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AN ANALOGUE FOR SEMIGROUPS OF A GROUP PROBLEM OF P. ERDÖS AND B.H. NEUMANN

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For Bernhard Neumann with respect and affection

In response to a question posed by P. Erdös, B.H. Neumann showed that in a group with every subset of pairwise noncommuting elements finite there is a bound on the size of these sets. Recently, H.E. Bell, A.A. Klein and the first author showed that a similar result holds for rings. However in the case of semigroups, finiteness of subsets of pairwise noncommuting elements does not assure the existence of a bound for their size. The largest class of semigroups in which we found Neumann's result valid are cancellative semigroups.

1. INTRODUCTION

In 1975, Paul Erdös posed the following problem:

> Let G be a group in which every set of pairwise noncommuting elements is finite; is there then a finite bound on the cardinality of sets of pairwise noncommuting elements?

In [7], B.H. Neumann answered the question in the affirmative, characterising these groups as those with a centre of finite index. It appears to be only natural to discuss the problem in the more general setting of semigroups. This is the topic of this paper. In line with [7], we make the following definition.

DEFINITION 1.1: A semigroup is called a PE-semigroup if every set of pairwise noncommuting elements is finite.

The following example shows that in the setting of semigroups we cannot expect results as general as in the case of groups. The example given here is one of a large number of different ones. There are semigroups where the constituent groups are p-groups of nilpotency class two and the index of their centres is increasing.

EXAMPLE 1.2: Let S_n be the symmetric group on n letters, $n \geqslant 1$. Consider the disjoint union $S = \bigcup_{n=1}^{\infty} S_n \cup \{0\}$ with a product defined as follows: For $a \in S_n$, $b \in S_m$,

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let $a \cdot b$ be as in S_n if $n = m$, and $a \cdot b = 0$ if $n \neq m$; also $0 \cdot x = x \cdot 0 = 0$ for all $x \in S$. Then S is a PE-semigroup but the size of sets of pairwise noncommuting elements is not bounded.

This raises the question under what conditions on the PE-semigroup there exists a bound on the size of sets of pairwise noncommuting elements. In [1], PE-rings were investigated, that is rings in which every set of pairwise noncommuting elements is finite, and it was shown that a ring is a PE-ring if and only if the centre has finite index in the ring, thus resulting in a bound for the size of sets of pairwise noncommuting elements. In the context of semigroups this result can be reformulated in the following way.

THEOREM 1.3. [1] *For every PE-semigroup which can occur as the multiplicative semigroup of a ring there exists a bound on the size of sets of pairwise noncommuting elements.*

In this paper we shall show that for every PE-semigroup which can be embedded into a group, or, more generally, which is cancellative, there exists a bound on the size of sets of pairwise noncommuting elements. But before we proceed with the proofs, we take a look at a related commutativity condition. In [4], the authors establish commutativity for an infinite group G with the property that for each pair X, Y of infinite subsets of G, there exist $x \in X$ and $y \in Y$ such that x and y commute. In [1] the same result is established for rings with this property. In context with semigroups we make the following definition.

DEFINITION 1.4: A semigroup S is a PE^*-semigroup if for any infinite subsets X, Y of S there exist $x \in X$ and $y \in Y$ such that x and y commute.

It can be easily seen that any PE^*-semigroup is a PE-semigroup. But for semigroups we cannot expect results as strong as for groups or rings, since the semigroup S of Example 1.2 is also a PE^*-semigroup, but not commutative. This leaves us again searching for conditions on the semigroup implying that a PE^*-semigroup is commutative. In the next section we shall show that a cancellative PE^*-semigroup is commutative.

2. CANCELLATIVE PE^*-SEMIGROUPS

A semigroup S is called cancellative if for $a, x, y \in S$, $ax = ay$ or $xa = ya$ implies that $x = y$. The importance of cancellative semigroups in this context comes from the observation that we can conclude that xA is an infinite set if A is an infinite subset of a cancellative semigroup S and $x \in S$, a fact frequently used in our arguments here. Likewise, the following characterisation of PE-semigroups is of importance throughout this paper. In [7], a proof was given in the case of groups using a graph-theoretical argument. However, the proof in [1, Lemma 2.2] does not require such an argument, and works equally well for semigroups.

Lemma 2.1. *The semigroup S is a PE-semigroup if and only if every infinite subset of S contains an infinite set of pairwise commuting elements of S.*

The main result of this section will be a consequence of the next two lemmas.

