cdot, \cdot \rangle$ to $\mathcal{G}$. Recall (see [3]) that bi-invariance of a left-invariant metric $\langle \cdot, \cdot \rangle$ is equivalent to the condition

$$\langle [X, Y], Z \rangle = \langle [Z, X], Y \rangle, \quad \text{for all } X, Y, Z \in \mathcal{G}. \tag{2.1}$$

Now let $I \subseteq \mathbb{R}$ be an open interval. Given any C^∞ curve $x : I \to G$, we can define a C^∞ curve $V : I \to \mathcal{G}$ by

$$V(t) = \left(dL_{x(t)^{-1}}\right)_{x(t)} \dot{x}(t), \tag{2.2}$$

where $L_g : G \to G$ is left multiplication by $g \in G$, namely $L_g(h) = gh$, and $(dL_g)_h : T_hG \to T_{gh}G$ is the derivative of L_g at $h \in G$. As noted in [7], allowing for the opposite sign convention to [6] for the Riemannian curvature R, [6, Theorem 21.3] gives the following result.

LEMMA 2.1. *Let $x : I \to G$ be a C^∞ curve, with $V : I \to \mathcal{G}$ defined by* (2.2). *Then, for all $t \in I$,*

(i) $(dL_{x(t)^{-1}})_{x(t)} \nabla_{d/dt} \dot{x} = \dot{V}(t)$,
(ii) $(dL_{x(t)^{-1}})_{x(t)} \nabla^2_{d/dt} \dot{x} = \ddot{V}(t) + 1/2[V(t), \dot{V}(t)]$,
(iii) $(dL_{x(t)^{-1}})_{x(t)} \nabla^3_{d/dt} \dot{x} = \frac{d^3V}{dt^3}(t) + [V(t), \ddot{V}(t)] + 1/4\big[V(t), [V(t), \dot{V}(t)]\big]$,
(iv) $(dL_{x(t)^{-1}})_{x(t)} R(\nabla_{d/dt} \dot{x}, \dot{x})\dot{x} = -1/4\big[V(t), [V(t), \dot{V}(t)]\big]$.

We now obtain the following characterisation of elastic curves in G.

THEOREM 2.2. *A C^∞ curve $x : I \to G$ is an elastic curve if and only if the curve $V : I \to \mathcal{G}$ defined by* (2.2) *satisfies*

$$\ddot{V}(t) = \big[\dot{V}(t), V(t)\big] - \left(\big\|\dot{V}(t)\big\|^2 + \langle V(t), C \rangle\right) V(t) + C, \tag{2.3}$$

$$\|V(t)\|^2 = 1, \tag{2.4}$$

for some constant $C \in \mathcal{G}$ and all $t \in I$.

PROOF. First suppose x is an elastic curve. Since x satisfies (1.1), left-invariance of the Riemannian metric gives (2.4):

$$1 = \|\dot{x}(t)\|^2 = \left\|\left(dL_{x(t)^{-1}}\right)_{x(t)} \dot{x}(t)\right\|^2 = \|V(t)\|^2 . \tag{2.5}$$

By Theorem 1.2, x satisfies (1.3) for some $\tilde{b}$. Applying $(dL_{x(t)^{-1}})_{x(t)}$ to (1.3) and using (2.2), Lemma 2.1(i,iii,iv) and left-invariance, we find

$$\frac{d^3V}{dt^3} = [\ddot{V}, V] - \frac{d}{dt}\left(\left(\frac{3}{2}\|\dot{V}(t)\|^2 + \tilde{b}\right)V\right).$$

Integrating once, we have, for some constant $C \in \mathscr{G}$,

$$\ddot{V} = [\dot{V}, V] - \left(\frac{3}{2}\|\dot{V}\|^2 + \tilde{b}\right)V + C. \tag{2.6}$$

It remains to write (2.6) in the form (2.3). First note that, by (2.4), the first two derivatives of $\|\dot{V}\|^2$ vanish. So

$$\langle \dot{V}, V\rangle = 0, \tag{2.7}$$

$$\langle \ddot{V}, \dot{V}\rangle + \|\dot{V}\|^2 = 0. \tag{2.8}$$

Taking the inner product of (2.6) with $\dot{V}$ and applying (2.1) and (2.7), we have $\langle \ddot{V}, \dot{V}\rangle = \langle \dot{V}, C\rangle$. So for some constant $b \in \mathbb{R}$,

$$\|\dot{V}\|^2 = 2\langle V, C\rangle + b. \tag{2.9}$$

Similarly, taking the inner product of (2.6) with V gives

$$\langle \ddot{V}, V\rangle = -\frac{3}{2}\|\dot{V}\|^2 - \tilde{b} + \langle V, C\rangle .$$

Therefore, and by (2.8) and (2.9), we have $\tilde{b} = -b/2$ and thus

$$\frac{3}{2}\|\dot{V}(t)\|^2 + \tilde{b} = \|\dot{V}(t)\|^2 + \langle V(t), C\rangle . \tag{2.10}$$

Substitution into (2.6) gives (2.3), as required. Now suppose V satisfies (2.3) and (2.4). Then (1.1) holds, by (2.5), and it remains to show that x satisfies (1.2) for some λ. Writing $\lambda = \|\dot{V}\|^2 + \langle V, C\rangle$ and differentiating (2.3) gives

$$\frac{d^3V}{dt^3} = [\ddot{V}, V] - \frac{d}{dt}(\lambda V). \tag{2.11}$$

Applying (2.2) and Lemma 2.1(i,iii,iv), we have, for all $t \in I$,

$$\left(dL_{x(t)^{-1}}\right)_{x(t)}\left(\nabla^3_{d/dt}\dot{x} + R\left(\nabla_{d/dt}\dot{x}, \dot{x}\right)\dot{x} + \nabla_{d/dt}(\lambda\dot{x})\right) = \mathbf{0}.$$

Since $(dL_{x(t)^{-1}})_{x(t)}$ is an isomorphism, x satisfies (1.2), as claimed. $\square$

By analogy with the *Lie quadratics* of [7–9] and [11] (unconstrained solutions of $\ddot{V} = [\dot{V}, V] + C$), a curve $V : I \to \mathcal{G}$ satisfying (2.3) and (2.4) for some $C \in \mathcal{G}$ and all $t \in I$ will be called an *elastic Lie quadratic* with *constant* C. If $x : I \to G$ is an elastic curve, the curve V defined by (2.2) is an elastic Lie quadratic; it will be called the elastic Lie quadratic *associated* with x. For later reference, we re-state the observation (2.9) from the preceding proof.

