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Representation of Banach Ideal Spaces and Factorization of Operators

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Abstract. Representation theorems are proved for Banach ideal spaces with the Fatou property which are built by the Calderón–Lozanovskiĭ construction. Factorization theorems for operators in spaces more general than the Lebesgue L^p spaces are investigated. It is natural to extend the Gagliardo theorem on the Schur test and the Rubio de Francia theorem on factorization of the Muckenhoupt A_p weights to reflexive Orlicz spaces. However, it turns out that for the scales far from L^p-spaces this is impossible. For the concrete integral operators it is shown that factorization theorems and the Schur test in some reflexive Orlicz spaces are not valid. Representation theorems for the Calderón–Lozanovskiĭ construction are involved in the proofs.

0 Introduction

Let (X_0, X_1) be a compatible couple of Banach spaces and let $F(X_0, X_1)$ be a Banach space intermediate between X_0 and X_1 generated by an construction F (maybe interpolation one). The so-called *inverse interpolation problem* is the problem whether the space $F(X_0, X_1)$ may be obtained by the same functor F from other compatible Banach couples having some additional properties.

Consider also a *uniqueness problem* for the fixed construction or interpolation method F, that is, the following problem: Does the equality $F(X_0, X_1) = F(Y_0, Y_1)$ with equivalent norms for arbitrary Banach spaces X_0, X_1, Y_0, Y_1, or for the spaces from a given class, imply that $X_0 = Y_0$ and $X_1 = Y_1$ with equivalence of their norms?

The above two problems are difficult to solve, therefore it makes sense to consider their simpler versions. In this paper we consider a *uniqueness problem* for the fixed construction or interpolation method F, sometimes also called the *representation of the space* $F(\cdot)$, by asking the following question:

Does the equality $F(X_0, X_1) = F(X_0, X_2)$ *hold with equivalent norms for arbitrary Banach spaces* X_0, X_1, X_2, *or for the spaces from a given class, imply that* $X_1 = X_2$ *with equivalent norms?*

The non-uniqueness of the real and the complex method are well known. Already, Grisvard, Seeley and others (see [53], pages 320–321 for references) have considered

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real and complex interpolation with boundary conditions and proved that for $0 < \theta < 1/2$,

$$[L^2(\Omega), W^{1,2}(\Omega)]_\theta = [L^2(\Omega), W_0^{1,2}(\Omega)]_\theta = W^{\theta,2}(\Omega),$$

where $\Omega \subset R^n$ is a bounded C^∞-domain and $W_0^{\theta,2}(\Omega) = \{x \in W^{\theta,2}(\Omega) : u_{|\partial\Omega} = 0\}$, with $W^{\theta,2}$ being the usual Sobolev space.

For the real interpolation construction, Lions–Magenes (see [28, Theorem 11.1, p. 55]) proved that for $0 < \theta < 1/2$,

$$\big(L^2(\Omega), W^{1,2}(\Omega)\big)_{\theta,2} = \big(L^2(\Omega), W_0^{1,2}(\Omega)\big)_{\theta,2} = W^{\theta,2}(\Omega),$$

where $\Omega = \{(x,y) : x^2 + y^2 < 1\}$. For the real interpolation method we can also take Triebel's example connected with the negative answer for the problem of interpolation of intersections. For the weight function $w(t) = \min(t, 1-t)^{-1/2}$, $t \in (0,1)$ and the spaces on $(0,1)$ we have for $1/2 < \theta < 1$,

$$(L^2, W^{1,2} \cap L^2_w)_{\theta,2} = (L^2, W_0^{1,2})_{\theta,2} = W_0^{\theta,2},$$

where $W_0^{\theta,2}$ is the closure of $C_0^\infty(0,1)$ in $W^{\theta,2}$. Moreover, Wallsten has given an example of a space M (*cf.* [54]) for which $(M, L^\infty)_{\theta,p} = (L^1, L^\infty)_{\theta,p}$ for $1/2 < \theta < 1$.

In the seventies, Fefferman–Stein, Rivière–Sagher, Hanks, Bennett–Sharpley and others (see [2, § 5.6–5.7] for results and references) proved equalities for the complex interpolation method:

$$[H^1, L^p]_\theta = L^q = [L^1, L^p]_\theta \text{ and } [L^p, \mathrm{BMO}]_\theta = L^{\frac{p}{1-\theta}} = [L^p, L^\infty]_\theta$$

for all $0 < \theta < 1$, $1 < p < \infty$ and $1/q = 1 - \theta + \theta/p$, and for the real interpolation method:

$$\begin{aligned}(L^1, L^\infty)_{\theta,p} &= (\operatorname{Re} H^1, L^\infty)_{\theta,p} = (L^1, \mathrm{BMO})_{\theta,p} \\ &= (\operatorname{Re} H^1, L^\infty)_{\theta,p} = (\operatorname{Re} H^1, \mathrm{BMO})_{\theta,p} = L^{\frac{1}{1-\theta},p}\end{aligned}$$

for all $0 < \theta < 1$, $1 \le p \le \infty$, where $L^{\frac{1}{1-\theta},p}$ are classical Lorentz spaces. We also mention that for an arbitrary couple of Banach spaces X_0 and X_1, the equalities

$$[\tilde{X}_0, \tilde{X}_1]_\theta = [X_0, X_1]_\theta = [X_0^0, X_1^0]_\theta \text{ and } (\tilde{X}_0, \tilde{X}_1)_{\theta,p} = (X_0, X_1)_{\theta,p}$$

hold isometrically for all $0 < \theta < 1$, $1 \le p \le \infty$, where $\tilde{X}_i$ is the Gagliardo completion of X_i and X_i^0 is the closure of $X_0 \cap X_1$ in X_i for $i = 0, 1$.

Let us point out that the situation may be quite different if we assume from the beginning that all the spaces in the problem have some common structure, for example, they are Banach lattices on a given measure space (Banach ideal spaces). This phenomenon, that some problem has negative solution for general Banach spaces but positive answer in the class of ideal Banach spaces with the Fatou property, was first observed in [33, 34] in connection to Peetre's problem on interpolation of intersections.

Cwikel and Nilsson [14] showed the uniqueness theorem for the Calderón construction $F(X_0, X_1) = X_0^{1-\theta} X_1^{\theta}$ when X_0, X_1, X_2 are Banach ideal spaces and all have the Fatou property. Their arguments used in the proof are related to ideas in a theorem of Pisier [45]. Some related results for finite dimensional Banach spaces were considered by Rochberg [47].

