BULL. AUSTRAL. MATH. SOC. MOS 50D99

VOL. 4 (1971), 155-158.

A homomorphism theorem for projective planes

Don Row

We prove that a non-degenerate homomorphic image of a projective plane is determined to within isomorphism by the inverse image of any one point. An application gives conditions for the preservation of central collineations by a homomorphism.

Except for papers by D.R. Hughes [3] and L.A. Skornjakov [7], a growing list of examples [4, 5, 6], and some non-existence results [1, 2, 3], little general information concerning homomorphisms of projective planes is available. The aim of this note is to give a standard fundamental isomorphism theorem for these homomorphisms, and from it a simple coordinate-free derivation of Hughes' conditions [3; Theorems 4.1, 4.2, 4.3] for the preservation of central collineations by a homomorphism.

ISOMORPHISM THEOREM. *A non-degenerate homomorphic image of a projective plane is determined to within isomorphism by the inverse image of any one point.*

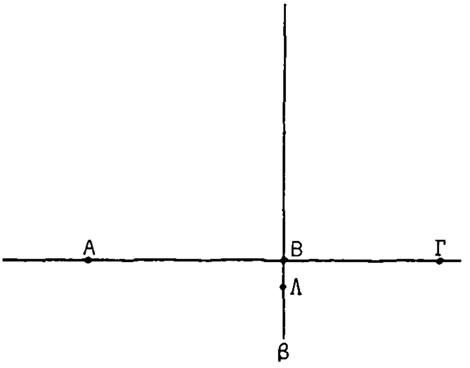
Proof. Consider homomorphisms $h_i : \pi \to \pi_i$ of a plane π into planes π_i , $i = 1, 2$. Let $A, B, \Gamma, \ldots$ be the points and $\alpha, \beta, \gamma, \ldots$ the lines of π . For convenience we identify each line with the set of points incident with it. Denoting the inverse image, or coset, with respect to h_i , containing any element $\mathcal{B}$ by $[B]_i$ we assume $[A]_1 = [A]_2$ for some point $A \in \pi$ (in which case we write $[A]$ for both $[A]_i$).

Received 23 September 1970. This work was supported by National Research Council of Canada grant no. A-7251.

https://doi.org/10.1017/S0004972700046402 Published online by Cambridge University Press

We now prove $[B]_1 = [B]_2$ for any $B \notin [A]$, remembering that $[B]_i = \bigcup_\beta \left([B]_i \cap \beta\right)$, the union being taken over all $\beta \supset B$.

Each line of $h_i\pi$ contains at least three points and we select $\Gamma \in AB$ so that $\Gamma \notin [A]$, $\Gamma \notin [B]_1$. Thus for any $\Lambda \in \beta \cap [B]_1$, $h_1\Lambda\Gamma = h_1AB$ ensuring that $\Lambda\Gamma \cap [A]$ is non-empty and $h_2\Gamma\Lambda = h_2Ah_2\Gamma = h_2AB$. If $\beta \cap [A]$ is empty then $h_2AB \neq h_2\beta$, giving $h_2\Lambda = h_2\Lambda\Gamma . h_2\beta = h_2B$ (even if $h_2\Gamma = h_2B$).



On the other hand, if $\beta \cap [A]$ is non-empty we first choose a point Δ satisfying $h_1\Delta \notin h_1\beta$, $h_2\Delta \notin h_2\beta$ as follows: select $\delta \supset B$ with $\delta \cap [A]$ empty and $\Delta \in \delta$ so that $\Delta \notin [B]_2$, and consequently by the above argument, $\Delta \notin [B]_1$. There is a line $\beta' \supset B$ whose image under h_1 does not contain either h_1A or $h_1\Delta$. Thus any $\Lambda \in \beta \cap [B]_1$ is perspective from Δ to some point of $\beta' \cap [B]_1$. Again by the preceding argument, $\beta' \cap [B]_1 \subseteq \beta' \cap [B]_2$ and thus $\Lambda \in \beta \cap [B]_2$ (even if $h_2\Delta \subset h_2\beta'$).

