Proceedings of the Edinburgh Mathematical Society (1991) 34, 19–29 ©

ON THE EXTENSIONS OF SOME CLASSICAL DISTRIBUTIONS*

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(Received 13th December 1988)

Some properties of polynomials associated with strong distribution functions are given, including conditions for the polynomials to satisfy a three term recurrence relation. Strong distributions that are extensions to the four classical distributions are given as examples.

1980 *Mathematics subject classification* (1985 Revision): 30B70, 33A65

1. Introduction

We consider distribution functions whose moments exist for positive and negative values. That is functions $\psi(t)$ which are bounded and non-decreasing in $(-\infty,\infty)$ and for which the moments

$$\mu_n=\int_{-\infty}^{\infty} t^n\, d\psi(t)$$

are finite for $n=0,\ \pm 1,\ \pm 2,\dots$. Such functions have been described as strong distribution functions because they arise as solutions of strong moment problems (see [1, 2]). The distribution is called symmetric if all the odd order moments are zero and is called a positive half distribution if all the points of increase are on the positive real axis.

The Hankel determinants are defined by

$$H_r^{(m)}=\begin{vmatrix} \mu_m & \mu_{m+1} & \cdots & \mu_{m+r-1} \\ \mu_{m+1} & \mu_{m+2} & \cdots & \mu_{m+r} \\ \vdots & \vdots & & \vdots \\ \mu_{m+r-1} & \mu_{m+r} & & \mu_{m+2r-2} \end{vmatrix}.$$

for all positive and negative m and $r\geqq 1$, with

*This work was supported by a grant from FAPESP of Brazil.

https://doi.org/10.1017/S0013091500004971 Published online by Cambridge University Press

$$H_{-1}^{(m)}=0 \quad \text{and} \quad H_0^{(m)}=1.$$

For any strong distribution

$$H_r^{(2m)}>0, \quad r \geqq 0, \quad m=0, \pm 1, \pm 2, \ldots .$$

In the case of a positive half distribution we also have

$$H_r^{(2m+1)}>0, \quad r \geqq 0, \quad m=0, \pm 1, \pm 2, \ldots,$$

while for symmetric distributions

$$H_{2r+1}^{(2m+1)}=0 \quad \text{and} \quad (-1)^r H_{2r}^{(2m+1)}>0, \quad r \geqq 0, \quad m=0, \pm 1, \pm 2, \ldots .$$

The first of these latter results is because the columns of $H_j^{(k)}$ are linearly dependent if both j and k are odd. The second follows from the well known Jacobi identity

$$\{H_r^{(m)}\}^2 - H_r^{(m-1)} H_r^{(m-1)} + H_{r+1}^{(m-1)} H_{r-1}^{(m+1)} = 0.$$

2. Polynomials related to strong distributions

Given a strong distribution function $\psi(t)$ we define the polynomials $\{Q_n(z)\}_0^\infty$ by

$$\int_{-\infty}^{\infty} t^{-2[n/2]+s} Q_n(t)\, d\psi(t) = 0 \qquad 0 \leqq S \leqq n-1$$

$$= \gamma_n, \qquad s=n \tag{2.1}$$

for $n \geqq 1$, with $Q_0(z)=1$, and $[x]$ denotes integer part of x.

In monic form the polynomials can be expressed as

$$Q_{2n}(z)=\frac{1}{H_{2n}^{(-2n)}} \begin{vmatrix} \mu_{-2n} & & \cdots & \mu_0 \\ \vdots & & & \vdots \\ \mu_{-1} & & \cdots & \mu_{2n-1} \\ 1 & z & \cdots & z^{2n} \end{vmatrix}$$

$$Q_{2n+1}=\frac{1}{H_{2n+1}^{(-2n)}} \begin{vmatrix} \mu_{-2n} & & \cdots & \mu_1 \\ \vdots & & & \vdots \\ \mu_0 & & \cdots & \mu_{2n-1} \\ 1 & z & \cdots & z^{2n+1} \end{vmatrix}$$

and, further,

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$$\gamma_{2n}=H_{2n+1}^{(-2n)}/H_{2n}^{(-2n)},\quad \gamma_{2n+1}=H_{2n+2}^{(-2n)}/H_{2n+1}^{(-2n)}.$$

The existence of the polynomials is guaranteed by the positivity of $H_r^{(2m)}$, $r\geqq 0$, $m=0,\pm 1,\ldots$, and clearly all γ_k are positive. It is not difficult to show that the zeros of $Q_n(z)$ are real and distinct, for all values of $n\geqq 1$.