The inverse interpolation problem for the real method of interpolation $F = (\cdot)_\Phi$ on some class of Banach ideal spaces was investigated in [5]. One of the results shows that if X is a symmetric space on $(0, \infty)$ and $(X, L^\infty)_{\theta,p} = (L^p, L^\infty)_{\theta,p}$, then the fundamental function of X is equivalent to $t^{1/p}$.

Cwikel–Nilsson [14, p. 45] and Asekritova–Krugljak [1, p. 114] proved the following uniqueness theorem for the real interpolation method:

Let X_0, X_1, Y_0, Y_1 be Banach ideal spaces. If $(X_0, X_1)_{\theta_i,p_i} = (Y_0, Y_1)_{\theta_i,p_i}$, $i = 0, 1$, for some $\theta_0, \theta_1 \in (0, 1), \theta_0 \neq \theta_1$ and $p_0, p_1 \in [1, \infty]$, then $(X_0, X_1)_{\theta,p} = (Y_0, Y_1)_{\theta,p}$ for all $\theta \in (0, 1)$ and $p \in [1, \infty]$.

In this paper we consider uniqueness results (representation theorems) for the Calderón–Lozanovskiĭ construction $F(X_0, X_1) = \varphi(X_0, X_1)$ with a general function $\varphi \in \mathcal{U}$.

These results have applications in the factorization of operators between Banach ideal spaces. In the theory of integral operators with positive kernels a special role is played by the so-called *Schur lemma* or *Schur test* (see [23, p. 37] or [52, p. 42]), which says that an integral operator $Kx(t) = \int k(t, s)x(s)\, ds$ with a positive kernel $k(t, s) \geq 0$ is bounded in L^p for $1 < p < \infty$ if and only if there exists a positive function u such that

$$Ku^{p'}(t) \leq Cu^{p'}(t) \text{ and } K'u^p(t) \leq Cu^p(t),$$

where K' is a formally associate operator and $1/p' + 1/p = 1$. We can rewrite this in the factorization way: there exists a positive function u (weight function u) such that

$$K\colon L^1_{u^p} \to L^1_{u^p} \text{ and } K\colon L^\infty_{u^{-p'}} \to L^\infty_{u^{-p'}} \text{ is bounded.}$$

In the eighties, interest in statements like the Schur lemma increased after the solution of the factorization problem of Muckenhoupt's A_p-condition on weight by Jones [21], and even stronger after the Rubio de Francia elementary proof of Jones' theorem [13, 48]. These studies were later developed in [9, 12, 16–18, 20, 49]. All these papers contain the factorization problem of various classical operators in weighted L^p spaces.

We will extend factorization theorems from weighted L^p spaces to weighted Banach ideal $X^{(p)}$ spaces, and the factorization will be proved first through the weighted L^∞ spaces. The main factorization problem is to have factorization through the weighted L^1 and weighted L^∞ spaces and this question will be also discussed here.

We prove the factorization result for a sufficiently large class of positive sublinear bounded operators T between L^p spaces through the Lorentz and Marcinkiewicz spaces generated by a certain weight function. Then we show that factorization of the symmetric space $X^{(p)}$ through weighted X and L^∞ is not true for the positive sublinear Hardy operator.

The failure of the main factorization theorem in Calderón–Lozanovskiĭ spaces generated by a non-power function is proved for the Volterra operator and the averaging operator. This shows that we cannot go far from the scale of L^p spaces with the factorization theorems.

Finally, we show that the Schur lemma is not true in some reflexive Orlicz spaces for the classical Hardy operator; that is, we can construct reflexive Orlicz spaces in which the classical Hardy operator is bounded (it is bounded even in any reflexive Orlicz space) but the factorization through weighted L^1 and weighted L^∞ spaces is not possible.

Let us mention that a quite different question, called also the *Lions problem*, about the effective dependence of a given family of spaces on its function parameter φ, was considered for the complex method of interpolation by Stafney [51], for the real method of interpolation in [1, 8, 19], for the Calderón–Lozanovskiĭ construction $\varphi(\cdot)$ in [6], and for the Gustavsson–Peetre construction $G_\varphi(\cdot)$ in [7]. The question for the Calderón–Lozanovskiĭ construction is: when are the spaces $\varphi_0(X_0, X_1)$ and $\varphi_1(X_0, X_1)$ different for $\varphi_0 \neq \varphi_1$?

The content of the paper is as follows: In Section 1 we define the Banach ideal spaces and the Calderón–Lozanovskiĭ construction and collect their properties including the Lozanovskiĭ factorization theorem.

In Section 2 we prove representation theorems, called also uniqueness theorems, for the Calderón–Lozanovskiĭ construction generated by different Banach ideal spaces or the weighted L^p spaces. The main representation theorems (Theorems 1–4) show that under some little assumption on φ the equality $\varphi(X_0, X_1) = \varphi(X_0, X_2)$ with equivalent norms implies that $X_1 = X_2$ with equivalent norms, and the equality $\varphi(L^1_u, L^\infty_v) = \varphi(L^1_w, L^\infty)$ with equivalent norms implies the equivalence of weights $w^\theta \approx u^\theta v^{1-\theta}$ for some $0 \leq \theta \leq 1$.

Section 3 contains pointwise estimates for positive sublinear operators. Factorization results in weighted $X^{(p)}$ spaces are presented there. They are extensions of the corresponding results of Hernández [17, 18] for weighted L^p spaces. The main tool in the proofs is Lemma 6 of the Gagliardo and Rubio de Francia type.

For a large class of positive sublinear operators T which are bounded between L^p spaces we show a factorization of T through the Lorentz and Marcinkiewicz spaces generated by a certain weight function.

We also prove that the positive sublinear Hardy operator bounded between symmetric spaces $X^{(p)}$ cannot be factorized by a weighted space X and weighted L^∞ when the upper Boyd index of the space X is 1. This example of the Hardy positive sublinear operator shows that without any additional assumptions on an operator the factorization theorem through weighted L^1 and weighted L^∞ spaces cannot be true.

In Section 4, representation theorems are used to show that the factorization problem in Calderón–Lozanovskiĭ spaces generated by a non-power function is not true, in general, for the Volterra operator and in Section 5 the same is done for the averaging operator.

Section 6 contains a counter-example showing that the classical Hardy operator between some reflexive Orlicz spaces cannot be factorized through weighted L^1 and weighted L^∞ spaces. This also shows that the Schur lemma for positive integral operators between some reflexive Orlicz spaces is false. Detailed proofs of the constructions of the functions in the counter-example are collected in an appendix.

