ANALYTIC-NUMERICAL SOLUTIONS OF RESTRICTED NON-RESONANCE PLANAR THREE-BODY PROBLEM

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We consider a restricted planar circular three-body problem (Sun–Jupiter–asteroid) in a non-resonance case. There are two new algorithms developed for construction of a quasi-periodic solution in a trigonometric form by means of computer algebra. The first corresponds to classical method of simple iterations leading to series in powers of small mass m_J, the second, to iterations with rapid (quadratic) convergence, but having ordinary type and not involving a successive coordinate transformations. All these iterations require a realization of algebraic operations on trigonometric polynomials with the help of computers of high capacity. It would be interesting to compare the solutions obtained with the two algorithms and to estimate the domain of their practical convergence.

1. Let us consider the following equations of the given problem

$$\begin{aligned} dp/d\theta &= \mu\Phi_1(p,e,G,L), \qquad dG/d\theta = 1-\mu F_1(p,e,G,L), \\ de/d\theta &= \mu\Phi_2(p,e,G,L), \qquad dL/d\theta = 1-F_2(p,e,G,L), \end{aligned} \tag{1}$$

where θ is the longitude of asteroid, p is the orbital parameter, e is the eccentricity, G is the true anomaly, L is the difference between θ and Jupiter's longitude. Φ_1, Φ_2 are uneven and F_1, F_2 even functions of angular variables G, L. Units of mass and time are defined such that $k^2 = 1$ (gravitational constant), $m_S + m_J = 1$, $m_J = \mu$ and Jupiter's semimajor axis and mean motion are $a_J = 1$ and $n_J = 1$. These equations and the expressions of their right-hand sides are well known. In particular, $\Phi_1 = 2p^3[(1+e\cos G)^{-3} - A^{-3/2}]\sin L$, where $A = p^2 + (1+e\cos G)^2 - 2p(1+e\cos G)\cos L$. Introducing vectors $x = (p,e)$, $y = (G,L)$, $\Phi = (\mu\Phi_1, \mu\Phi_2)$, $F(\mu F_1, F_2)$ we will seek the solution of system (1) in the form:

$$x = x_0 + \sum_{||k||=1}^{N} U_k \cos(k,\psi), \qquad y = \psi + \sum_{||k||=1}^{N} V_k \sin(k,\psi), \tag{2}$$

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where $\psi = (\psi_1, \psi_2)$, $\psi_j = \omega_j\theta + \psi_{j0}, j = 1, 2$, $k = (k_1, k_2)$ is a vector with integer components k_1, k_2. $x_0 = (p_0, e_0)$ and $\omega = (\omega_1, \omega_2)$ are vectors of the mean values p, e and frequencies ω_1, ω_2, respectively; $\|k\| = |k_1| + |k_2|$, $(k, \psi) = k_1\psi_1 + k_2\psi_2$, $(k, \omega) = k_1\omega_1 + k_2\omega_2$ and the number N is a given maximal (sufficiently high) order of harmonics. Coefficients U_k, V_k (two-dimensional vectors) and ω_1, ω_2 are unknown; p_0, e_0 are also unknown, but we fix their numerical values; vector $\psi_0 = (\psi_{10}, \psi_{20})$ is left arbitrary. We will obtain the solution (2) with numerical coefficients (i.e. with numerical components of vectors U_k, V_k, ω) and we also fix the value of mass μ. Substituting (2) into (1) we obtain the corresponding relations and our main equations in U_k, V_k, ω follows:

$$\omega = 1 - \tilde{F}_0(U, V), \quad -(k, \omega)U_k = \tilde{\Phi}_k(U, V), \quad (k, \omega)V_k = -\tilde{F}_k(U, V), \tag{3}$$

where $1 \le \|k\| \le N$ and $\tilde{F}_0\tilde{F}_k, \tilde{\Phi}_k$ are coefficients of Fourier expansions:

$$\Phi(x, y) = \sum \tilde{\Phi}_k(U, V)\sin(k, \psi)$$

$$F(x, y) = \tilde{F}_0(U, V) + \sum \tilde{F}_k(U, V)\cos(k, \psi). \tag{4}$$

U, V denote vectors whose components are all U_k, V_k respectively. Coefficients $\tilde{\Phi}_k, \tilde{F}_k$ are theoretically certain expressions in different coefficients U_k, V_k.