COROLLARY 2.3. *Let $V : I \to \mathcal{G}$ be an elastic Lie quadratic. Then for some constant $b \in \mathbb{R}$ and all $t \in I$,*

$$b = \left\| \dot{V}(t) \right\|^2 - 2 \langle V(t), C \rangle .$$

COROLLARY 2.4. *Let $V : I \to \mathcal{G}$ be an elastic Lie quadratic and define $W : I \to \mathcal{G}$ by $W(t) = \ddot{V}(t) + \big(||\dot{V}(t)||^2 + \langle V(t), C\rangle\big) V(t)$. Then*

$$\dot{W}(t) = [W(t), V(t)], \tag{2.12}$$

for all $t \in I$, and $||W(t)||$ is constant.

PROOF. As noted in the proof of Theorem 2.2, differentiating (2.3) gives (2.11) (with $\lambda = ||\dot{V}||^2 + \langle V, C\rangle$), which can be written in the form (2.12). Therefore, and by (2.1), $d||W||^2/dt = 2\langle [W, V], W\rangle = 0$. □

Differential equations of the form (2.12) are called *Lax equations*. They are important in the theory of integrable systems, since if a matrix differential equation can be written in the form $\dot{W} = [W, V]$ then the spectrum of W is preserved by the flow (see [2]). In the present situation, the Lax equation (2.12) is crucial to the solution of (2.2), or equivalently

$$\dot{x}(t) = \left(dL_{x(t)}\right)_e V(t), \tag{2.13}$$

for an elastic curve x in terms of its elastic Lie quadratic V. In section 6, we construct this solution in the case $G = SO(3)$, using results of our related paper [12], in which (2.13) is solved in both $SO(3)$ and $SO(1, 2)$ subject to an *arbitrary* constraint of the form $\dot{W} = [W, V]$. Solutions of (2.13) subject to $\dot{W} = [W, V]$ in arbitrary semisimple Lie groups have been developed in [10].

3. Compact G: Extendibility of Elastica to $\mathbb{R}$

In this section we prove the following result.

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THEOREM 3.1. *Suppose G is compact. Then for any $\tilde{b}, t_0 \in \mathbb{R}$, $x_0 \in G$, and $v_0, v_1, v_2 \in T_{x_0}G$ satisfying*

$$\|v_0\|^2 = 1, \quad \langle v_1, v_0\rangle = 0, \quad \langle v_2, v_0\rangle + \|v_1\|^2 = 0, \tag{3.1}$$

there exists a unique solution $x : \mathbb{R} \to G$ of (1.3) *satisfying*

$$x(t_0) = x_0, \quad \dot{x}(t_0) = v_0, \quad \nabla_{d/dt}\dot{x}\big|_{t=t_0} = v_1, \quad \nabla^2_{d/dt}\dot{x}\big|_{t=t_0} = v_2. \tag{3.2}$$

The proof is by means of two lemmas, the first of which applies to any G.

LEMMA 3.2. *Given $\tilde{b}$, $t_0 \in \mathbb{R}$, $x_0 \in G$ and $v_0, v_1, v_2 \in T_{x_0}G$ satisfying* (3.1), *there exists a constant $K \geq 0$ such that, for any open interval I containing t_0 and any solution $x : I \to G$ of* (1.3) *satisfying* (3.2), *we have*

$$\left\|\nabla_{d/dt}\dot{x}\right\|, \left\|\nabla^2_{d/dt}\dot{x}\right\| \leq K, \tag{3.3}$$

for all $t \in I$.

PROOF. Let I be an open interval containing t_0 and $x : I \to G$ a solution of (1.3) satisfying (3.2). By Theorem 1.2, x is an elastic curve. Let V be the associated elastic Lie quadratic. By Corollary 2.3 and (2.4), we have, for some constants $b \in \mathbb{R}$ and $C \in \mathscr{G}$,

$$\|\dot{V}\| = \sqrt{2\langle V, C\rangle + b} \leq \sqrt{2\|C\| + |b|}. \tag{3.4}$$

So by Lemma 2.1(i) and left-invariance of the Riemannian metric,

$$\left\|\nabla_{d/dt}\dot{x}\right\| \leq \sqrt{2||C|| + |b|}.$$

By Corollary 2.4, $||\ddot{V} + \big(||\dot{V}||^2 + \langle V, C\rangle\big)V|| = \sigma$ for some constant $\sigma \geq 0$. In particular, $\sigma \geq ||\ddot{V}|| - ||\big(||\dot{V}||^2 + \langle V, C\rangle\big)V||$. So by (2.4) and (3.4),

$$\left\|\ddot{V}\right\| \leq \sigma + \left\|\dot{V}\right\|^2 + ||C|| \leq \sigma + |b| + 3||C||. \tag{3.5}$$

By (2.3), (2.4), (3.4) and (3.5),

$$\left\|[V, \dot{V}]\right\| \leq \left\|\ddot{V}\right\| + \left\|\dot{V}\right\|^2 + |\langle V, C\rangle| + ||C|| \leq \sigma + 2|b| + 7||C||.$$

Therefore, and by Lemma 2.1(ii), left-invariance and (3.5), we have

$$\left\|\nabla^2_{d/dt}\dot{x}\right\| = \left\|\ddot{V} + \frac{1}{2}[V, \dot{V}]\right\| \leq \left\|\ddot{V}\right\| + \frac{1}{2}\left\|[V, \dot{V}]\right\| \leq \frac{1}{2}(3\sigma + 4|b| + 13||C||).$$

Setting $K = \max\{\sqrt{2||C|| + |b|}, 1/2(3\sigma + 4|b| + 13||C||)\}$, (3.3) holds on I. It remains to show that K depends only on $\tilde{b}$, x_0, v_0, v_1 and v_2, so that the same bounds (3.3) hold for *any* solution $x : I \to G$ of (1.3), (3.2) defined on *any* open interval $I \ni t_0$. By (2.2) and Lemma 2.1, $V(t_0)$, $\dot{V}(t_0)$ and $\ddot{V}(t_0)$ depend only on x_0, v_0, v_1 and v_2. Therefore, and by (2.3) and (2.10) (which holds here, as in the proof of Theorem 2.2), C depends only on $\tilde{b}$, x_0, v_0, v_1 and v_2. Similarly, b, σ and therefore K depend only on $\tilde{b}$, x_0, v_0, v_1 and v_2. □

LEMMA 3.3. *Suppose G is compact. Then for any $l \geq 0$, there exists $\delta_l > 0$ such that given any $\tilde{b}, t_0 \in \mathbb{R}$, $x_0 \in G$ and $v_0, v_1, v_2 \in T_{x_0}G$ satisfying* (3.1) *and $||v_1||, ||v_2|| \leq l$, there exists a unique solution $x : (t_0 - \delta_l, t_0 + \delta_l) \to G$ of* (1.3) *satisfying* (3.2).

