

# THE MONGE-AMPÈRE EQUATION AND WARPED PRODUCTS OF HIGHER RANK

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## Abstract

We show that a warped product  $M_f = \mathbb{R}^n \times_f \mathbb{R}$  has higher rank and nonpositive curvature if and only if  $f$  is a convex solution of the Monge-Ampère equation. In this case we show that  $M$  contains a Euclidean factor.

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## 1. Introduction

A Riemannian manifold has higher rank if along each of its geodesics there are at least two linearly independent parallel Jacobi fields. For nonpositively curved manifolds this seems to be a rather restrictive condition. An irreducible Hadamard manifold of higher rank is symmetric if it additionally admits a cocompact discontinuous group  $\Gamma$  of isometries. In fact it suffices if  $\Gamma$  satisfies the duality condition (see [1, 2, 3, 6]). It is an open question whether this additional condition may be entirely omitted ('Rank Rigidity Conjecture').

In this note we consider warped products  $M = \mathbb{R}^n \times_f \mathbb{R}$ . It turns out that nonpositive sectional curvature on  $M$  translates to convexity of  $f$  and that higher rank of  $M$  requires  $f$  to satisfy the Monge-Ampère equation. We show in Proposition 3.2 that a positive convex solution  $f \in C^\infty(\mathbb{R}^n)$  of the Monge-Ampère equation has the form  $f(x) = q(Ax)$ , where  $q \in C^\infty(\mathbb{R}^k)$  is a positive convex function with  $k < n$  and  $A: \mathbb{R}^n \rightarrow \mathbb{R}^k$  is an orthogonal projection, that is  $AA^* = \text{id}_{\mathbb{R}^k}$ . Hence a Riemannian manifold  $M = \mathbb{R}^n \times_f \mathbb{R}$  of nonpositive curvature and higher rank splits as a Riemannian product  $(\mathbb{R}^k \times_q \mathbb{R}) \times \mathbb{R}^{n-k}$  (Theorem 2.1).

### 2. Warped Products of Higher Rank

Let  $M$  be a Riemannian manifold with curvature tensor  $R$ . The Jacobi operator  $J_v$  of a tangent vector  $v \in T_pM$  at  $p \in M$  is the symmetric endomorphism  $x \mapsto R(x, v)v$  of  $T_pM$ .  $M$  has higher infinitesimal rank if the kernel of the Jacobi operator of each  $v \in TM$  is at least 2-dimensional. By the Jacobi equation, manifolds with higher rank have this property. On the other hand, the semisymmetric spaces of conullity 2 considered in [5], [8] and [4, Theorem 2] provide examples of manifolds of rank 1 which have higher infinitesimal rank.

Given a smooth real positive function  $f$  on  $\mathbb{R}^n$  we let  $M$  denote  $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$  with the warped product metric given at a point  $(x, t) \in M$  by

$$g = dx^2 \oplus f(x)^2 dt^2,$$

where  $dx^2, dt^2$  denote the standard metrics on  $\mathbb{R}^n, \mathbb{R}$  respectively. We show that rank rigidity holds for these warped products.

**THEOREM 2.1.** *The warped product  $M = \mathbb{R}^n \times_f \mathbb{R}, n > 0$ , has nonpositive curvature and higher infinitesimal rank if and only if  $f(x) = q(Ax)$ , where  $q \in C^\infty(\mathbb{R}^k)$  is a positive convex function and  $A: \mathbb{R}^n \rightarrow \mathbb{R}^k$  is an orthogonal projection with  $k < n$ . Thus  $M$  is isometric to the product  $(\mathbb{R}^k \times_q \mathbb{R}) \times \mathbb{R}^{n-k}$ .*

**PROOF.** This follows from Lemma 2.2 and Proposition 3.2. □

**LEMMA 2.2.**  *$M = \mathbb{R}^n \times_f \mathbb{R}$  has nonpositive curvature and higher infinitesimal rank if and only if  $f$  is a positive convex solution of the Monge-Ampère equation (3.1).*