Preliminary versions of Theorems 4, 8 and 9 were announced without proofs in [4].

1 Banach Ideal Spaces and the Calderón–Lozanovskiĭ Construction

Let (Ω, μ) be a complete σ-finite measure space and let $L^0(\mu)$ or $L^0(\Omega)$ denote, as usual, the space of all equivalence classes of measurable functions on Ω with the topology of convergence in measure on μ-finite sets. The order $|x| \le |y|$ means that $|x(t)| \le |y(t)|$ for μ-almost all $t \in \Omega$.

A Banach subspace $X = (X, \|\cdot\|_X)$ of $L^0(\mu)$ such that there exists $u \in X$ with $u > 0$ μ-a.e. on Ω and $\|x\|_X \le \|y\|_X$ whenever $|x| \le |y|$ is called a *Banach ideal space* on Ω or on (Ω, μ).

If X is a Banach ideal space on Ω and $w \in L^0(\mu)$ is a weight function on Ω, that is, $w > 0$ a.e. on Ω, we define the *weighted space* X_w by $\|x\|_{X_w} := \|xw\|_X$.

The *associated space* X' to X is the space of all $x \in L^0(\mu)$ such that

$$\int_\Omega |x(t)y(t)|\, d\mu < \infty$$

for every $y \in X$ endowed with the norm

$$\|x\|_{X'} = \sup\Big\{ \int_\Omega |x(t)y(t)|\, dt : \|y\|_X \le 1 \Big\}.$$

X' is a Banach ideal space.

A Banach ideal space X with a norm $\|\cdot\|_X$ has the *Fatou property* if for any increasing positive sequence (x_n) in X with $\sup_n \|x_n\|_X < \infty$ we have that $\sup_n x_n \in X$ and $\|\sup_n x_n\|_X = \sup_n \|x_n\|_X$.

For every Banach ideal space X we have the embedding $X \subset X''$ with $\|x\|_{X''} \le \|x\|_X$ for any $x \in X$. Moreover, $X = X''$ with equality of the norms if and only if X has the Fatou property (*cf.* [25, 27]).

Let $\bar{X} = (X_0, X_1)$ be a couple of Banach ideal spaces on Ω and let $\mathcal{U}$ denote the set of all non-negative, concave and positively homogeneous continuous functions $\varphi\colon [0,\infty) \times [0,\infty) \to [0,\infty)$ such that $\varphi(0,0) = 0$. Then the *Calderón–Lozanovskiĭ construction* or the *Calderón–Lozanovskiĭ spaces* $\varphi(\bar{X}) = \varphi(X_0, X_1)$ consists of all $x \in L^0(\mu)$ such that $|x| \le \lambda\varphi(|x_0|, |x_1|)$ for some $x_i \in X_i$ with $\|x_i\|_{X_i} \le 1$, $i = 0, 1$. The spaces $\varphi(\bar{X})$ are Banach ideal spaces on Ω equipped with the norm

$$\|x\|_\varphi = \inf\{\lambda > 0; |x| \le \lambda\varphi(|x_0|, |x_1|), \|x_0\|_{X_0} \le 1, \|x_1\|_{X_1} \le 1\}$$

(see [30]). In the case of power functions $\varphi_\theta(s,t) = s^{1-\theta}t^\theta$ with $0 \le \theta \le 1$, $\varphi_\theta(\bar{X})$ are the well known Calderón spaces $X_0^{1-\theta}X_1^\theta$ (see [11]). The particular case $X^\theta(L^\infty)^{1-\theta} = X^{(p)}$ for $\theta = 1/p$, $1 \le p < \infty$, is known as the p-convexification of X (see [27, 38]).

The properties of $\varphi(\bar{X})$ were studied by Lozanovskiĭ in [30, 31] (see also [35]), where among other facts is proved the duality result

$$\varphi(X_0, X_1)' = \hat{\varphi}(X_0', X_1')$$

with equivalent norms. Here, for $\varphi \in \mathcal{U}$, the conjugate function $\hat{\varphi}$ is defined by

$$\hat{\varphi}(s,t) := \inf\Big\{ \frac{\alpha s + \beta t}{\varphi(\alpha,\beta)} : \alpha, \beta > 0 \Big\}, \quad s,t \geq 0.$$

We have $\hat{\varphi} \in \mathcal{U}$ and $\hat{\hat{\varphi}} = \varphi$ (see [31, 32] and [35, Lemma 15.8]). Note that

$$\text{(1)} \qquad \frac{1}{\varphi(\frac{1}{s},\frac{1}{t})} = \inf_{\alpha,\beta>0} \frac{\max(\alpha s, \beta t)}{\varphi(\alpha,\beta)} \leq \hat{\varphi}(s,t) \leq \frac{2}{\varphi(\frac{1}{s},\frac{1}{t})}.$$

Lozanovskiĭ also showed that $(X_0^{1-\theta}X_1^{\theta})' = (X_0')^{1-\theta}(X_1')^{\theta}$ with equality of the norms ([29], Theorem 2). Using this equality for $\theta = 1/2$ it was shown in [29] that $X^{1/2}(X')^{1/2} = L^2$ isometrically. From this result follows the Lozanovskiĭ factorization theorem, proved in [29, Theorem 6] (see also [35, p. 185] and [46]):

Theorem A *Let X be a Banach ideal space. Then for every $0 \leq z \in L^1$ and $\varepsilon > 0$ we can find $0 \leq x \in X$ and $0 \leq y \in X'$ such that $z = xy$ and*

$$\|x\|_X \|y\|_{X'} \leq (1+\varepsilon)\|z\|_1.$$

If X has the Fatou property, we may take $\varepsilon = 0$ in the above inequality.

Calderón–Lozanovskiĭ spaces are closely related to Orlicz spaces. Let $M\colon [0,\infty) \to [0,\infty]$ be a nondecreasing, convex and left-continuous function, not identical 0 or ∞ on $(0,\infty)$, with $M(0) = 0$. Let $\varphi \in \mathcal{U}$ be defined by $\varphi(s,t) = tM^{-1}(s/t)$ if $t > 0$ and 0 if $t = 0$, where M^{-1} is the right continuous inverse of M. Then for any Banach ideal space X on Ω, the Calderón–Lozanovskiĭ space $\varphi(X, L^{\infty})$ is the Banach ideal space

$$X^M = \{x \in L^0(\mu); M(|x|/\lambda) \in X \text{ for some } \lambda > 0\}$$

equipped with the norm

$$\|x\|_{X^M} = \inf\{\lambda > 0; \|M(|x|/\lambda)\|_X \leq 1\}.$$

In particular, $\varphi(L^1, L^{\infty})$ coincides isometrically with the Orlicz space L^M (see [10, 35, 44]).