PROOF. Picard's theorem on local unique solvability of ordinary differential equations almost asserts this, but with δ_l depending on x_0, v_0, v_1 and v_2. But x_0 and v_0 lie in compact sets, so restricting v_1 and v_2 to also lie in a compact set permits a uniform choice of δ_l. □

PROOF OF THEOREM 3.1. By Picard's theorem, for some $\delta > 0$, there exists a unique solution $x : I \to G$, where $I = (t_0 - \delta, t_0 + \delta)$, of (1.3) satisfying (3.2). Let K be given by Lemma 3.2, so that (3.3) holds on I, and let δ_K be given by Lemma 3.3. Then, taking $\epsilon > 0$ with $\epsilon < \delta, \delta_K$ and setting $t_0^+ := t_0 + \delta - \epsilon$, Lemma 3.3 says that x can be extended uniquely to the interval $(t_0 - \delta, t_0^+ + \delta_K)$, which contains I. By Lemma 3.2, this extension also satisfies (3.3). It follows that x can be extended uniquely to $(t_0 - \delta, \infty)$. Similarly, x can be extended uniquely to $(-\infty, t_0 + \delta)$ and thus to $\mathbb{R}$. □

By Theorems 1.2 and 3.1, when G is compact all elastic curves in G extend uniquely to $\mathbb{R}$. So there is no loss of generality in restricting our attention to elastic curves defined on the whole real line. The rest of the paper constructs the elastic curves $x : \mathbb{R} \to SO(3)$ in terms of quadratures.

4. $G = SO(3)$: Solution for $\langle V(t), C\rangle$

From now on we take $G = SO(3)$. Then $\mathcal{G} = so(3)$, the set of all skew-symmetric real 3×3 matrices. Recall that E^3 is a Lie algebra with Lie bracket the cross product $\times$. The map $B : E^3 \to so(3)$ defined by $B(v)w = v \times w$ is a Lie algebra isomorphism. Since the Euclidean inner product is, up to a positive multiple, the unique inner product on $\mathbb{R}^3$ satisfying (2.1), we can assume without loss of generality that B is an isometry.

Let us also denote the Euclidean inner product and corresponding norm on E^3 by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$. For later reference, recall that, for all $w_1, w_2, w_3 \in E^3$,

$$(w_1 \times w_2) \times w_3 = \langle w_1, w_3\rangle\, w_2 - \langle w_2, w_3\rangle\, w_1. \tag{4.1}$$

Let $x : \mathbb{R} \to SO(3)$ be an elastic curve and $\tilde{V} : \mathbb{R} \to so(3)$ the associated elastic Lie quadratic. Let $\tilde{C}$ be the constant of $\tilde{V}$. Now define $V = B^{-1}(\tilde{V}) : \mathbb{R} \to E^3$ and $C = B^{-1}(\tilde{C})$. This change of notation is made for convenience. Since B is a Lie algebra isomorphism and isometry, V satisfies (2.4) and

$$\ddot{V}(t) = \dot{V}(t) \times V(t) - \left(\left\|\dot{V}(t)\right\|^2 + \langle V(t), C\rangle\right) V(t) + C, \tag{4.2}$$

for all $t \in \mathbb{R}$. That is, V is an elastic Lie quadratic, with constant C, in the Lie algebra $(E^3, \times)$. We work with V rather than $\tilde{V}$, solving (4.2) with (2.4). The following result is easily verified.

LEMMA 4.1. *For any $A \in SO(3)$ and any $t_0 \in \mathbb{R}$,*

(i) *$t \mapsto A(V(t))$ is an elastic Lie quadratic in E^3 with constant $A(C)$,*
(ii) *$t \mapsto V(t - t_0)$ is an elastic Lie quadratic in E^3 with constant C.*

So we can assume without loss of generality that

$$C = \begin{bmatrix} 0 & 0 & c \end{bmatrix}^T \quad \text{for some } c \in \mathbb{R},\ V_1(0) = 0, \tag{4.3}$$

where, here and throughout, we write $V(t) = [V_1(t) \quad V_2(t) \quad V_3(t)]^T$.

EXAMPLE 1. Suppose $C = [0 \quad 0 \quad c]^T$ and, for some $h \in \mathbb{R}$, $V(h) = [0 \quad 0 \quad 1]^T$ and $\dot{V}(h) = \mathbf{0}$. Set $V_0(t) = [0 \quad 0 \quad 1]^T$ for all $t \in \mathbb{R}$. Then $V_0(t)$ satisfies (4.2) for all $t \in \mathbb{R}$, with $\|\dot{V}_0(t)\|^2 + \langle V_0(t), C\rangle = c$, $V_0(h) = V(h)$ and $\dot{V}_0(h) = \dot{V}(h)$. So $V(t) = V_0(t)$ for all t sufficiently near h, by local uniqueness in Picard's theorem. Since neither (4.2) nor the values of $V_0(h)$ and $\dot{V}_0(h)$ are affected by the choice of h, $V(t) = V_0(t) = [0 \quad 0 \quad 1]^T$ for all $t \in \mathbb{R}$. Similarly, $V(t)$ is constant if $V(h) = [0 \quad 0 \quad -1]^T$ and $\dot{V}(h) = \mathbf{0}$.

When $C = \mathbf{0}$, V is said to be *null*. Null Lie quadratics in E^3 (solutions of $\ddot{V} = \dot{V} \times V$) have been studied by the second author in [7]. Null *elastic* Lie quadratics in E^3 admit a much simpler, closed form description.