**PROOF.** We first express the curvature tensor of  $M$  in terms of  $f$ . To that end, let  $X, Y$  and  $T$  be vector fields on  $M = \mathbb{R}^n \times_f \mathbb{R} \cong \mathbb{R}^{n+1}$  which are parallel with respect to the Euclidean metric on  $\mathbb{R}^{n+1}$  and such that  $X, Y$  are tangent to the first factor and  $T = \partial/\partial t$  is tangent to the second factor. We have  $g(T, T) = f^2$ . For the covariant derivative we compute

$$\nabla_X Y = 0, \quad \nabla_X T = \nabla_T X = \frac{Xf}{f} T \quad \text{and} \quad \nabla_T T = -f \text{ grad } f.$$

The sectional curvature is given by

$$\begin{aligned} (2.1) \quad K(X + T, Y) &= \langle R(X + T, Y)Y, T + X \rangle = -\langle \nabla_Y \nabla_T Y, X + T \rangle \\ &= -\left\langle \nabla_Y \frac{Yf}{f} T, X + T \right\rangle = -(YYf)f = -f d^2 f(Y). \end{aligned}$$

For  $n = 1$ ,  $M$  has higher infinitesimal rank if and only if  $f$  is constant. Let now  $n \geq 2$ . By (2.1),  $M$  has nonpositive curvature if and only if the Hessian  $d^2 f$  is everywhere nonnegative. For any  $X$  the Jacobi operator of  $X$  has at least  $n$ -dimensional kernel. The Jacobi operator of  $X + T$  has at least 2-dimensional kernel if and only if  $\langle R(X + T, Y)Y, T + X \rangle = -f d^2 f(Y) = 0$  for some  $Y \neq 0$ . But this is equivalent to  $\det d^2 f = 0$ . □

### 3. Global Convex Solutions of the Monge-Ampère Equation

Henceforth  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  always denotes a smooth nonnegative convex solution of the Monge-Ampère equation. Thus the Hessian  $d^2 f$  of  $f$  satisfies

$$(3.1) \quad \det d_x^2 f = 0 \quad \text{and} \quad d_x^2 f \geq 0$$

for all  $x \in \mathbb{R}^n$ . On a strictly convex domain  $\Omega \subset \mathbb{R}^n$  we have from [7] that

$$(3.2) \quad f(x) = \sup_L \{L(x)\}, \quad x \in \Omega,$$

where  $L$  runs through all affine linear functions on  $\mathbb{R}^n$  with  $L \leq f$  on the boundary  $\partial\Omega$ .

LEMMA 3.1. *For any  $q \in \mathbb{R}^n$  with  $f(q) = 0$  and any  $\rho > 0$  we have that  $q$  is contained in the convex hull of*

$$f^{-1}(0) \cap (q + \rho S^{n-1}) = \{z \in \mathbb{R}^n : f(z) = 0, |z - q| = \rho\}.$$

*In particular there are  $z_0, \dots, z_n \in f^{-1}(0) \cap (q + \rho S^{n-1})$  and  $\lambda_0, \dots, \lambda_n \geq 0$  with  $\sum_{i=0}^n \lambda_i = 1$  such that  $\sum_{i=0}^n \lambda_i z_i = q$ .*