2. It is essential that we can solve equations (3) by iterations in absence of analytical expressions for $\tilde{\Phi}_k(U, V), \tilde{F}_k(U, V)$.

The zero-approximation (corresponds to known formulae of unperturbed motion): $x^0 = (p_0, e_0)$, $y^0 = (G^0, L^0)$, where

$$G^0 = \omega_1^0\theta + \psi_{10}, \quad L^0 = \omega_2^0\theta + \psi_{20} + \sum_{j=1}^{N} S_j^0 \cos(jG^0), \tag{5}$$

$\omega_1^0 = 1$ and ω_2^0, Sj^0 are known expressions in p_0, e_0; hence, $U^0 = 0, V^0 = (0, S_j^0)$. If numbers p_0, e_0 are given, then we obtain corresponding numbers ω_2^0, S_j^0. We assume that there is no acute resonance between frequencies ω_1^0, ω_2^0.

The first approximation

$$\omega^{(1)} = 1 - \tilde{F}_0(U^0, V^0), \quad \psi^{(1)} = \omega^{(1)}\theta + \psi_0,$$

$$U_k^{(1)} = -\frac{1}{(k, \omega^{(1)})}\tilde{\Phi}_k(U^0, V^0), \quad V_k^{(1)} = -\frac{1}{(k, \omega^{(1)})}\tilde{F}_k(U^0, V^0). \tag{6}$$

It is possible to compute components of vectors $\tilde{\Phi}_k(U^0, V^0), \tilde{F}_k(U^0, V^0)$ in the following way. For example, quantities $\tilde{\Phi}_{1k}(U^0V^0)$ are Fourier coefficients of the function

$$\Phi_1(x^0, y^0) = 2p_0^3 \left[(1 + e_0 \cos G^0)^{-3} - (A_0)^{-3/2}\right],$$

where $A_0 = p_0^2 + (1+e_0 \cos G^0)^2 - 2p_0(1+e_0 \cos G^0) \cos L^0$, $G^0 = \psi_1^0$ and L^0 is represented by (5). The algebraic manipulations of Fourier polynomials done with the help of computer algebra lead us to the expansion of form (4) with coefficients $\tilde{\Phi}^0_{1k}$. Similarly, we obtain other components of mentioned vectors. Afterwards, we calculate $\omega^{(1)}, U_k^{(1)}, V_k^{(1)}$ and obtain $x^{(1)}, y^{(1)}$ in form (2) with numerical coefficients.

We can use for construction of subsequent approximations a)simple iterations and b)iterations with quadratic convergence.

Simple iterations $\quad \omega^{(2)} = 1 - \tilde{F}_0(U^{(1)}, V^{(1)})$,

$$U_k^{(2)} = -\frac{1}{(k, \omega^{(2)})}\tilde{\Phi}_k(U^{(1)}, V^{(1)}), \quad V_k^{(2)} = -\frac{1}{(k, \omega^{(2)})}\tilde{F}_k(U^{(1)}, V^{(1)}). \quad (7)$$

The calculations of $\tilde{F}_k(U^{(1)}, V^{(1)}), \tilde{\Phi}_k(U^{(1)}, V^{(1)})$ are reduced to obtaining Fourier expansions of functions $F(x^{(1)}, y^{(1)}), \Phi(x^{(1)}, y^{(1)})$, where $x^{(1)}, y^{(1)}$ are known expansions of form (2) with numerical coefficients. Subsequent approximations are defined similarly.

Iterations with quadratic convergence

In accordance with Newton's method we put in (3)

$$\omega = \omega^{(1)} + \nu^{(1)}, \quad U_k = U_k^{(1)} + u_k^{(1)}, \quad V_k = V_k^{(1)} + v_k^{(1)}$$

and form linearized algebraic equations in $\nu^{(1)}, u_k^{(1)}, v_k^{(1)}$ (corrections to the first approximation) introducing vectors $u = \{u_k\}, v = \{v_k\}$, whose components are sets of all u_k, v_k correspondingly with $1 \leq \|k\| \leq N$. These equations are the following