PROPOSITION 4.2. *Suppose V is null and satisfies* (4.3). *Then*

$$V(t) = \begin{bmatrix} a\sin(\omega t) & a\cos(\omega t) & \sqrt{1-a^2} \end{bmatrix}^T$$

for all $t \in \mathbb{R}$, where $a = \sqrt{b/(b+1)}$ and $\omega = 1/\sqrt{1-a^2}$.

Here, as before, b is the constant of Corollary 2.3. For the rest of this section and the next, assume $C \neq \mathbf{0}$, so that $c \neq 0$ in (4.3). Set

$$y(t) = \langle V(t), C \rangle ,$$

for all $t \in \mathbb{R}$. By (2.4), $|y(t)| \leq ||C||$, and by (4.3), $V_3(t) = y(t)/c$. If y is known, (4.2) reduces to the following system of *linear* differential equations:

$$\ddot{V}_1(t) = \dot{V}_2(t)\frac{y(t)}{c} - V_2(t)\frac{\dot{y}(t)}{c} - (3y(t) + b)V_1(t), \tag{4.4}$$

$$\ddot{V}_2(t) = -\dot{V}_1(t)\frac{y(t)}{c} + V_1(t)\frac{\dot{y}(t)}{c} - (3y(t) + b)V_2(t). \tag{4.5}$$

We now show that y satisfies a differential equation from the theory of elliptic functions. First, we have an additional integral for elastica in $SO(3)$.

LEMMA 4.3. *$\langle \dot{V}(t) \times V(t), C \rangle + y(t)$ is constant.*

PROOF. Take the cross product of (4.2) with V. Then by (2.4), which also implies $\langle \dot{V}, V \rangle = 0$, and (4.1), we have $\ddot{V} \times V + \dot{V} = C \times V$. So

$$\frac{d}{dt}\left\langle \dot{V} \times V + V, C \right\rangle = \langle C \times V, C \rangle = 0. \qquad \square$$

Now define two constants (note that the second is just $b + 1$):

$$\begin{aligned} k_1 &= \left\langle \dot{V}(t) \times V(t), C \right\rangle + y(t) + ||C||^2, \\ k_2 &= \left\| \dot{V}(t) \right\|^2 - 2y(t) + 1. \end{aligned} \tag{4.6}$$

THEOREM 4.4. *The function $y : \mathbb{R} \to [-||C||, ||C||]$ satisfies*

$$\dot{y}(t)^2 = 2k_3 + 2k_1 y(t) - k_2 y(t)^2 - 2y(t)^3, \tag{4.7}$$

for some constant $k_3 \in \mathbb{R}$ and all $t \in \mathbb{R}$.

PROOF. Recall that $||\dot{V}||^2 + y = 3y + k_2 - 1$ and take the inner product of (4.2) with C, giving $\ddot{y} + (3y + k_2)y = k_1$. Now integrate to get (4.7). $\square$

By left-invariance of the Riemannian metric, Lemma 2.1(i) and since B is an isometry, $||\dot{V}||^2$ is the squared geodesic curvature of the elastic curve x:

$$\left\| \nabla_{d/dt}\dot{x} \right\|^2 = \left\| \left(dL_{x(t)^{-1}}\right)_{x(t)} \ddot{x}(t) \right\|^2 = \left\| \dot{\tilde{V}} \right\|^2 = \left\| B(\dot{V}) \right\|^2 = \left\| \dot{V} \right\|^2 .$$

By Corollary 2.3 and Theorem 4.4, $||\dot{V}||^2$ also satisfies a differential equation of the form (4.7). So we recover Jurdjevic's prior result for the squared geodesic curvature of elastica in S^3 [5, Chapter 14, Theorem 3]. If y is constant then so is $||\dot{V}||^2$, and V admits a closed form description.

PROPOSITION 4.5. *Suppose V satisfies* (4.3) *and $y(t)$ is constant. Then*

$$V(t) = \left[a \sin(\omega t) \quad a \cos(\omega t) \quad V_3(0)\right]^T, \tag{4.8}$$

for all $t \in \mathbb{R}$, where $a = \sqrt{1 - V_3(0)^2}$ and $\omega = ||\dot{V}(0)||/a$. □

To find y when y is non-constant, we first consider the possible roots of the cubic polynomial in y given by the right hand side of (4.7). Define $p : [-||C||, ||C||] \to \mathbb{R}$ by $p(z) = 2k_3 + 2k_1 z - k_2 z^2 - 2z^3$.

LEMMA 4.6. *If $y(t)$ is non-constant then p has three real roots.*

PROOF. Suppose p has only one real root. Then by (4.7), $\dot{y}(t) = 0$ for at most one value of $y(t)$. Since $\dot{y}(t)$ is not identically 0, $\dot{y}(t) = 0$ for at most one value of t, say $\dot{y}(t_*) = 0$. Without loss of generality, we can assume $\dot{y}(t) < 0$ for all $t > t_*$. Therefore, and since $|y(t)| \le ||C||$ for all $t \in \mathbb{R}$, the limit $L = \lim_{t\to\infty} y(t)$ exists. We claim that for any $m \in \mathbb{Z}^+$, there exists t_m such that $-1/m \le \dot{y}(t) \le 0$ for all $t > t_m$. If not, there exists $\epsilon > 0$ with $\dot{y}(t) < -\epsilon$ for all $t \in \mathbb{R}$. But then the Mean Value Theorem gives $y(n+1) < y(n) - \epsilon$ for all $n \in \mathbb{Z}^+$, which is a contradiction since $y(t)$ is bounded below. So the claim is true, and thus $\lim_{m\to\infty} \dot{y}(t_m) = 0$. Taking $t = t_m$ in (4.7) and letting $m \to \infty$, we have $0 = p(\lim_{m\to\infty} y(t_m))$. So $L = \lim_{t\to\infty} y(t) = \lim_{m\to\infty} y(t_m)$ is a root of p. But $L \ne y(t_*)$ since $\dot{y}(t) < 0$ for all $t > t_*$. This contradicts our assumption, so p must have three real roots. □

We now use Theorem 4.4 to find y. Denote the roots of p by $\gamma \le \beta \le \alpha$. We consider the four possible cases for these inequalities separately.