PROOF. We assume the contrary and suppose  $q = 0$  and  $\rho = 1$ , since (3.1) is invariant under translations and homotheties. The convex hull of  $f^{-1}(0) \cap S^{n-1}$  is an intersection of halfspaces of  $\mathbb{R}^n$ . Hence after a suitable choice of coordinates we may further suppose that it is contained in  $\{x_1 \leq -2\delta\}$  for some  $\delta > 0$ . We apply (3.2) to the unit ball  $\Omega = D^n$ . Since  $S^{n-1} \cap \{x_1 \geq -\delta\}$  is compact, there is  $\epsilon > 0$  such that  $f|_{S^{n-1} \cap \{x_1 \geq -\delta\}} \geq \epsilon$ . Let  $\alpha$  be the affine linear function on  $\mathbb{R}^n$  given by

$$\alpha(x_1, \dots, x_n) = (\delta + x_1) \frac{\epsilon}{\delta + 1}.$$

Clearly  $\alpha|_{\partial\Omega} \leq f|_{\partial\Omega}$ , hence  $f(0) \geq \alpha(0) = \delta\epsilon/(\delta + 1) > 0$ , contradicting the assumption that  $f(0) = 0$ . □

PROPOSITION 3.2. *Let  $f \in C^\infty(\mathbb{R}^n)$  be a smooth positive function with  $d_x^2 f \geq 0$  and  $\det d_x^2 f = 0$  for all  $x \in \mathbb{R}^n$ . Then there is a  $k < n$ , an orthogonal projection  $A: \mathbb{R}^n \rightarrow \mathbb{R}^k$  and a  $q \in C^\infty(\mathbb{R}^k)$  such that  $f(x) = q(Ax)$  for all  $x$ .*

PROOF. We may assume  $f \geq 0$  and  $f(0) = 0$ , replacing  $f(x)$ , if necessary, by  $f(x) - f(0) - d_0 f x$ . For  $\rho > 0$  let  $z_i(\rho), i = 0, \dots, n$ , be as in Lemma 3.1, so that  $f(z_i(\rho)) = 0, |z_i| = \rho$  and  $\sum_{i=0}^n \lambda_i z_i = 0$  for some  $\lambda_i \geq 0$  with  $\sum_{i=0}^n \lambda_i = 1$ . Choosing a suitable sequence  $\rho_k \rightarrow \infty$ , we get convergent sequences  $z_i(\rho_k)/|z_i(\rho_k)| \rightarrow z_i(\infty)$ . Then  $U = \text{span}\{z_0(\infty), \dots, z_n(\infty)\}$  is a subspace of  $\mathbb{R}^n$  of dimension  $n - k > 0$ . We have

$$f(\lambda z_i(\infty)) = \lim_{k \rightarrow \infty} f\left(\frac{\lambda z_i(\rho_k)}{|z_i(\rho_k)|}\right) \leq \lim_{k \rightarrow \infty} \left( \left(1 - \frac{\lambda}{\rho_k}\right) f(0) + \frac{\lambda}{\rho_k} f(z_i(\rho_k)) \right) = 0$$

for each  $\lambda \in \mathbb{R}^+$ , since  $f$  is convex and  $f(0) = 0$ . For  $\lambda \rightarrow \infty$  the convex hulls of  $\{\lambda z_0(\infty), \dots, \lambda z_n(\infty)\}$  exhaust  $U$ . Hence  $f|_U = 0$ .

By convexity

$$(3.3) \quad f((1 - \mu)p + u) \leq (1 - \mu)f(p) + \mu f(u/\mu) = (1 - \mu)f(p)$$

for any  $p \in \mathbb{R}^n, 0 < \mu \leq 1$  and  $u \in U$ . In the limit  $\mu \rightarrow 0$  the left hand side of (3.3) tends to  $f(p + u)$ , hence  $f(p + u) \leq f(p)$ . Since  $p$  and  $p + u$  are arbitrary elements of the affine space  $p + U$  we conclude that  $f$  is constant on the affine spaces  $p + U$ . We choose an isometry of the orthogonal complement  $U^\perp$  of  $U$  with  $\mathbb{R}^k$ . Let  $A$  be the orthogonal projection  $\mathbb{R}^n \rightarrow U^\perp \cong \mathbb{R}^k$  and  $q$  be the restriction of  $f$  to  $U^\perp \cong \mathbb{R}^k$ .  $\square$

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