Lemma 2.2. *Let S be a cancellative PE^*-semigroup, $x \in S$, $C_S(x)$ the centraliser of x in S, and B an infinite set of pairwise commuting elements. Then $C_S(x) \cap B$ is infinite.*

Proof: By our previous remark it follows that xB is infinite if B is infinite. Since S is a PE^*-semigroup there exist $b, b' \in B$ such that $xb \cdot b' = b' \cdot xb$, or $xb' \cdot b = b'x \cdot b$. By cancellation, it follows that $xb' = b'x$. Thus $b' \in B \cap C_S(x)$. Consider $B \backslash \{b'\}$. Continuing in this manner, we arrive at an infinite set of commuting elements contained in $B \cap C_S(x)$. ∎

In particular, by Lemmas 2.1 and 2.2, in an infinite cancellative PE^*-semigroup, the centraliser of every element contains an infinite set of pairwise commuting elements. Hence, by Lemma 2.2, we get the following result immediately:

Lemma 2.3. *Let S be an infinite cancellative PE^*-semigroup. Then $C_S(x) \cap C_S(y)$ contains an infinite subset of commuting elements for any $x, y \in S$.*

We are now ready to prove the main result of this section.

Theorem 2.4. *Let S be an infinite cancellative PE^*-semigroup. Then S is commutative.*

Proof: Let $x, y \in S$ and let B be an infinite subset of pairwise commuting elements contained in $C_S(x) \cap C_S(y)$, which exists by Lemma 2.3. The sets xB and yB are infinite, since S is cancellative. Now S being a PE^*-semigroup implies that there exist $a, b \in B$ such that $xa \cdot yb = yb \cdot xa$. Since $a, b \in C_S(x) \cap C_S(y)$ and a, b commute, it follows that $xyab = yxab$. But by cancellation $xy = yx$, the desired result. ∎

3. PE-semigroups embeddable into groups

The following characterisation of PE-groups is due to B.H. Neumann [7] and Baer [8, Theorem 4.16].

Theorem 3.1. *For a group G the following are equivalent:*

(i) *G is a PE-group;*

(ii) *G is central-by-finite;*

(iii) *G is the union of finitely many Abelian subgroups.*

We should mention here that in [2] an extended list of equivalences can be found and a similar characterisation for rings appears in [1]. The goal of this section is to give a characterisation of PE-semigroups which are embeddable into groups.

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If S is a semigroup embeddable into a group G, we always can assume without loss of generality that $G = \langle S \rangle$, the group generated by S. Thus we shall write $\langle S \rangle$ for this group from now on. Every $g \in \langle S \rangle$ can be written as $g = s_1 t_1^{-1} \dots s_k t_k^{-1}$, where $s_i, t_i \in S$ and s_1 and/or t_k^{-1} may be equal to 1.

Denote by $R(g) = \{s_1, \dots, s_k, t_1, \dots, t_k\}$ a set of elements in S needed for the presentation of $g \in \langle S \rangle$. More generally, for any subset of elements $\{g_\lambda; \lambda \in \Lambda\}$ in $\langle S \rangle$, where Λ is an index set, denote by $R(g_\lambda; \lambda \in \Lambda)$ a set of elements in S needed for the presentation of the elements g_λ, $\lambda \in \Lambda$.

For later use, we prove the next lemma in the more general case of cancellative semigroups.

Lemma 3.2. *Let S be a finitely generated cancellative PE-semigroup and let $b \in S$ be a nonperiodic element. Then there exists a positive integer k such that $b^k \in Z(S)$.*

Proof: Since b is nonperiodic, it follows that $\{b, b^2, b^3, \dots\}$ is an infinite set. Then $B_x = \{b^i x\,; i \in \mathbb{N}\}$ is infinite for all $x \in S$. Since S is a PE-semigroup, there exist $r, s \in \mathbb{N}$ with $r > s$ such that $b^r x b^s x = b^s x b^r x$. By cancellation, we obtain that $b^{r-s} x = x b^{r-s}$. It follows that there exists $k \in \mathbb{N}$ such that b^k commutes with all generators of S. Hence $b^k \in Z(S)$. ∎

For finitely generated PE-semigroups we can find a special presentation of the elements in $\langle S \rangle$.