The Calderón–Lozanovskiĭ construction is an exact interpolation method for positive linear or positive sublinear operators (see Berezhnoi [3], Shestakov [50], Maligranda [32]; *cf.* also [35, Theorem 15.13]). For arbitrary linear operators (not necessarily positive) on Banach ideal spaces with the Fatou property, this was proved by Ovchinnikov [43] (see also [10, 35, 42, 44] for the class of quasi-Banach ideal spaces). Some other properties of Calderón–Lozanovskiĭ spaces were investigated in [6, 22, 26].

The equivalence of two weights $u \approx v$ on Ω or $u(t) \approx v(t)$ on Ω will mean that there exists a constant $C > 0$ such that $\frac{1}{C}u(t) \leq v(t) \leq Cu(t)$ for all $t \in \Omega$ μ-a.e. Also $u \approx v$ simply means $u(t) \approx v(t)$ for all $t > 0$.

Equality of two Banach spaces $X = Y$ means equality of X and Y as the sets and also equivalence of their norms.

2 Representation of Calderón–Lozanovskiĭ Spaces

In the proof of the first representation theorem we will need the following lemma:

Lemma 1 *Let X, Y be two Banach ideal spaces with the Fatou property. If the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ are equivalent on $X \cap Y$, i.e., there exists a constant $C > 0$ such that*

$$\frac{1}{C}\|x\|_X \le \|x\|_Y \le C\|x\|_X \quad \textit{for all } x \in X \cap Y,$$

then $X = Y$ with equivalent norms.

Proof Let $x \in X$ with the norm $\|x\|_X \le 1$. In the Banach ideal space $X \cap Y$ with the natural norm $\|x\|_{X \cap Y} = \max\{\|x\|_X, \|x\|_Y\}$ we take a unit function u, that is, the function $u \in X \cap Y$ such that $u(t) > 0$ μ-a.e. on Ω. Then for the sequence of functions defined by

$$x_n(t) = \min\{|x(t)|, nu(t)\}, \quad n \in N$$

we have $x_n \in X \cap Y$, $\|x_n\|_Y \le C$, $0 \le x_n \le x_{n+1}$ and $\lim_{n\to\infty} x_n(t) = |x(t)|$ μ-a.e.

Using the Fatou property of Y we obtain that $x \in Y$ and $\|x\|_Y \le C$. Therefore, $X \subset Y$ and $\|x\|_Y \le C\|x\|_X$ for all $x \in X$. Similarly we can prove the reverse imbedding, and Lemma 1 is proved. ■

Theorem 1 *Let $\varphi \in \mathfrak{U}$. Assume that X_0, X_1 and X_2 are Banach ideal spaces on the same σ-finite measure space (Ω, μ), and suppose that all of the spaces have the Fatou property. If*

$$\varphi(X_0, X_1) = \varphi(X_0, X_2) \tag{2}$$

and

$$\lim_{n\to\infty} \inf_{s>0} \frac{\varphi(1, sn)}{\varphi(1, s)} = \infty, \tag{3}$$

then $X_1 = X_2$.

Proof Assume that $X_1 \ne X_2$. Then, by Lemma 1, we can find a sequence $x_n \in X_1 \cap X_2$ such that $\|x_n\|_{X_1} \le 1$ and $\|x_n\|_{X_2} > n$. Since X_2 has the Fatou property it follows that

$$\sup_{\|y\|_{X_2'}=1} \int_\Omega |x_n(t)y(t)|\, d\mu = \|x_n\|_{X_2''} = \|x_n\|_{X_2} > n,$$

and so we can find a sequence $y_n \in X_2'$, $\|y_n\|_{X_2'} \le 1$ such that $\int_\Omega |x_n(t)y_n(t)|\, d\mu = n$.

Now, by the Lozanovskiĭ factorization theorem (Theorem A), we can find sequences $0 \le \xi_n \in X_0$, $0 \le \eta_n \in X_0'$ such that

$$\|\xi_n\|_{X_0} = 1,\ \|\eta_n\|_{X_0'} = 1 \text{ and } \frac{1}{n}|x_n(t)y_n(t)| = \xi_n(t)\eta_n(t) \quad \mu\text{-a.e. on } \Omega.$$

We have $\varphi(\xi_n(t), |x_n(t)|) \in \varphi(X_0, X_1)$ with $\|\varphi(\xi_n, |x_n|)\|_{\varphi(X_0,X_1)} \leq 1$, and

$$\hat{\varphi}(\eta_n(t), |y_n(t)|) \in \hat{\varphi}(X_0', X_2')$$

with $\|\hat{\varphi}(\eta_n, |y_n|)\|_{\hat{\varphi}(X_0',X_2')} \leq 1$.

Since $\hat{\varphi}(X_0', X_2') = [\varphi(X_0, X_2)]' = [\varphi(X_0, X_1)]'$ it follows that

$$\begin{aligned} A_n &:= \int_\Omega \varphi(\xi_n(t), |x_n(t)|)\hat{\varphi}\big(\eta_n(t), |y_n(t)|\big)\, d\mu \\ &\leq \|\varphi(\xi_n, |x_n|)\|_{\varphi(X_0,X_1)} \|\hat{\varphi}(\eta_n, |y_n|)\|_{\varphi(X_0,X_1)'} \\ &\leq \|\hat{\varphi}(\eta_n, |y_n|)\|_{\varphi(X_0,X_1)'} \\ &\leq C\|\hat{\varphi}(\eta_n, |y_n|)\|_{\hat{\varphi}(X_0',X_2')} \leq C, \end{aligned}$$

which gives that $\sup_{n\in\mathbf{N}} A_n < \infty$.