A second sequence of polynomials is then defined in the usual way by

$$P_n(z)=\int_{-\infty}^{\infty}\frac{Q_n(z)-Q_n(t)}{z-t}\,d\psi(t),\quad n\geqq 0 \tag{2.2}$$

and clearly $P_n(z)$ is a polynomial of degree $n-1$ with leading coefficient μ_0.

Strong positive half distributions and strong symmetric distributions belong to those distribution functions for which the following result holds.

Theorem. *Let $\psi(t)$ be a strong distribution function such that*

$$H_{2n}^{(-2n+1)}\neq 0,\quad n\geqq 0.$$

The polynomials $Q_n(z)$ and $P_n(z)$ each satisfy the three term recurrence relations

$$\begin{aligned} R_{2n}(z)&=(z-\beta_{2n})R_{2n-1}(z)-\alpha_{2n}R_{2n-2}(z)\\ R_{2n+1}(z)&=\{(1+\alpha_{2n+1})z-\beta_{2n+1}\}R_{2n}(z)-\alpha_{2n+1}z^2R_{2n-1}(z)\end{aligned} \tag{2.3}$$

for $n\geqq 1$ with $Q_0(z)=1, Q_1(z)=z-\mu_1/\mu_0$, $P_0(z)=0$ and $P_1(z)=\mu_0$. The coefficients are given by

$$\alpha_{2n}=\left\{\frac{H_{2n}^{(-2n+1)}}{H_{2n-1}^{(-2n+2)}}\right\}^2\frac{H_{2n-2}^{(-2n+2)}}{H_{2n}^{(-2n)}},\quad \beta_{2n}=\frac{H_{2n}^{(-2n+1)}H_{2n-1}^{(-2n+1)}}{H_{2n-1}^{(-2n+2)}H_{2n}^{(-2n)}}$$

$$\alpha_{2n+1}=\frac{H_{2n+1}^{(-2n)}H_{2n-1}^{(-2n+2)}}{\{H_{2n}^{(-2n+1)}\}^2},\quad \beta_{2n+1}=\frac{H_{2n+1}^{(-2n+1)}H_{2n}^{(-2n)}}{H_{2n+1}^{(-2n)}H_{2n}^{(-2n+1)}}$$

for $n\geqq 1$.

Proof. First for the odd index, write

$$A(z)=\{Q_{2n+1}(z)-zQ_{2n}(z)\}-\alpha_{2n+1}z\{Q_{2n}(z)-zQ_{2n-1}(z)\},$$

a polynomial of degree $2n$ at most, as

$$A(z)=-\beta_{2n+1}Q_{2n}(z)+B(z),$$

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where $B(z)$ is some polynomial of degree $2n-1$ at most. Hence from (2.1) it follows that

$$\int_{-\infty}^{\infty} t^s B(t)\, d\psi(t) = \begin{cases} 0, & s=-2n, -2n+1, \dots, -2 \\ -\gamma_{2n}-\alpha_{2n+1}(\gamma_{2n}-\gamma_{2n-1}) & s=-1. \end{cases}$$

Since $H_{2n}^{(-2n)}$ is non zero, then choosing α_{2n+1} such that

$$\gamma_{2n}+\alpha_{2n+1}(\gamma_{2n}-\gamma_{2n-1})=0$$

means that $B(z)$ is identically zero. This gives the required three term relation. Further, as γ_{2n} is positive, choosing α_{2n+1} in this way is possible only if $\gamma_{2n}-\gamma_{2n-1}\neq 0$. Expressing γ_{2n} and γ_{2n-1} in terms of the Hankel determinants and using the Jacobi identity we find that $\gamma_{2n}-\gamma_{2n-1}\neq 0$ if $H_{2n}^{(-2n+1)}\neq 0$. In this case α_{2n+1} can be given as in the theorem. With this choice of α_{2n+1} the value of β_{2n+1} can be found by considering the integral equation

$$\int_{-\infty}^{\infty} t^{-2n-1} A(z)\, d\psi(t) = -\beta_{2n+1} \int_{-\infty}^{\infty} t^{-2n-1} Q_{2n}(z)\, d\psi(t).$$