Lemma 3.3. *Let S be a finitely generated PE-semigroup embeddable into a group. Then every $g \in \langle S \rangle$ can be written as $g = sz$, where $s \in S$ and $z \in Z(\langle S \rangle)$.*

Proof: Let $S = \langle x_1, \dots, x_m \rangle$ as a semigroup. If x_i has finite order, then $x_i^{-1} \in S$. If x_i has infinite order then, by Lemma 3.2, we can assume that there exists an integer $\alpha_i \geqslant 1$ such that $x_i^{\alpha_i} \in Z(S)$. Setting $z_i = x_i^{\alpha_i}$, we obtain that $x_i^{-1} = x_i^{\alpha_i - 1} \cdot z_i^{-1}$, where $z_i^{-1} \in Z(\langle S \rangle)$. Now let $g \in \langle S \rangle$. Then g is a product of powers of the x_i and their inverses. By the above, every x_i^{-1} either lies in S or can be replaced by the product of a nonnegative power of x_i and a central element. Our claim follows. ∎

The preceding lemma gives rise to the following corollary.

Corollary 3.4. *Let S be a PE-semigroup embeddable into a group and $g \in \langle S \rangle$. Then every conjugate of g can be written in the form g^t where $t \in S$.*

Proof: Let g^h be any conjugate of g under $\langle S \rangle$, and consider $H = \langle R(g,h) \rangle$. Then, by Lemma 3.3, $h = tz$ with $t \in S$ and $z \in Z(H)$. We conclude $g^h = g^{tz} = g^t$, the desired result. ∎

For the rest of this section we proceed in a similar manner as in [7]. However we have to argue with infinite subsets of S. The next lemma follows along the lines of [7, Lemma 1]. However, we shall apply Lemma 2.1 instead of Ramsey's Theorem as used by B.H. Neumann.

Call an element of a group an *FC-element* if it has finitely many conjugates, and a group an *FC-group* if all of its elements are FC-elements.

LEMMA 3.5. *Let S be a PE-semigroup embeddable into a group. Then $\langle S\rangle$ is an FC-group.*

PROOF: We may assume without loss of generality that S is infinite. Suppose that $\langle S\rangle$ is not an FC-group. Then there exists $g \in \langle S\rangle$ such that g has infinitely many conjugates. We can assume that $g \in S$, since g is a product of finitely many elements of S and their inverses and products of FC-elements are FC-elements. Corollary 3.4 implies that there exists an infinite subset T of S such that $g^t \neq g^{t'}$, whenever t and t' are distinct elements in T.

By Lemma 2.1, there exists an infinite subset U of T consisting of commuting elements. Consider the set $gU = \{gu; u \in U\}$. Since $g \in S$ and $U \subseteq S$, we have $gU \subseteq S$ and gU is infinite, since S is embeddable into a group. For u, v distinct elements of U we have $[gu, gv] = (gu)^{-1}(gv)^{-1}gugv = u^{-1}g^{-1}v^{-1}ugv = (g^u)^{-1} \cdot g^v \neq 1$, since $g^u \neq g^v$. Thus no two distinct elements of the infinite set gU commute, a contradiction since S is a PE-semigroup. We conclude that $\langle S\rangle$ is an FC-group. $\square$

The next lemma follows along the lines of [7, Lemma 4]. However the two sequences in question have to be selected in S instead of $\langle S\rangle$.

LEMMA 3.6. *Let S be an infinite PE-semigroup in a group $\langle S\rangle$ and suppose that $\langle S\rangle$ is an FC-group which is not central-by-finite. Furthermore, suppose that S contains two finite sequences of elements $(a_1, \dots, a_n)$, $(b_1, \dots, b_n)$ with the following properties:*

(i) *if $i \neq j$, then $a_ia_j \neq a_ja_i$;*
(ii) *if $i \neq j$, then $a_ib_j = b_ja_i$;*
(iii) *$a_ib_i \neq b_ia_i$ for all i;*
(iv) *$b_ib_j = b_jb_i$ for all i, j.*

Then S contains two further elements a_{n+1}, b_{n+1} such that (i) *through* (iv) *remain valid for the sequences $(a_1, \dots, a_{n+1})$ and $(b_1, \cdots, b_{n+1})$ of length $n+1$.*