On the other hand, by an estimate in (1), we have $\hat{\varphi}(1, u)\varphi(1, \frac{1}{u}) \geq 1$ and

$$\begin{aligned} A_n &= \int_\Omega \xi_n(t)\eta_n(t)\varphi\left(1, \frac{|x_n(t)|}{\xi_n(t)}\right)\hat{\varphi}\left(1, \frac{|y_n(t)|}{\eta_n(t)}\right) d\mu \\ &\geq \int_\Omega \xi_n(t)\eta_n(t)\varphi\left(1, \frac{|x_n(t)|}{\xi_n(t)}\right)\frac{1}{\varphi(1, \frac{\eta_n(t)}{|y_n(t)|})}\, d\mu \\ &= \int_\Omega \xi_n(t)\eta_n(t)\frac{\varphi(1, \frac{n\eta_n(t)}{|y_n(t)|})}{\varphi(1, \frac{\eta_n(t)}{|y_n(t)|})}\, d\mu \\ &\geq \inf_{s>0}\frac{\varphi(1, ns)}{\varphi(1, s)}\int_\Omega \xi_n(t)\eta_n(t)\, d\mu = \inf_{s>0}\frac{\varphi(1, ns)}{\varphi(1, s)}, \end{aligned}$$

that is,

$$\sup_{n\in\mathbf{N}} A_n \geq \sup_{n\in\mathbf{N}}\inf_{s>0}\frac{\varphi(1, ns)}{\varphi(1, s)} \geq \lim_{n\to\infty}\inf_{s>0}\frac{\varphi(1, sn)}{\varphi(1, s)} = \infty,$$

which gives a contradiction. Therefore $X_1 = X_2$ and the norms are equivalent. ■

Theorem 1, used in the case of power function $\varphi_\theta(s, t) = s^{1-\theta}t^\theta$ with $0 < \theta < 1$, gives the following corollary, which was proved differently by Cwikel and Nilsson [14, Theorem 3.5].

Corollary 1 *Let $0 < \theta < 1$. If $X_0^{1-\theta}X_1^\theta = X_0^{1-\theta}X_2^\theta$ for Banach ideal spaces X_0, X_1 and X_2 on (Ω, μ) with the Fatou property, then $X_1 = X_2$.*

Remark 1 For concrete spaces, the assumption (3) on φ can be weakened, as we will prove in Theorem 4. Let $\varphi \in \mathcal{U}$ and $\lim_{t\to\infty}\varphi(t, 1) = \infty$. Assume that the measure space (Ω, μ) is nonatomic. If

$$\varphi(L_u^1, L^\infty) = \varphi(L_w^1, L^\infty)$$

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with equivalent norms for some weight functions u, w on Ω, then $u(t) \approx w(t)$ on Ω.

Using Theorem 1 we can prove the following uniqueness theorem for two couples $\bar{X} = (X_0, X_1)$ and $\bar{Y} = (Y_0, Y_1)$ of Banach ideal spaces with the Fatou property.

Theorem 2 *Let $\bar{X} = (X_0, X_1)$ and $\bar{Y} = (Y_0, Y_1)$ be two couples of Banach ideal spaces on the same measure space (Ω, μ) with all spaces having the Fatou property. Suppose that for $\varphi_0, \varphi_1 \in \mathfrak{U}$ we can find $\varphi \in \mathfrak{U}$ such that either*

$$\varphi(\varphi_0(s,1),1) = \varphi_1(s,1) \text{ for all } s > 0 \text{ or } \varphi(1, \varphi_0(1,t)) = \varphi_1(1,t) \text{ for all } t > 0.$$

Assume also that φ satisfies (3) *and either φ_0 or φ_1 satisfies*

$$\lim_{n\to\infty} \inf_{s>0} \frac{\varphi_i(sn,1)}{\varphi_i(s,1)} = \infty. \tag{4}$$

If $\varphi_0(X_0, X_1) = \varphi_0(Y_0, Y_1)$ and $\varphi_1(X_0, X_1) = \varphi_1(Y_0, Y_1)$, then $X_0 = Y_0$ and $X_1 = Y_1$.

Proof By the reiteration formulas (see [35, pp. 180–181]) it yields that

$$\varphi\big(\varphi_0(X_0, X_1), X_1\big) = \varphi_1(X_0, X_1) \text{ and } \varphi\big(\varphi_0(Y_0, Y_1), Y_1\big) = \varphi_1(Y_0, Y_1).$$

From the equalities in the assumption we obtain that

$$\varphi(X, X_1) = \varphi(X, Y_1) \text{ with } X = \varphi_0(X_0, X_1).$$

Using Theorem 1 we obtain that $X_1 = Y_1$ with equivalent norms. Now, if φ_i satisfies (4) for $i = 0$ or $i = 1$, then from the first or the second equality in the assumption and from the just proved equality $X_1 = Y_1$ we have

$$\varphi_i(X_0, X_1) = \varphi_i(Y_0, X_1), \quad i = 0, 1,$$

or

$$\tilde{\varphi}_i(X_1, X_0) = \tilde{\varphi}_i(X_1, Y_0), \quad i = 0, 1,$$

where $\tilde{\varphi}_i(s,t) = \varphi_i(t,s)$. Since the condition (4) for φ_i means the condition (3) for $\tilde{\varphi}_i$ we obtain from Theorem 1 that $X_0 = Y_0$ with equivalent norms, and the proof is complete. ■

As a corollary, we obtain the result proved by Cwikel and Nilsson [14, Theorem 3.1] for the power functions φ_{θ_0} and φ_{θ_1}.

Corollary 2 *If $X_0^{1-\theta_0} X_1^{\theta_0} = Y_0^{1-\theta_0} Y_1^{\theta_0}$ and $X_0^{1-\theta_1} X_2^{\theta_1} = Y_0^{1-\theta_1} Y_1^{\theta_1}$ for some $\theta_0, \theta_1 \in [0,1]$ with $\theta_0 \neq \theta_1$ and for Banach ideal spaces X_0, X_1, Y_0, Y_1 on (Ω, μ), all with the Fatou property, then $X_0 = Y_0$ and $X_1 = Y_1$.*

Theorem 3 *Let X, Y be two Banach ideal spaces on Ω and u, v two weights on Ω. Then for $0 \le \theta \le 1$ we have equality*

$$X^{1-\theta} Y^{\theta} = (X_u)^{1-\theta} (Y_v)^{\theta}$$

if and only if $u(t)^{1-\theta} v(t)^{\theta} \approx C$ on Ω.

Proof Let $a \le u(t)^{1-\theta} v(t)^{\theta} \le b$ for some $a, b > 0$ and all $t \in \Omega$ μ-a.e.