Hence $[B]_1 \subseteq [B]_2$ and, interchanging the roles of h_1 and h_2, $[B]_1 = [B]_2$. Thus the two homomorphisms have identical point cosets and, as $[AB]_1 = \{A'B' \mid A' \in [A], B' \in [B]\} = [AB]_2$ whenever $[A] \neq [B]$, identical line cosets. Consequently the map $h_1\lambda \to h_2\lambda$, $h_1\Lambda \to h_2\Lambda$; $\lambda, \Lambda \in \pi$ is well defined and an isomorphism, $h_1\pi \to h_2\pi$. //

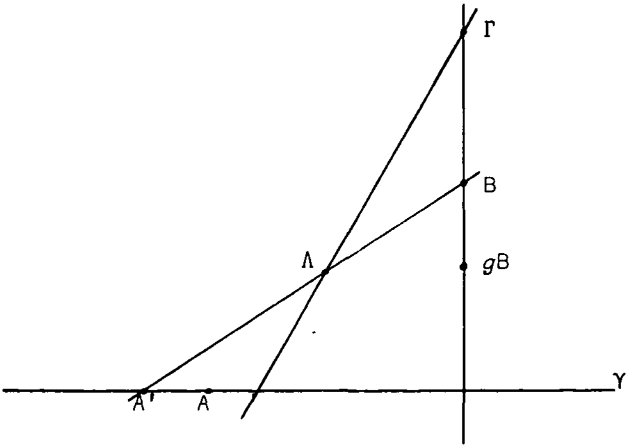
By considering h_1 and defining h_2 by $[A]_2 = [A]_1$, $[B]_2 = \{\Lambda | \Lambda \in \pi, \Lambda \notin [A]_1\}$, $[AB]_2 = \{\lambda | \lambda \in \pi\}$ for any two points A, B in distinct cosets of h_1 we see that the theorem fails if one of the $h_i\pi$ is degenerate.

https://doi.org/10.1017/S0004972700046402 Published online by Cambridge University Press

THEOREM (Hughes). *A homomorphism having a non-degenerate image preserves a central collineation* g *if and only if there are points* B, gB *having images distinct from the image of the centre, and not incident with the image of the axis, of the collineation.*

Proof. Consider $h : \pi \to \pi'$, and write $h_1 = h$, $h_2 = hg$. To show g is preserved it suffices to show $[A]_1 = [A]_2$ for any $A \in \gamma$ satisfying $hA \notin hB\Gamma$ where Γ is the centre, and γ the axis, of g.

If $\Lambda \in [A]_1$, writing $A' = \Lambda B.\gamma$, we have $hA' = hA$ and consequently $hg\Lambda = h\Gamma\Lambda.h(A'gB) = h\Gamma\Lambda.h(\Lambda gB) = hA = hgA$, that is $\Lambda \in [A]_2$. As $hA \notin hB\Gamma \iff hgA \notin hgB\Gamma$, we similarly consider g^{-1} and show $[A]_1 = [A]_2$.



The converse is apparent. //

References

[1] Johannes André, "Über Homomorphismen projektiver Ebenen", *Abh. Math. Sem. Univ. Hamburg* 34 (1969), 98-114.

[2] Peter Dembowski, "Homomorphismen von λ-Ebenen", *Arch. Math.* 10 (1959), 46-50.

[3] D.R. Hughes, "On homomorphisms of projective planes", *Proc. Sympos. Appl. Math.* 10, 45-52, (Amer. Math. Soc., Providence, Rhode Island, 1960).

https://doi.org/10.1017/S0004972700046402 Published online by Cambridge University Press

[4] Wilhelm Klingenberg, "Projektive Geometrien mit Homomorphismus", *Math. Ann.* 132 (1956), 180-200.

[5] Günter Pickert, *Projektive Ebenen* (Die Grundlehren der mathematischen Wissenschaften, Band 80, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1955).

[6] Don Row, "Sharply transitive projective planes and their homomorphisms", (Tech. Report Math. Dept., University of Tasmania, 15, 1967).

[7] L.A. Skornyakov, "Homomorphisms of projective planes and *T*-homomorphisms of ternary rings" (Russian), *Mat. Sb. N.S.* 43 (85) (1957), 285-294.

University of Tasmania,
Hobart, Tasmania.