PROOF: We begin by choosing a_1, b_1 to be any pair of noncommuting elements in S. Then, assuming that we already have $a_1, \dots, a_n$ and $b_1, \dots, b_n$ satisfying (i) - (iv), let $A = C_{\langle S\rangle}(a_1, \dots, a_n, b_1, \dots, b_n)$. Now $\langle S\rangle$ is an FC-group. It follows that A has finite index in $\langle S\rangle$. But $\langle S\rangle$ is not Abelian and so [7, Corollary 3] implies that A is not Abelian. Thus there exist $a, b \in A$ such that $[a, b] \neq 1$. Consider $H = \langle R(a_1, \dots, a_n, b_1, \dots, b_n, a, b)\rangle$. By Lemma 3.3, there exist $s, t \in S$ and z, z' in $Z(H)$ such that $a = sz$ and $b = tz'$. Setting $a_{n+1} = sb_1 \dots b_n$ and $b_{n+1} = t$, we have $a_{n+1}, b_{n+1} \in S$. Observing $a_{n+1} = az^{-1}b_1 \dots b_n$ and $b_{n+1} = bz'^{-1}$, it can be easily verified that (i) - (iv) are valid for the sequences $(a_1, \dots, a_{n+1})$ and $(b_1, \dots, b_{n+1})$. Thus the lemma follows. $\square$

Now we are ready to state and prove a theorem analogue to Theorem 3.1 in the case of semigroups.

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Theorem 3.7. *Let S be a semigroup embeddable into a group. Then the following conditions are equivalent:*

(i) *S is a PE-semigroup;*

(ii) *$\langle S\rangle$ is central-by-finite;*

(iii) *S is the set-theoretic union of finitely many commutative subsemigroups.*

Proof: First assume (i). Then $\langle S\rangle$ is an FC-group by Lemma 3.5. If $\langle S\rangle$ is not central-by-finite, an infinite sequence $(a_1, a_2, a_3, \ldots)$ of pairwise noncommuting elements in S can be constructed by Lemma 3.6. This is a contradiction, since S is a PE-semigroup. Hence (i) implies (ii).

Next, assume (ii). By Theorem 3.1 we have that $\langle S\rangle$ is the union of finitely many Abelian subgroups, say $U_1, \ldots, U_n$. Then S is the union of the commutative subsemigroups $U_1 \cap S, \ldots, U_n \cap S$. Hence (ii) implies (iii).

Finally, assume (iii), that is S is the union of finitely many commutative subsemigroups $H_1, \ldots, H_n$. Consider X, an infinite subset of S. Since X is infinite, at least one of the intersections $X \cap H_1, \ldots, X \cap H_n$ is infinite. This implies that X contains an infinite set of pairwise commuting elements. Since X was arbitrary, it follows by Lemma 2.1 that S is a PE-semigroup. We conclude (iii) implies (i). ∎

The following corollary is now immediate.

Corollary 3.8. *The size of the sets of pairwise noncommuting elements in a PE-semigroup which can be embedded into a group is bounded.*

4. Cancellative PE-semigroups

There exist cancellative semigroups which cannot be embedded into groups. Examples of such semigroups were given for example, by Malcev in [5]. Thus the class of cancellative semigroups is larger than the class of semigroups embeddable into groups. The topic of this section is the investigation of cancellative PE-semigroups. We shall show that for this class of semigroups there exists a bound on the size of the sets of pairwise noncommuting elements. However, no new approach is needed. We shall reduce the proof of our claim to the preceding section by showing that a cancellative PE-semigroup can be embedded into a group. We start with some preparatory results.

Lemma 4.1. *Let S be a cancellative semigroup with nonempty centre. Then S can be embedded into a cancellative semigroup with unit element whose centre is a group.*

Proof: We define a relation on $S \times Z(S)$ by $(b,u) \approx (c,v)$ if $bv = uc$. It can be shown that this relation is an equivalence relation on $S \times Z(S)$. Set

$$a/z = \{(b,u) \in S \times Z(S);\ \ (b,u) \approx (a,z)\},$$

an equivalence class for "$\approx$", and let

$$T = \{a/z;\ a \in S,\ z \in Z(S)\}$$

denote the set of equivalence classes. We define a multiplication on T by $(a/z) \cdot (b/u) = ab/zu$. It can easily be verified that this multiplication is well-defined.