If $x \in X^{1-\theta} Y^{\theta}$ with norm < 1, then

$$|x| \le |x_0|^{1-\theta} |x_1|^{\theta} \text{ with } \|x_0\|_X \le 1 \text{ and } \|x_1\|_Y \le 1,$$

which we can rewrite as

$$|x| \le b \Big|\frac{x_0}{u}\Big|^{1-\theta} \Big|\frac{x_1}{v}\Big|^{\theta} = b |x_0'|^{1-\theta} |x_1'|^{\theta}$$

with $\|x_0'\|_{X_u} = \|x_0\|_X \le 1$ and $\|x_1'\|_{Y_v} = \|x_1\|_Y \le 1$. This means that

$$x \in (X_u)^{1-\theta} (Y_v)^{\theta}$$

with norm $\le b$.

Conversely, if $x \in (X_u)^{1-\theta} (Y_v)^{\theta}$ with norm < 1, then

$$|x| \le |x_0|^{1-\theta} |x_1|^{\theta} \text{ with } \|x_0\|_{X_u} \le 1 \text{ and } \|x_1\|_{Y_v} \le 1,$$

which gives

$$|x| \le \frac{1}{a} |x_0 u|^{1-\theta} |x_1 v|^{\theta} = \frac{1}{a} |x_0'|^{1-\theta} |x_1'|^{\theta}$$

with $\|x_0'\|_X = \|x_0\|_{X_u} \le 1$ and $\|x_1'\|_Y = \|x_1\|_{Y_v} \le 1$, that is, $x \in X^{1-\theta} Y^{\theta}$ with norm $\le \frac{1}{a}$.

To prove the reverse implication assume that $X^{1-\theta} Y^{\theta} = (X_u)^{1-\theta} (Y_v)^{\theta}$. Then, by the duality theorem,

$$(X')^{1-\theta} (Y')^{\theta} = (X'_{1/u})^{1-\theta} (Y'_{1/v})^{\theta},$$

and for any non-negative functions $x_0 \in X$, $x_1 \in Y$, $y_0 \in X'$, $y_1 \in Y'$ from the unit balls, we have

$$\begin{aligned} \int_{\Omega} x_0(t)^{1-\theta} x_1(t)^{\theta} y_0(t)^{1-\theta} y_1(t)^{\theta} u(t)^{1-\theta} v(t)^{\theta} \, d\mu & \\ \le \|x_0^{1-\theta} x_1^{\theta}\|_{X^{1-\theta} Y^{\theta}} \|(y_0 u)^{1-\theta} (y_1 v)^{\theta}\|_{(X^{1-\theta} Y^{\theta})'} & \\ = \|x_0^{1-\theta} x_1^{\theta}\|_{X^{1-\theta} Y^{\theta}} \|(y_0 u)^{1-\theta} (y_1 v)^{\theta}\|_{(X')^{1-\theta} (Y')^{\theta}} & \\ \le C \|x_0^{1-\theta} x_1^{\theta}\|_{X^{1-\theta} Y^{\theta}} \|(y_0 u)^{1-\theta} (y_1 v)^{\theta}\|_{({X'}_{1/u})^{1-\theta} ({Y'}_{1/v})^{\theta}} & \\ \le C \|x_0\|_X^{1-\theta} \|x_1\|_Y^{\theta} \|y_0 u\|_{{X'}_{1/u}}^{1-\theta} \|y_1 v\|_{{Y'}_{1/v}}^{\theta} \le C. & \end{aligned}$$

By the factorization theorem (Theorem A), for any $0 \le z \in L^1(\mu)$ we can find non-negative $x_0 \in X$, $y_0 \in X'$ and $x_1 \in Y$, $y_1 \in Y'$ such that $x_0 y_0 = z$ and $x_1 y_1 = z$. Then

$$\int_{\Omega} z(t) u(t)^{1-\theta} v(t)^{\theta} \, d\mu = \int_0^{\infty} x_0(t)^{1-\theta} x_1(t)^{\theta} y_0(t)^{1-\theta} y_1(t)^{\theta} u(t)^{1-\theta} v(t)^{\theta} \, d\mu \le C,$$

from which we obtain that $u(t)^{1-\theta} v(t)^{\theta} \in L^{\infty}(\mu)$, or equivalently $u(t)^{1-\theta} v(t)^{\theta} \le C$ for all $t \in \Omega$ μ-a.e., since the following implication is true:

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If $0 \le w \notin L^\infty(\mu)$*, then we can find* $0 \le z \in L^1(\mu)$ *for which* $\int_\Omega z(t)w(t)\,d\mu = \infty$.

Now, for the spaces $X_0 = X_u$ and $X_1 = Y_v$ we have by the formula in the assumption that

$$((X_0)_{1/u})^{1-\theta}((X_1)_{1/v})^\theta = X^{1-\theta}Y^\theta = (X_u)^{1-\theta}(Y_v)^\theta = X_0^{1-\theta}X_1^\theta.$$

Therefore the equality $\big((X_0)_{1/u}\big)^{1-\theta}\big((X_1)_{1/v}\big)^\theta = X_0^{1-\theta}X_1^\theta$ holds, from which together with the proof as above we obtain that $(\frac{1}{u})^{1-\theta}(\frac{1}{v})^\theta \in L^\infty(\mu)$ or, equivalently, $u(t)^{1-\theta}v(t)^\theta \ge c > 0$ for all $t \in \Omega$ μ-a.e. Thus $u(t)^{1-\theta}v(t)^\theta$ is equivalent to a constant function on Ω. ■

Corollary 3 *Let X, Y be two Banach ideal spaces on Ω and u, v two weights on Ω. If we have equalities*

$$X^{1-\theta_0}Y^{\theta_0} = (X_u)^{1-\theta_0}(Y_v)^{\theta_0} \text{ and } X^{1-\theta_1}Y^{\theta_1} = (X_u)^{1-\theta_1}(Y_v)^{\theta_1}$$

for some $\theta_0, \theta_1 \in [0,1]$ with $\theta_0 \ne \theta_1$, then $u(t) \approx v(t) \approx C$ on Ω.

Proof From Theorem 3, used twice, we have that $u^{1-\theta_0}v^{\theta_0} = u(\frac{v}{u})^{\theta_0} \approx C_0$ on Ω and $u^{1-\theta_1}v^{\theta_1} = u(\frac{v}{u})^{\theta_1} \approx C_1$ on Ω. Therefore, $u(t) \approx v(t) \approx C$ on Ω. ■

The next theorem on the representation or the inverse interpolation problem will have only weighted L^1 and L^∞ spaces but then we can change the spaces on both places. We again need a lemma.