The expression for the even index is verified in a similar fashion by considering

$$Q_{2n}(z)-zQ_{2n-1}(z) = -\beta_{2n}Q_{2n-1}(z)-\alpha_{2n}Q_{2n-2}(z)+B(z),$$

where $B(z)$ is some polynomial of degree $2n-3$ at most.

Having established the recurrence relations for the $Q_n(z)$, we then use the definition (2.2) of $P_n(z)$ to show that they also satisfy the relations. □

The above recurrence relations indicate that the ratios $P_n(z)/Q_n(z)$ are, for $n=1,2,3,\dots$, the successive convergents of the continued fraction.

$$\frac{\mu_0}{z-\beta_1} - \frac{\alpha_2}{z-\beta_2} - \frac{\alpha_3 z^2}{(1+\alpha_3)z-\beta_3} - \frac{\alpha_4}{z-\beta_4} - \frac{\alpha_5 z^2}{(1+\alpha_5)z-\beta_5} - \frac{\alpha_6}{z-\beta_6} - \cdots.$$

From the definition of $P_n(z)$ we see that

$$\frac{P_n(z)}{Q_n(z)} = \int_{-\infty}^{\infty} \frac{1}{z-t}\, d\psi(t) - \frac{1}{Q_n(z)} \int_{-\infty}^{\infty} \frac{Q_n(t)}{z-t}\, d\psi(t).$$

Expanding the integrand in the second integral in inverse powers of z and using the orthogonality properties of $Q_n(z)$ yields

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$$\frac{P_{2n}(z)}{Q_{2n}(z)}=\int_{-\infty}^{\infty}\frac{1}{z-t}\,d\psi(t)+O\left(\frac{1}{z^{2n+1}}\right)\quad n\geqq 1$$

and (2.4)

$$\frac{P_{2n+1}(z)}{Q_{2n+1}(z)}=\int_{-\infty}^{\infty}\frac{1}{z-t}\,d\psi(t)+O\left(\frac{1}{z^{2n+3}}\right)\quad n\geqq 0.$$

The symbol $O(1/z^r)$ denotes a power series in inverse powers of z starting with $1/z^r$.

Since

$$Q_{2n}(0)=\frac{H_{2n}^{(-2n+1)}}{H_{2n}^{(-2n)}}$$

then under the condition of the above theorem, $Q_{2n}(0)\neq 0$. On the other hand $Q_{2n+1}(0)$ may be zero, but, if it is, we can show from the linear system of equations yielded by (2.1) that $Q'_{2n+1}(0)\neq 0$. With these results we can expand the ratio $P_n(z)/Q_n(z)$ in powers of z and obtain

$$\frac{P_{2n}(z)}{Q_{2n}(z)}=\int_{-\infty}^{\infty}\frac{1}{z-t}\,d\psi(t)+O(z^{2n})\quad n\geqq 1$$

and (2.5)

$$\frac{P_{2n+1}(z)}{Q_{2n+1}(z)}=\int_{-\infty}^{\infty}\frac{1}{z-t}\,d\psi(t)+O(z^{2n-1})\quad n\geqq 0.$$

3. Examples

1. The strong Tchebycheff distribution. We first consider the distribution function $\psi_T(t)$ given by

$$d\psi_T(t)=\frac{|t|}{\sqrt{b^2-t^2}\,\sqrt{t^2-a^2}},\quad t\in B\equiv[-b,-a]\cup[a,b]$$

$$=0,\qquad\qquad t\notin B$$

with $0<a<b<\infty$.