PROPOSITION 4.7. *If $\gamma = \beta = \alpha$ or $\gamma < \beta = \alpha$ then $y(t)$ is constant.*

PROOF. If $\gamma = \beta = \alpha$ then Lemma 4.6 applies. If $\gamma < \beta = \alpha$ then (4.7) reads $\dot{y}^2 = -2(y+\alpha)^2(y-\gamma)$. So $y(t) \le \gamma$ for all $t \in \mathbb{R}$, and thus $\dot{y}(t) = 0$ for at most one value of $y(t)$. Now the same argument as in the proof of Lemma 4.6 shows that $y(t)$ is constant. □

EXAMPLE 2. Suppose $V(0) = [1 \;\; 0 \;\; 0]^T$, $\dot{V}(0) = [0 \;\; 1 \;\; 0]^T$ and $C = [0 \;\; 0 \;\; 1]^T$. Then $k_2 = 2$, $k_1 = k_3 = 0$ and thus $\dot{y}(t)^2 = -2y(t)^2(y(t)+1)$ for all $t \in \mathbb{R}$. So $y(t) = 0$ and thus $V(t) = [\cos(t) \quad \sin(t) \quad 0]^T$ for all $t \in \mathbb{R}$.

PROPOSITION 4.8. *Suppose $\gamma = \beta < \alpha$. If $y(0) = \beta$ then $y(t)$ is constant. Otherwise, for some real constant k and all $t \in \mathbb{R}$,*

$$y(t) = (\beta - \alpha)\tanh^2\left(t\sqrt{\frac{\alpha-\beta}{2}} + k\right) + \alpha. \tag{4.9}$$

In this case, $\lim_{t\to\infty} V_3(t) = \beta/c$ *and* $\lim_{t\to\infty} ||\dot{V}(t)||^2 = 2\beta + b$.

PROOF. Equation (4.7) reads

$$\dot{y}(t)^2 = -2(y(t) - \alpha)(y(t) - \beta)^2. \tag{4.10}$$

If $y(0) = \beta$ then $y(t)$ is constant. Otherwise, integrating (4.10) shows that $y(t)$ is one of

$$y_{\pm}(t) = (\beta - \alpha)\tanh^2\left(\pm t\sqrt{\frac{\alpha - \beta}{2}} + k\right) + \alpha,$$

where $(y(0) - \alpha)/(\beta - \alpha) = \tanh^2(k)$. We can choose $y = y_+$ and k such that $\dot{y}_+(0) = \dot{y}(0)$. Since $y(t) \le \alpha$, by (4.10), k is real. So $y(t)$ can be written in the form (4.9). The limits follow from (4.3) and Corollary 2.3. $\square$

So if $y(0) = \beta$ then V is given by (4.8). If $\beta < y(0) < \alpha$ then as $t \to \infty$, $V(t)$ converges either to a constant (if $2\beta + b = 0$) or (otherwise) to the curve given by (4.8) with $a = \sqrt{1 - (\beta/c)^2}$ and $\omega = \sqrt{2\beta + b}/a$.

EXAMPLE 3. Suppose $V(0) = [1 \ \ 0 \ \ 0]^T$, $\dot{V}(0) = [0 \ \ 1 \ \ 0]^T$ and $C = [0 \ \ 0 \ \ 1/2]^T$. Then $k_1 = -1/4$, $k_2 = 2$ and $k_3 = 0$, giving $\gamma = \beta = -1/2$ and $\alpha = 0$. So $\lim_{t\to\infty} y(t) = -1/2$. Also $b = ||\dot{V}(0)||^2 - 2y(0) = 1$. That is, $2\beta + b = 0$. So $\lim_{t\to\infty} V(t) = [0 \ \ 0 \ \ -1]^T$, since $||V(t)|| = 1$.

EXAMPLE 4. Suppose $V(0) = [0 \ \ 0 \ \ 1]^T$, $C = [0 \ \ 0 \ \ c]^T \neq \mathbf{0}$ and $\gamma = \beta < \alpha$. Then $\dot{y}(0) = 0$ since $\langle \dot{V}, V\rangle = 0$. So $y(0) \in \{\alpha, \beta\}$. If $c < 0$ then $y(0) = \beta$ and $y(t)$ is constant by Proposition 4.8. We claim that $c > 0$ contradicts the assumption $\gamma = \beta < \alpha$. After evaluating the k_i, (4.7) reads

$$\dot{y}(t)^2 = c^2\left(\|\dot{V}(0)\|^2 - 2c - 1\right) + 2\left(c + c^2\right)y(t) - \left(\|\dot{V}(0)\|^2 - 2c + 1\right)y(t)^2 - 2y(t)^3.$$

But $y(0) = \alpha$ since $c > 0$. So $\dot{y}(t)^2 = -2(y(t) - c)(y(t) - \beta)^2$. Comparing coefficients, $\beta^2 = c\big((||\dot{V}(0)||^2 + 1)/2 - c - 1\big)$ and $2\beta = -(||\dot{V}(0)||^2 + 1)/2$. So $\big((||\dot{V}(0)||^2 + 1)/2 - 2c\big)^2 + 4c = 0$, contradicting $c > 0$. Similarly, if $V(0) = [0 \ \ 0 \ \ -1]^T$ and $\gamma = \beta < \alpha$ then $c > 0$ and $y(t)$ is constant.

When $\gamma < \beta < \alpha$, y can be expressed in terms of the Jacobi elliptic function $\mathrm{sn}(u|m)$, where $u \in \mathbb{R}$ and $0 < m < 1$ (see [1] for the definition).

PROPOSITION 4.9. *Suppose $\gamma < \beta < \alpha$. Then*

$$y(t) = (\beta - \alpha)\ \mathrm{sn}^2\left(t\sqrt{\frac{\alpha - \gamma}{2}} + k \,\middle|\, m\right) + \alpha, \tag{4.11}$$

for all $t \in \mathbb{R}$, where $m = (\alpha - \beta)/(\alpha - \gamma)$ and k is a real constant.

PROOF. Equation (4.7) reads

$$\dot{y}(t)^2 = -2(y(t)-\alpha)(y(t)-\beta)(y(t)-\gamma).$$

Since $\gamma < \beta < \alpha$, we have $0 < m < 1$. So, by integrating, we find that $y(t)$ is one of

$$y_\pm(t) = (\beta-\alpha)\,\mathrm{sn}^2\left(\pm t\sqrt{\frac{\alpha-\gamma}{2}} + k \,\middle|\, m\right) + \alpha,$$

where $(y(0)-\alpha)/(\beta-\alpha) = \mathrm{sn}^2(k|m)$. We can again choose $y = y_+$ and k such that $\dot{y}_+(0) = \dot{y}(0)$. Since $\gamma < \beta \le y(0) < \alpha$, k is real. $\square$

If y is known, V can be found, as shown in the next section.