Consider $z \in Z(S)$. We have $(z/z) \cdot (b/u) = (b/u) \cdot (z/z) = bz/uz = b/u$, since $(bz, uz) \approx (b, u)$. Thus T contains an identity. It can easily be shown that $Z(T) = \{u/v; u, v \in Z(S)\}$. Let $u/v \in Z(T)$. Then $(u/v) \cdot (v/u) = uv/uv = z/z$, hence $(u/v)^{-1} = v/u$ and $v/u \in Z(T)$. It follows that $Z(T)$ is a group.

Next we show S can be isomorphically embedded into T. Choose any z in $Z(S)$ and define $\phi : S \to T$ by $\phi(b) = bz/z$ and let $\overline{S} = \{bz/z; b \in S\}$. It can easily be shown that ϕ is a one-one homomorphism of S onto $\overline{S}$.

Finally, we show T is a cancellative semigroup. Suppose $(a/z) \cdot (b/u) = (a/z) \cdot (c/v)$. This implies $abzv = aczu$. By cancellation, we obtain $bv = cu$ and conclude $b/u = c/v$. Similarly, $(b/u) \cdot (a/z) = (c/v) \cdot (a/z)$ implies $b/u = c/v$. ∎

PROPOSITION 4.2. *Let S be a finitely generated cancellative PE-semigroup containing a nonperiodic element. Then S can be embedded into a group.*

PROOF: By Lemma 3.2, there exists a nonidentity element in $Z(S)$. Thus, by Lemma 4.1, S can be isomorphically embedded into a cancellative semigroup T in which $Z(T)$ forms a group.

We shall show now that T is a group, that is, every element in T has an inverse. Let $g \in T$ and write $g = a/z$, $a \in S$, $z \in Z(S)$.

If a is periodic, then we have in T that $a^k = 1$ for some k. Since $g^k = 1/z^k$ is invertible as an element of $Z(T)$, it follows that g is invertible.

If a is nonperiodic, then, by Lemma 3.2, $a^k \in Z(S)$ for some positive integer k. Hence $g^k = a^k/z^k \in Z(T)$ and g^k is invertible by Lemma 4.1. It follows that g itself is invertible. ∎

We are now ready to prove the main result of this section.

THEOREM 4.3. *Let S be a cancellative PE-semigroup. Then S can be embedded into a group.*

PROOF: Consider S_0, a finitely generated subsemigroup of S. First, let every element in S_0 be periodic. By [**6**, Lemma 2.1], S_0 contains a unit element. It follows that every element in S_0 has finite order and thus has an inverse. We conclude that S_0 is a group.

Next, suppose that S_0 contains a nonperiodic element. By Proposition 4.2, S_0 can be embedded into a group.

Thus every finitely generated subsemigroup of S can be embedded into a group. We can apply now [**3**, Theorem 12.6], stating that a semigroup is embeddable into a group if and only if every finitely generated subsemigroup is so embeddable. We conclude that S is embeddable into a group, the desired result. ∎

Now we can apply Theorem 3.7 to cancellative PE-semigroups and obtain all the results we had for semigroups embeddable into groups. We state just one corollary which answers Erdös' original question.

Corollary 4.4. *The size of sets of pairwise noncommuting elements in a cancellative PE-semigroup is bounded.*

References

[1] H. Bell, A. Klein and L.C. Kappe, 'An analogue for rings of a group problem of P. Erdős and B.H. Neumann', *Acta Math. Hungar.* **77** (1997), 57–67.

[2] M.A. Brodie and L.C. Kappe, 'Finite coverings by subgroups with a given property', *Glasgow Math. J.* **35** (1993), 179–188.

[3] A.H. Clifford and G.B Preston, *The algebraic theory of semigroups Vol. 2* (American Mathematical Society, Providence RI, 1967).

[4] P.S. Kim, A. Rhemtulla and H. Smith, 'A characterization of infinite metabelian groups', *Houston J. Math.* **17** (1991), 429–437.

[5] A. Malcev, 'On the immersion of an algebraic ring into a field', *Math. Ann.* **113** (1937), 686–691.

[6] B.H. Neumann, 'Some remarks on cancellative semigroups', *Math. Z.* **117** (1970), 97–111.

[7] B.H. Neumann, 'A problem of Paul Erdős on groups', *J. Austral. Math. Soc. Ser. A* **21** (1976), 467–472.

[8] D.J.S. Robinson, *Finiteness conditions and generalized soluble groups*, Part I (Springer Verlag, Berlin, Heidelberg, New York, 1972).

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