In the limit as $a\to 0$ and $b\to 1$ the distribution becomes the Tchebycheff distribution and so we may view it as an extension to this distribution. Further, since $a>0$, the function has finite moments of negative order and thus we refer to $\psi_T(t)$ as a strong Tchebycheff distribution. We have the following result.

Theorem. *For the strong Tchebycheff distribution function $\psi_T(t)$ defined above the polynomials $Q_n(z)$ and $P_n(z)$ satisfy the three term recurrence relation (2.3) with α_n and β_n given by*

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$$\beta_n = 0, \qquad \alpha_{2n} = \gamma, \qquad n \geqq 1,$$

$$\alpha_3 = \frac{1}{2}\frac{\lambda^2}{\gamma}, \qquad \alpha_{2n+1} = \frac{1}{4}\frac{\lambda^2}{\gamma}, \qquad n \geqq 2,$$

where $\gamma = ab$ and $\lambda = (b-a)$.

Proof. Consider the continued fraction

$$\frac{\mu_0^T}{z} - \frac{a_2}{z} - \frac{2a_1 z^2}{(1+2a_1)z} - \frac{a_2}{z} - \frac{a_1 z^2}{(1+a_1)z} - \frac{a_2}{z} - \cdots \tag{3.1}$$

in which $a_2 = \gamma$ and $a_1 = \lambda^2/(4\gamma)$.

As the coefficients of (3.1) are bounded then the continued fraction coverges uniformly to an analytic function over every bounded closed region in the upper half plane $\mathrm{Im}(z) > 0$. See [4, Theorem 9]. Denoting this function by $F(z)$ then

$$F(z) = \frac{\mu_0^T}{z} - \frac{a_2}{z} - \frac{2a_1 z^2}{(1+2a_1)z} - f(z),$$

where $f(z)$ is a 2-periodic continued fraction which can be written as

$$f(z) = \frac{a_2}{z} - \frac{a_1 z^2}{(1+a_1)z} - f(z).$$

Solving for $f(z)$ yields

$$f(z) = \frac{(z^2+a_2) \pm \sqrt{(z^2+a_2)^2 - 4a_2(1+a_1)z^2}}{2z}.$$

If we now choose

$$a = \sqrt{a_2}\,\{\sqrt{1+a_1} - \sqrt{a_1}\},$$

and

$$b = \sqrt{a_2}\,\{\sqrt{1+a_1} + \sqrt{a_1}\},$$

then clearly $a_2 = \gamma$ and $a_1 = \lambda^2/(4\gamma)$ and we have

$$f(z) = \frac{1}{2z}\{(z^2+ab) \pm \sqrt{z^2-b^2}\,\sqrt{z^2-a^2}\}.$$

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The function $f(z)$ has two values but only one of them is appropriate since $F(z)$ must take one value only. We note that $\operatorname{Im} F(z)<0$ whenever $\operatorname{Im}(z)>0$, see [4]. Consequently

$$f(z)=\frac{1}{2z}\{(z^2+ab)-\sqrt{z^2-b^2}\sqrt{z^2-a^2}\}$$

and

$$F(z)=\mu_0^T z/\{\sqrt{z^2-b^2}\sqrt{z^2-a^2}\}. \tag{3.2}$$

The function $F(z)$ can be written alternatively as

$$F(z)=\frac{\mu_0^T}{\pi}\int_B \frac{1}{z-t}\frac{|t|}{\sqrt{(b^2-t^2)}\sqrt{(t^2-a^2)}}\,dt,$$

a result given in Van Assche [5].

We can show that $\mu_0^T=\pi$ and hence

$$F(z)=\int_{-\infty}^{\infty}\frac{1}{z-t}\,d\psi_T(t). \tag{3.3}$$

Hence the continued fraction converges to the Stieltjes function of the strong Tchebycheff distribution. Using results given in [3] we can then show that the convergents $P_n(z)/Q_n(z)$ of (3.1) satisfy (2.4) and (2.5) for this distribution. It is then easy to show that $Q_n(z)$ and $P_n(z)$ satisfy (2.1) and (2.2) respectively, see [2]. This completes the proof.