5. $G = SO(3)$: Solution for $V(t)$

When $y(t)$ is constant, V is given by (4.8). We now assume $y(t)$ is non-constant, continue with the assumption $C = [0 \quad 0 \quad c]^T \neq \mathbf{0}$, but drop the assumption that $V_1(0) = 0$. As before, for all $t \in \mathbb{R}$,

$$V(t) = \left[V_1(t) \quad V_2(t) \quad \tfrac{y(t)}{c}\right]^T,$$

with $y(t)$ given by either (4.9) or (4.11). In this section we solve the system (4.4), (4.5) by quadratures for V_1 and V_2. Let $\|\cdot\|$ denote the Euclidean norm on either $\mathbb{R}^2$ or $\mathbb{R}^3$, as necessary. Set

$$Z(t) = \left[V_1(t) \quad V_2(t)\right]^T,$$

$$r(t) = \sqrt{1 - \frac{y(t)^2}{c^2}}, \tag{5.1}$$

for all $t \in \mathbb{R}$, so that $\|Z(t)\|^2 = r(t)^2$, by (2.4).

First, suppose $I \subseteq \mathbb{R}$ is an open interval with $r(t) \neq 0$ for all $t \in I$. If we choose $t_0 \in I$ and $\theta_0 \in \mathbb{R}$ such that $Z(t_0) = r(t_0)[\cos\theta_0 \quad \sin\theta_0]^T$ then, since $r(t) \neq 0$ on I, there exists a unique C^∞ map $\theta : I \to \mathbb{R}$ satisfying $\theta(t_0) = \theta_0$ and $Z(t) = r(t)[\cos\theta(t) \quad \sin\theta(t)]^T$, for all $t \in I$. By (4.6),

$$r(t)^2\dot{\theta}(t) = V_1(t)\dot{V}_2(t) - \dot{V}_1(t)V_2(t) = \frac{y(t)-k_1}{c} + c, \tag{5.2}$$

and thus

$$\theta(t) = \theta_0 + \int_{t_0}^{t} \frac{1}{r(t)^2}\left(\frac{y(t)-k_1}{c} + c\right)dt,$$

for all $t \in I$. In particular, if the set

$$\zeta(r) = \{t \in \mathbb{R} : r(t) = 0\}$$

is empty then we can solve (4.4), (4.5) on the whole real line in this fashion. If $\zeta(r)$ is non-empty we also need another method, described after the following two lemmas.

LEMMA 5.1. *Suppose $\zeta(r)$ is non-empty. Then*

(i) *if $\dot{Z}(h) = \mathbf{0}$ for some $h \in \mathbb{R}$ then $V(t)$ is constant,*
(ii) *if $\gamma = \beta < \alpha$ then $y(t)$ is constant.*

PROOF. Choose $t_* \in \zeta(r)$. Then $V(t_*) = [0 \quad 0 \quad \pm 1]^T$. First suppose $V(t_*) = [0 \quad 0 \quad 1]^T$. Then taking $t = t_*$ in (4.6) gives

$$k_1 = c + c^2. \tag{5.3}$$

Since $\langle \dot{V}, V \rangle = 0$, $y(h)\dot{y}(h) = 0$. If $y(h) = 0$ then $k_1 = c^2$ by (4.6), contradicting $c \neq 0$. So $\dot{y}(h) = 0$ and thus $k_1 = y(h) + c^2$. So by (5.3), $y(h) = c$ and thus $Z(h) = 0$. Therefore, $V(t)$ is constant by Example 1 and Lemma 4.1(ii). Similarly, $V(t)$ is constant if $V(t_*) = [0 \quad 0 \quad -1]^T$, completing the proof of (i). For (ii), if $\gamma = \beta < \alpha$ then $y(t)$ is constant by Example 4 and Lemma 4.1(ii). □

Now let $\kappa : \mathbb{R} \to [0, \infty)$ be the curvature of Z, regarded as a function of t (not of arc length).

LEMMA 5.2. *If $V(t)$ is non-constant and $\zeta(r)$ is non-empty then, for all $t \in \mathbb{R}$,*

$$\kappa(t) = \frac{\left| -\frac{y(t)^2}{c} + \left(\frac{k1}{c} - c\right)(3y(t) + b) \right|}{\left\|\dot{Z}(t)\right\|^3}. \tag{5.4}$$

PROOF. Since $\dot{Z}(t) \neq \mathbf{0}$, by Lemma 5.1(i), we have

$$\kappa(t) = \frac{\left|\ddot{V}_1(t)\dot{V}_2(t) - \dot{V}_1(t)\ddot{V}_2(t)\right|}{\left\|\dot{Z}(t)\right\|^3},$$

for all $t \in \mathbb{R}$. By (4.4) and (4.5),

$$\ddot{V}_1\dot{V}_2 - \dot{V}_1\ddot{V}_2 = \left\|\dot{Z}\right\|^2 \frac{y}{c} - \left(\dot{V}_1 V_1 + \dot{V}_2 V_2\right)\frac{\dot{y}}{c} + (3y + b)\left(\dot{V}_1 V_2 - V_1 \dot{V}_2\right).$$

By Corollary 2.3, $||\dot{Z}||^2 = 2y + b - \dot{y}^2/c^2$. Since $\langle \dot{V}, V \rangle = 0$ we have

$$\dot{V}_1 V_1 + \dot{V}_2 V_2 = -\frac{\dot{y}y}{c^2}.$$

Together with (5.2), these give (5.4). □

Now assume $\zeta(r)$ is non-empty. Suppose $I \subseteq \mathbb{R}$ is an open interval with $\kappa(t) \neq 0$ for all $t \in I$. Choose $t_0 \in I$ and define $\tilde{s} : I \to [0, \infty)$ to be the arc length of Z, that is, set