$$\begin{aligned}
(k, \nu^{(1)})U_k^{(1)} + (k, \omega^{(1)})u_k^{(1)} + \left(\partial\tilde{\Phi}_k/\partial U\right)_1 u^{(1)} + \left(\partial\tilde{\Phi}_k/\partial V\right)_1 v^{(1)} &= \Delta\tilde{\Phi}^{(1)}_{(k)} \\
(k, \nu^{(1)})V_k^{(1)} + (k, \omega^{(1)})v_k^{(1)} + \left(\partial\tilde{F}_k/\partial U\right)_1 u^{(1)} + \left(\partial\tilde{F}_k/\partial V\right)_1 v^{(1)} &= \Delta\tilde{F}_k^{(1)} \\
\nu^{(1)} + \left(\partial\tilde{F}_0/\partial U\right)_1 u^{(1)} + \left(\partial\tilde{F}_0/\partial V\right)_1 v^{(1)} &= \Delta\tilde{F}_0^{(1)}
\end{aligned}$$

where $1 \leq \|k\| \leq N, \Delta\tilde{\Phi}_k^{(1)} = \tilde{\Phi}_k^0 - \tilde{\Phi}_k^{(1)}$ etc. and $\tilde{\Phi}_k^{(j)} = \tilde{\Phi}_k(U^{(j)}, V^{(j)})$ etc. The derivative $\left(\partial\tilde{\Phi}_k/\partial U\right)_1 = \partial\tilde{\Phi}_k/\partial U\big|_{U=U^{(1)}, V=V^{(1)}}$ is the block matrix $[C_{l1,l2}]$ with $C_{l1,l2}$ being $2 \times 2-$ matrices $(\partial\tilde{\Phi}_k/\partial U_{l1,l2})_1$.

According to the expansion of $\Phi(x^{(1)}, y^{(1)})$ we have

$$\Phi(x^{(1)}, y^{(1)}) = \sum_{\|k\|=1}^{N} \tilde{\Phi}_k(U^{(1)}, V^{(1)}) \sin(k, \psi),$$

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and for fixed vector $l = (l1, l2)$

$$\frac{\partial \Phi}{\partial U_{l1,l2}} = \frac{\partial \Phi}{\partial x} cos(l,\psi) = \sum_{||k||=1}^{N} \frac{\partial \tilde{\Phi}_k}{\partial U_{l1,l2}} \sin(k,\psi).$$

Hence, $\left(\partial \tilde{\Phi}_k / \partial U_{l1,l2}\right)_1$ for different vectors k are Fourier matrix-coefficients for the function $\Phi^\star = (\partial \Phi / \partial x)_1 \cos(l,\psi)$.

We obtain Fourier expansion for this function by means of manipulations considered above. Others derivatives are calculated similarly. Certainly, these calculations are very cumbersome, but they are feasible by using computers of sufficient capacity.

Having calculated $\nu^{(1)}, u_k^{(1)}, v_k^{(1)}$, we obtain the second approximation for U_k, V_k, ω and the second approximation for x, y in form (2).

Algebraic equations for the corrections $u_k^{(2)}, v_k^{(2)}, \nu^{(2)}$ are formed in the similar way. The left-hand sides of these equations differ from the left-hand sides of equations in $u^{(1)}, v^{(1)}, \nu^{(1)}$ only in their super- or subscripts: (2) instead (1); on the right-hand sides we obtain the following functions:

$$\tilde{\Phi}_k(U^{(2)}, V^{(2)}) - \tilde{\Phi}_k(U^{(1)}, V^{(1)}) - \left(\partial \tilde{\Phi}_k / \partial U\right)_1 u^{(1)} - \left(\partial \tilde{\Phi}_k / \partial V\right)_1 v^{(1)},$$

$$\tilde{F}_k(U^{(2)}, V(2)) - \tilde{F}_k(U^{(1)}, V^{(1)}) - \left(\partial \tilde{F}_k / \partial U)\right)_1 u^{(1)} - \left(\partial \tilde{F}_k / \partial V)\right)_1 v^{(1)}.$$

We obtain, after calculation of $u_k^{(2)}, v_k^{(2)}, \nu_k^{(2)}$, the third approximation, etc. The computations are ended when differences between two adjacent approximations for ω and for all U_k, V_k are less, in norm, than a given quantity δ. These iterations possess quadratic type of convergence in relation to small mass μ if we leave out of account possible small divisors $(k, \omega^{(j)})$. Taking into consideration results of KAM-theory we could hope that quadratic convergence will compensate above mentioned small divisors in absence of an acute resonance between ω_1^0, ω_2^0.

Certainly, it arises the question about practical effectiveness of proposed algorithms in the course of immediate computations. If results are positive, the algorithm may be complicated for the purpose of considering 3-dimensional three-body problem and also to leave arbitrary some quantities as, for example, the mass μ.

References

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