We can express a and b in terms of γ and λ,

$$b=\frac{\gamma}{a}=\lambda+\frac{\gamma}{\lambda+}\frac{\gamma}{\lambda+}\frac{\gamma}{\lambda+}\cdots.$$

Also, by expanding the right hand side of (3.2) we see that the moments of $\psi_T(t)$ satisfy

$$\mu_{2n}^T=\frac{\pi}{4^n}\sum_{j=0}^{n}\sigma_j\sigma_{n-j}(a^2)^j(b^2)^{n-j},$$

$$\mu_{-2n-1}^T=\mu_{2n+1}^T=0,\quad \mu_{-2n-2}^T=\mu_{2n}^T/(ab)^{2n+1},$$

for $n\geqq 0$, where $\sigma_j=(2j)!/(j!)^2$.

2. The strong Legendre distribution. Next we consider

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$$\begin{aligned} d\psi_{Le}(t) &= dt, \qquad t \in B \equiv [-b,-a] \cup [a,b] \\ &= 0, \qquad t \notin B \end{aligned}$$

again with $0<a<b<\infty$.

The moments $\mu_n^{Le}, n=0,\pm 1,\pm 2,\ldots$ of this distribution are easily found. As $\psi_{Le}(t)$ is a symmetric distribution function the polynomials $Q_n(z)$ and $P_n(z)$ each satisfy (2.3). Numerical evidence suggests that

$$\beta_n=0, \quad \alpha_{2n}=\gamma, \quad \alpha_{2n+1}=\frac{\lambda^2}{\gamma}\cdot\frac{n^2}{4n^2-1}, \quad n\geqq 1, \tag{3.4}$$

where $\gamma=ab$ and $\lambda=(b-a)$.

The coefficients of the continued fraction

$$\frac{\mu_0^{Le}}{z} - \frac{\alpha_2}{z} - \frac{\alpha_3 z^2}{(1+\alpha_3)z} - \frac{\alpha_4}{z} - \frac{\alpha_5 z^2}{(1-\alpha_5)z} - \cdots$$

are bounded and hence the continued fraction converges uniformly over every bounded closed domain in the upper half plane $\operatorname{Im}(z)>0$. (See [4]). Hence, if the values in (3.4) are correct then the continued fraction converges to

$$\int_B \frac{1}{z-t}\, d\psi_{Le}(t).$$

In the case when $z=i$ we would then have

$$\tan^{-1}\left(\frac{\lambda}{1+\gamma}\right)=\frac{\lambda}{1}+\frac{a_2}{1}+\frac{a_3}{1}+\frac{a_4}{1}+\cdots$$

where

$$a_{2n+2}=\frac{\gamma^2(4n^2-1)}{(\lambda^2+4\gamma)n^2-\gamma}, \quad n\geqq 0$$

$$a_{2n+1}=-\frac{\lambda^2 n^2}{(\lambda^2+4\gamma)n^2-\gamma}, \quad n\geqq 1.$$

Taking the even contraction leads, after some manipulation, to the well known expansion

$$\tan^{-1}x=\frac{x}{1}+\frac{1^2x^2}{3}+\frac{2^2x^2}{5}+\frac{3^2x^2}{7}+\cdots.$$

A second result in support of (3.4) is the asymptotic behaviour of α_n. From an analysis similar to that given in Van Assche [5], of the three term recurrence relation (2.3), we find that

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$$\sqrt{\alpha_{2n}}\,\{\sqrt{1+\alpha_{2n+1}}-\sqrt{\alpha_{2n+1}}\}\to a$$

$$\sqrt{\alpha_{2n}}\,\{\sqrt{1+\alpha_{2n+1}}+\sqrt{\alpha_{2n+1}}\}\to b$$

and clearly the expressions in (3.4) are compatible with these limits.