Lemma 2 *If $\varphi(t,1)$ is a strictly increasing function and for some $x \in X$ and some measurable set A we have $\|x\chi_A\|_X = 1$, then $\|\varphi(|x|, v^{-1})\chi_A\|_{\varphi(X,L_v^\infty)} = 1$.*

Proof Clearly $\|\varphi(|x|, v^{-1})\chi_A\|_{\varphi(X,L_v^\infty)} \le 1$. Assume therefore that it is strictly less than 1. Then, for some $\varepsilon > 0$,

$$\varphi(|x|, v^{-1})\chi_A \le (1-\varepsilon)\varphi(|x_0|, |x_1|)\chi_A$$

with $\|x_0\|_X \le 1$ and $\|x_1\|_{L_v^\infty} \le 1$. Hence

$$\varphi(|x|, v^{-1})\chi_A \le (1-\varepsilon)\varphi(|x_0|, v^{-1})\chi_A \le \varphi\big((1-\varepsilon)|x_0|, v^{-1}\big)\,\chi_A.$$

Since $\varphi(t,1)$ is strictly increasing it follows that $|x(t)|\chi_A \le (1-\varepsilon)|x_0|\chi_A$ and so $\|x\chi_A\|_X \le (1-\varepsilon)\|x_0\chi_A\|_X \le (1-\varepsilon) < 1$, which is a contradiction. ■

For a function $\varphi \in \mathfrak{U}$ consider a submultiplicative and quasi-concave function ρ_φ on $(0,\infty)$ defined by

$$\rho_\varphi(a) = \limsup_{t\to\infty} \frac{\varphi(at,1)}{\varphi(t,1)}, \quad a > 0.$$

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By the well-known theorem on submultiplicative functions on $(0,\infty)$ we can find numbers $0 \le \alpha \le \beta \le 1$, called also the *indices of* φ, such that

$$\rho_\varphi(a) \ge \max(a^\alpha, a^\beta).$$

Moreover, for any $\varepsilon > 0$ we have $\rho_\varphi(a) \le a^{\alpha-\varepsilon}$ for $a > 0$ sufficiently close to zero, $\rho_\varphi(a) \le a^{\beta+\varepsilon}$ for a sufficiently large, and

$$\alpha = \alpha_\varphi = \lim_{a\to 0^+} \frac{\ln \rho_\varphi(a)}{\ln a},\ \beta = \beta_\varphi = \lim_{a\to\infty} \frac{\ln \rho_\varphi(a)}{\ln a}$$

(see, *e.g.*, [25], Theorem 1.3 or [35], Theorem 11.3).

Theorem 4 *Let $\varphi \in \mathfrak{U}$ and $\varphi(t,1)$ be a strictly increasing function. Assume that the measure space (Ω,μ) is nonatomic. If*

$$\varphi(L^1_u, L^\infty_v) = \varphi(L^1_w, L^\infty) \tag{5}$$

for some weight functions u, v, w, then there exists $\theta \in [0,1]$ such that

$$w(t)^\theta \approx u(t)^\theta v(t)^{1-\theta} \quad \text{on } \Omega. \tag{6}$$

More precisely, if v is equivalent to a constant function, then we can take $\theta = 0$ and if v is not equivalent to a constant function on Ω, then the function φ has the same indices $\alpha_\varphi = \beta_\varphi$ and we can take $\theta = \alpha = \alpha_\varphi$.

Proof If v is equivalent to a constant function on Ω, *i.e.*,

$$c = \sup_{t\in\Omega} \frac{1}{v(t)} \sup_{t\in\Omega} v(t) < \infty,$$

then we can take $\theta = 0$.

Assume therefore that $\sup_{t\in\Omega} \frac{1}{v(t)} \sup_{t\in\Omega} v(t) = \infty$. For any $k \in \mathbf{N}$ define sets

$$U_k = \{t \in \Omega : 2^{-k-1} < u(t) \le 2^{-k}\},\ V_k = \{t \in \Omega : 2^{-k-1} < v(t) \le 2^{-k}\},$$
$$W_k = \{t \in \Omega : 2^{-k-1} < w(t) \le 2^{-k}\},\ P = \{(i,j,k) \in \mathbf{N}^3 : U_i \cap V_j \cap W_k \ne \varnothing\}.$$

Note that $\bigcup_{(i,j,k)\in P} U_i \cap V_j \cap W_k = \Omega$. If $0 < \mu(A) < \infty$, then $\|\frac{1}{u\mu(A)}\chi_A\|_{L^1_u} = 1$ and by Lemma 2 we have the equality

$$\Big\| \varphi\Big(\frac{1}{u\mu(A)}, \frac{1}{v}\Big) \chi_A \Big\|_{\varphi(L^1_u, L^\infty_v)} = 1.$$

If $A \subset U_i \cap V_j \cap W_k$, then

$$\Big\| \varphi\Big(\frac{2^i}{\mu(A)}, 2^j\Big) \chi_A \Big\|_{\varphi(L^1_u, L^\infty_v)} \le \Big\| \varphi\Big(\frac{1}{u\mu(A)}, \frac{1}{v}\Big) \chi_A \Big\|_{\varphi(L^1_u, L^\infty_v)} = 1,$$

and, by the assumption that the norms are equivalent,

$$\Big\| \varphi\Big(\frac{2^i}{\mu(A)}, 2^j\Big)\, \chi_A \Big\|_{\varphi(L^1_w, L^\infty)} \le C,$$

and so

$$\varphi\Big(\frac{2^i}{\mu(A)}, 2^j\Big)\, \chi_A(t) \le C\varphi\Big(\frac{|x(t)|}{w(t)}, 1\Big)\, \chi_A(t)$$

with $x \in L^1$, $\|x\|_{L^1} \le 1$ or

$$\varphi\Big(\frac{2^i}{\mu(A)}, 2^j\Big)\, \chi_A(t) \le 2C\varphi\big(|x(t)|2^k, 1\big)\, \chi_A(t).$$

Take $d = \operatorname{ess\,inf}_{t\in A} |x(t)|$. If $d > 0$, then

$$\varphi\Big(\frac{2^i}{\mu(A)}, 2^j\Big)\, \chi_A(t) \le 2C\varphi(d2^k, 1)\chi_A(t)$$

and $d\chi_A(t) \le |x(t)|\chi_A(t)$ gives $\|d\chi_A\|_{L^1} \le \|x\chi_A\|_{L^1} \le 1$ or $d \le \frac{1}{\mu(A)}$. Thus

$$\varphi\Big(\frac{2^i}{\mu(A)}, 2^j\Big)\, \chi_A(t) \le 2C\varphi\Big(\frac{1}{\mu(A)}2^k, 1\Big)\, \chi_A(t).$$

If $d = 0$, then $\lim_{t\to 0^+} \varphi(t, 1) > 0$, and the above estimate also holds.