$$\tilde{s}(t) = \int_{t_0}^{t} \|\dot{Z}(\xi)\| \, d\xi.$$

Reparameterise Z by arc length and let $'$ denote differentiation with respect to arc length. By Lemma 5.1(i), $\dot{Z}(t) \neq \mathbf{0}$ for all $t \in I$ and thus $||Z'(s)|| = 1$ for all $s \in \tilde{s}(I)$. Define $\tilde{\kappa} : \tilde{s}(I) \to [0, \infty)$ by $\tilde{\kappa} \circ \tilde{s} = \kappa$. Then $\tilde{\kappa}(s) \neq 0$ for all $s \in \tilde{s}(I)$. So the (planar) Serret-Frenet frame of Z, namely

$$T(s) = Z'(s), \qquad N(s) = \frac{T'(s)}{\|T'(s)\|},$$

is defined on $\tilde{s}(I)$, and $\tilde{\kappa}(s) = ||T'(s)||$ by definition. Therefore, the (planar) Serret-Frenet equations

$$T'(s) = \tilde{\kappa}(s)N(s), \qquad N'(s) = -\tilde{\kappa}(s)T(s) \tag{5.5}$$

hold for all $s \in \tilde{s}(I)$. Write $s_0 = s(t_0)$, $T(s) = [T_1(s) \;\; T_2(s)]^T$ and $N(s) = [N_1(s) \;\; N_2(s)]^T$. Then the Serret-Frenet equations have solution

$$T_i(s) = a_i \cos(\varphi(s)) + b_i \sin(\varphi(s)), \qquad N_i(s) = b_i \cos(\varphi(s)) - a_i \sin(\varphi(s)),$$

for all $s \in \tilde{s}(I)$, where $i = 1, 2$, the a_i, b_i are constants, and

$$\varphi(s) = \int_{s_0}^{s} \tilde{\kappa}(\xi) \, d\xi.$$

Then, for all $s \in \tilde{s}(I)$,

$$Z(s) = \int_{s_0}^{s} T(\xi) \, d\xi.$$

The following result guarantees that, given any $t_0 \in \mathbb{R}$, one of the above methods may be used to solve (4.4), (4.5) near $t = t_0$.

LEMMA 5.3. *Suppose $y(t)$ is non-constant and $\zeta(r)$ non-empty. Then there exists $\delta > 0$ such that given any $t_0 \in \mathbb{R}$, either*

$$r(t) \neq 0 \quad \textit{for all } t \in (t_0 - \delta, t_0 + \delta), \qquad \textit{or} \quad \kappa(t) \neq 0 \quad \textit{for all } t \in (t_0 - \delta, t_0 + \delta).$$

PROOF. Since $y(t)$ is non-constant, either $\gamma = \beta < \alpha$ or $\gamma < \beta < \alpha$, as noted in Section 3. By Lemma 5.1(ii), we must have $\gamma < \beta < \alpha$. So $y(t)$ has the form (4.11), and is therefore periodic [1]. Thus for any value η in the range of y, the set $\{t \in \mathbb{R} | y(t) = \eta\}$ is discrete and has no accumulation points in $\mathbb{R}$. Write $\zeta(\kappa) = \{t \in \mathbb{R} | \kappa(t) = 0\}$. We claim that the (Euclidean) distance

$$D(\zeta(r), \zeta(\kappa)) = \inf\{|t_1 - t_2| : t_1 \in \zeta(r),\ t_2 \in \zeta(\kappa)\}$$

between $\zeta(r)$ and $\zeta(\kappa)$ is nonzero. By (5.1) and (5.4), $\zeta(r)$ and $\zeta(\kappa)$ are both discrete with no accumulation points in $\mathbb{R}$. It remains to check that they have empty intersection. Let $t_* \in \zeta(r)$. Then $V(t_*) = [0 \quad 0 \quad \pm 1]^T$. Suppose $V(t_*) = [0 \quad 0 \quad 1]^T$. Then (5.3) holds. Therefore, and by Corollary 2.3, (5.4) reads

$$\kappa(t) = \frac{\left| -\frac{y(t)^2}{c} + y(t) + \left\| \dot{V}(t) \right\|^2 \right|}{\left\| \dot{Z}(t) \right\|^3}.$$

Since $y(t_*) = c$ we have $\kappa(t_*) = ||\dot{V}(t_*)||/||\dot{Z}(t_*)||^3$. Since $||\dot{Z}(t_*)|| \neq 0$ and thus $||\dot{V}(t_*)|| \neq 0$, by Lemma 5.1(i), $\kappa(t_*) \neq 0$. Similarly, $\kappa(t_*) \neq 0$ if $V(t_*) = [0 \quad 0 \quad -1]^T$. So the claim is true. Now take $\delta = D\big(\zeta(r), \zeta(\kappa)\big)/2$. □

With V known, it remains to solve $\tilde{V}(t) = (dL_{x(t)^{-1}})_{x(t)}\dot{x}(t)$ for x.

6. $G = SO(3)$: Solution for $x(t)$

In this section, $\langle \cdot, \cdot \rangle$ and $|| \cdot ||$ denote the inner product and norm on either $so(3)$ or E^3, as necessary. Define $\tilde{W} : \mathbb{R} \to so(3)$ by

$$\tilde{W}(t) = \ddot{\tilde{V}}(t) + \left(\left\| \dot{\tilde{V}}(t) \right\|^2 + \left\langle \tilde{V}(t), \tilde{C} \right\rangle \right) \tilde{V}(t).$$

By Corollary 2.4, $||\tilde{W}(t)|| = \sigma$ for some constant $\sigma \geq 0$ and all $t \in \mathbb{R}$. Since B is a Lie algebra isomorphism and isometry, the curve $W = B^{-1}(\tilde{W}) : \mathbb{R} \to E^3$ satisfies, for all $t \in \mathbb{R}$, (2.12) with $[\cdot, \cdot]$ replaced by $\times$, and $||W(t)|| = \sigma$.