3. The strong Hermite Distribution. Thirdly we set

$$d\psi_H(t)=e^{-(t^2+a^2/t^2)/2}\,dt \quad -\infty<t<\infty$$

with $0<a<\infty$. In this case the moments μ_n^H satisfy

$$\mu_0^H=\sqrt{2\pi}/e^a,\quad \mu^H_{-2n-1}=\mu^H_{2n+1}=0,$$

$$\mu^H_{-2n-2}=\mu^H_{2n}/a^{2n+1},$$

$$\mu^H_{2n+2}=(2n+1)\mu^H_{2n}+a^2\mu^H_{2n-2},$$

for $n\geqq 1$. We can also give μ^H_{2n} explicitly as

$$\mu^H_{2n}=\frac{\sqrt{2\pi}}{2^{3n}e^a}\sum_{r=0}^{n}\binom{2n+1}{2r+1}\sum_{s=0}^{r}\binom{r}{s}(8a)^s\frac{(2n-2s)!}{(n-s)!},\quad n\geqq 0.$$

The distribution function $\psi_H(t)$ is symmetric and hence the associated polynomials $Q_n(z)$ and $P_n(z)$ satisfy (2.3). Here computational evidence seems to suggest that

$$\beta_n=0,\quad \alpha_{2n}=a\quad \text{and}\quad \alpha_{2n+1}=\frac{n}{a},\quad n\geqq 1.$$

Again we do not have any analytic proof of this result. However as before, we are able to conjecture that it is true.

If the result is correct then

$$\int_{-\infty}^{\infty}\frac{1}{z-t}\,d\psi_H(t)=\frac{\mu_0^H}{z}\;\underset{-}{}\;\frac{a}{z}\;\underset{-}{}\;\frac{(1/a)z^2}{(1+1/a)z}\;\underset{-}{}\;\frac{a}{z}\;\underset{-}{}\;\frac{(2/a)z^2}{(1+2/a)z}\;\underset{-}{}\;\frac{a}{z}\;-\cdots.$$

This continued fraction is uniformly convergent over all bounded closed regions in the half plane $\mathrm{Im}(z)>0$. (See [4, Theorem 9]). Hence, by taking the even part of this continued fraction, we find

$$\int_{-\infty}^{\infty}\frac{1}{z-t}\,d\psi_H(t)=\frac{\mu_0^H}{z^2-a}\;\underset{-}{}\;\frac{z^2}{z^2-a}\;\underset{-}{}\;\frac{2z^2}{z^2-a}\;\underset{-}{}\;\frac{3z^2}{z^2-a}\;-\cdots. \tag{3.5}$$

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Then substituting $z/(z^2-a)=\pm i$, we get

$$1-\sqrt{\pi/2}\left\{\int_0^\infty \frac{1}{1+t^2}e^{-\frac{1}{2}t^2}\,dt\right\}=\frac{1}{1}+\frac{2}{1}+\frac{3}{1}+\frac{4}{1}+\cdots.$$

This expansion is correct, and can also be obtained from the J-fraction expansion of

$$\int_{-\infty}^{\infty} e^{-t^2/2}\,dt/(z-t),$$

by letting $z=i$.

4. A strong Laguerre distribution. Finally we consider

$$\begin{aligned} d\psi_{La}(t) &= t^{-\frac{1}{2}}\,e^{-(t+a^2/t)/2}\,dt, && 0<t<\infty \\ &=0, && -\infty<t\leqq 0 \end{aligned}$$

with $0<a<\infty$.

This is a positive half distribution and hence the polynomials $Q_n(z)$ and $P_n(z)$ do satisfy (2.3). It appears that

$$\begin{aligned} \beta_{2n-1} &= 2n-1+a, & \beta_{2n} &= a, \\ \alpha_{2n-1} &= (2n-2)/a, & \alpha_{2n} &= (2n-1)a, \end{aligned}$$

for $n\geqq 1$.

These results essentially follow from the strong Hermite case. Substituting $z^2=z$ and $t^2=u$ in (3.5) yields an M-fraction expansion for

$$\int_0^\infty t^{-1/2}\,e^{-1/2(t+a^2/t)}\,dt/(z-1).$$

The coefficients of this M fraction can then be used to derive the α_j and β_j.

The moments μ_n^{La} of this distribution function satisfy

$$\mu_n^{La}=\mu_{2n}^{H}$$

for all positive and negative n.

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