First suppose $\sigma \neq 0$. Let S^2 denote the unit sphere in E^3. Then setting $W_3(t) = W(t)/\sigma$, for all $t \in \mathbb{R}$, defines a map $W_3 : \mathbb{R} \to S^2$. There exists a C^∞ map $W_1 : \mathbb{R} \to S^2$ such that $\langle W_1(t), W_3(t) \rangle$ is identically 0. To see this, note that the set $\left\{(t, v) \in \mathbb{R} \times S^2 | \langle W_3(t), v \rangle = 0\right\}$ is a C^∞ fibre bundle over $\mathbb{R}$ with fibre S^1. Since $\mathbb{R}$ is contractible, the bundle is trivial, and W_1 can be defined using any cross-section. Having chosen W_1, define $W_2 : \mathbb{R} \to S^2$ by $W_2(t) = W_3(t) \times W_1(t)$. Now set $\phi(t) = \int_0^t \big\langle W_1(\xi), \dot{W}_2(\xi) + V(\xi) \times W_2(\xi) \big\rangle \, d\xi$,

$$U_1(t) = W_1(t)\cos(\phi(t)) + W_2(t)\sin(\phi(t)) \quad \text{and}$$
$$U_2(t) = W_2(t)\cos(\phi(t)) - W_1(t)\sin(\phi(t)),$$

for all $t \in \mathbb{R}$. Finally, define $U : \mathbb{R} \to SO(3)$ by $U(t) = [U_1(t) \quad U_2(t) \quad W_3(t)]$. By (2.12) and [12, Theorem 1], we have the following result.

THEOREM 6.1. *If $\sigma \neq 0$ then $x(t) = x(0)U(0)U(t)^T$ for all $t \in \mathbb{R}$.*

It remains to consider the case $\sigma = 0$. Given $g \in SO(3)$, let L_g and R_g be the left and right multiplications (respectively) by g. Define inner automorphisms $I_g = L_g \circ R_{g^{-1}}$ of $SO(3)$ and Lie algebra automorphisms $Ad_g := (dI_g)_e$ (here e is the identity, as in section 2). Recall (see [13]) that the derivative ad at e of the adjoint representation $Ad : g \mapsto Ad_g$ of $SO(3)$ is given by $ad_\xi(\eta) = [\xi, \eta]$, for $\xi, \eta \in so(3)$. Now define $\tilde{V}^* : \mathbb{R} \to so(3)$ by $\tilde{V}^*(t) = -Ad_{x(t)}\tilde{V}(t)$. Recall that an elastic Lie quadratic is *null* if its constant is $\mathbf{0}$. By [12, Theorem 10], we have the following result.

LEMMA 6.2. *Suppose $\sigma = 0$. Then $\tilde{V}^*$ is a null elastic Lie quadratic. The elastic curve $x^* : \mathbb{R} \to SO(3)$ with associated elastic Lie quadratic $\tilde{V}^*$ and $x^*(0) = x_0^*$ is given, for all $t \in \mathbb{R}$, by $x^*(t) = x_0^* x(0) x(t)^T$.*

Let $\tilde{C}^*$ be the constant of $\tilde{V}^*$. By Corollary 2.4, the curve $\tilde{W}^* : \mathbb{R} \to so(3)$ defined by

$$\tilde{W}^*(t) = \ddot{\tilde{V}}^*(t) + \left(\left\|\dot{\tilde{V}}^*(t)\right\|^2 + \left\langle \tilde{V}^*(t), \tilde{C}^* \right\rangle\right) \tilde{V}^*(t)$$

satisfies $\sigma^* = ||\tilde{W}^*(t)||$ for some constant $\sigma^* \geq 0$ and all $t \in \mathbb{R}$. By [12, Theorem 10], we have the following additional result.

LEMMA 6.3. *If $\sigma = 0$ and $\sigma^* = 0$ then $\tilde{V}^*(t)$ is constant.* □

So assume $\sigma = 0$. Then, by Lemma 6.2, to find x it suffices to find x^*. When $\sigma^* = 0$, $\tilde{V}^*(t)$ is constant and it is straightforward to solve $\tilde{V}^*(t) = (dL_{x^*(t)^{-1}})_{x^*(t)}\dot{x}^*(t)$ for x^*. When $\sigma^* \neq 0$, the null elastic Lie quadratic $\tilde{V}^*$ is found as in section 4, and then x^* is given by Theorem 6.1.

EXAMPLE 5. Suppose

$$V(0) = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^T, \quad \dot{V}(0) = \begin{bmatrix}0 & 0 & -1.25\end{bmatrix}^T, \quad C = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^T.$$

Then $y(t) = \langle V(t), C\rangle$ is given by (4.11) with (to 6 significant figures)

$$\alpha = 0.876515, \quad \beta = -0.556701, \quad \gamma = -1.60106.$$

Suppose $x(0) = e$. Representing x locally by a curve in E^3 is problematic, especially near singularities of the representation. Instead, Figure 1 shows the third column $x_3 : \mathbb{R} \to S^2$ of x for $t \in [0, 26]$.

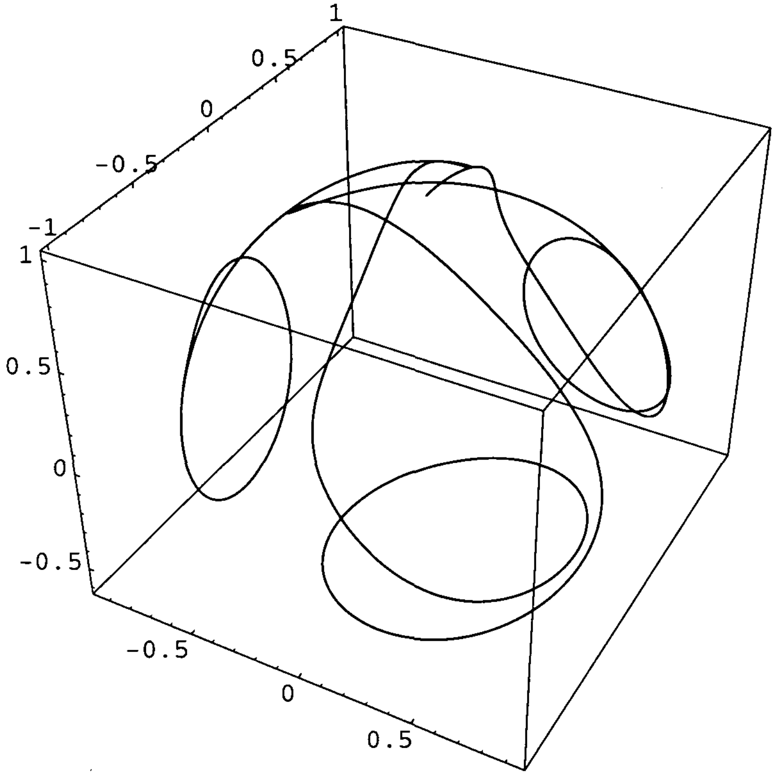

FIGURE 1. $x_3(t)$, for $t \in [0, 26]$, in Example 5.

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