Similarly, for $A \subset U_i \cap V_j \cap W_k$ with $0 < \mu(A) < \infty$ we have the estimate

$$\varphi\Big(\frac{1}{\mu(A)}2^k, 1\Big)\, \chi_A(t) \le 2C\varphi\Big(\frac{2^i}{\mu(A)}, 2^j\Big)\, \chi_A(t).$$

From the above estimates we have the inequalities

$$\frac{1}{2C}\varphi\Big(\frac{2^k}{\mu(A)}, 1\Big) \le 2^j\varphi(2^{i-j-k}\frac{2^k}{\mu(A)}, 1) \le 2C\varphi\Big(\frac{2^k}{\mu(A)}, 1\Big),$$

and by taking $\mu(A) \to 0^+$ (we can do this since measure is nonatomic) we have

$$2^j\rho_\varphi(2^{i-j-k}) \le 2C \text{ and } 2^{-j}\rho_\varphi(2^{j+k-i}) \le 2C.$$

Let $Q = \{p = i - j - k : (i, j, k) \in P\}$. If $\sup\{|p| : p \in Q\} < \infty$, then $2^j \le 2C$, $2^{-j} \le 2C$ and the weight $v(t)$ is equivalent to a constant function, which cannot happen. Thus we must have

$$\sup\{|p| : p \in Q\} = \infty.$$

Since

$$\rho_\varphi(2^{i-j-k})\rho_\varphi(2^{j+k-i}) \le 4C^2$$

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it follows that

$$\alpha = \beta,\ \rho_\varphi(a) \geq a^\alpha \text{ for all } a > 0, \text{ and } \rho_\varphi(2^p)\rho_\varphi(2^{-p}) \leq 4C^2 \text{ for all } p \in Q.$$

If $\sup\{p : p \in Q\} = \infty$ and $\limsup_{p\to\infty, p\in Q} \frac{\rho_\varphi(2^p)}{2^{p\alpha}} = \infty$, then

$$4C^2 \geq \limsup_{p\to\infty, p\in Q} \rho_\varphi(2^p)\rho_\varphi(2^{-p}) \geq \limsup_{p\to\infty, p\in Q} \rho_\varphi(2^p)(2^{-\alpha p}) = \infty.$$

If $\inf\{p : p \in Q\} = -\infty$ and $\limsup_{p\to-\infty, p\in Q} \frac{\rho_\varphi(2^p)}{2^{p\alpha}} = \infty$, then

$$4C^2 \geq \limsup_{p\to\infty, p\in Q} \rho_\varphi(2^p)\rho_\varphi(2^{-p}) \geq \limsup_{p\to\infty, p\in Q} \rho_\varphi(2^p)(2^{-\alpha p}) = \infty.$$

This means that $2^j 2^{(i-j-k)\alpha} \approx 1$ for all $(i, j, k) \in P$. Therefore on all sets $U_i \cap V_j \cap W_k$ we have $(\frac{1}{u})^\alpha(\frac{1}{v})^{1-\alpha} w^\alpha \approx 1$ or $w^\alpha \approx u^\alpha v^{1-\alpha}$, and since the sum of these sets is Ω, the proof is complete. ■

In equality (5) we can have four weights, but before we formulate it we prove the following lemma:

Lemma 3 *The equality $\varphi(L^1_{u_0}, L^\infty_{u_1}) = \varphi(L^1_{v_0}, L^\infty_{v_1})$ holds if and only if the equality $\varphi(L^1_{u_0 w}, L^\infty_{u_1 w}) = \varphi(L^1_{v_0 w}, L^\infty_{v_1 w})$ is true.*

Proof It is enough to show that

$$\varphi(L^1_{u_0}, L^\infty_{u_1}) \subset \varphi(L^1_{v_0}, L^\infty_{v_1}) \text{ implies } \varphi(L^1_{u_0 w}, L^\infty_{u_1 w}) \subset \varphi(L^1_{v_0 w}, L^\infty_{v_1 w})$$

with the same norms of embeddings. In fact, if $x \in \varphi(L^1_{u_0 w}, L^\infty_{u_1 w})$ with the norm < 1, then

$$|x| \leq \varphi(|x_0|, |x_1|) \text{ with } \|x_0\|_{L^1_{u_0 w}} \leq 1 \text{ and } \|x_1\|_{L^\infty_{u_1 w}} \leq 1,$$

and so

$$|x|w \leq \varphi(|x_0|w, |x_1|w) = \varphi(y_0, y_1)$$

with

$$\|y_0\|_{L^1_{u_0}} = \|x_0 w\|_{L^1_{u_0}} = \|x_0\|_{L^1_{u_0 w}} \leq 1 \text{ and } \|y_1\|_{L^\infty_{u_1}} = \|x_1 w\|_{L^\infty_{u_1}} = \|x_1\|_{L^\infty_{u_1 w}} \leq 1.$$

Thus $xw \in \varphi(L^1_{u_0}, L^\infty_{u_1})$ and, by the embedding assumption, $xw \in \varphi(L^1_{v_0}, L^\infty_{v_1})$, that is,

$$|x|w \leq C\varphi(|z_0|, |z_1|) \text{ with } \|z_0\|_{L^1_{v_0}} \leq 1 \text{ and } \|z_1\|_{L^\infty_{v_1}} \leq 1$$

or, equivalently,

$$|x| \leq C\varphi\left(\frac{|z_0|}{w}, \frac{|z_1|}{w}\right) = C\varphi(x'_0, x'_1)$$

with

$$\|x_0'\|_{L^1_{v_0w}} = \left\| \frac{z_0}{w} \right\|_{L^1_{v_0w}} = \|z_0\|_{L^1_{v_0}} \le 1 \text{ and } \|x_1'\|_{L^\infty_{v_1w}} = \left\| \frac{z_1}{w} \right\|_{L^\infty_{v_1w}} = \|z_1\|_{L^\infty_{v_1}} \le 1,$$

we obtain that $x \in \varphi(L^1_{v_0w}, L^\infty)_{v_1w}$ with the norm $\le C$. ■

Directly from Theorem 4 and Lemma 3 we obtain the following result:

Corollary 4 *Let φ and the measure space (Ω, μ) be the same as in Theorem 4. If*

$$\varphi(L^1_{u_0}, L^\infty_{u_1}) = \varphi(L^1_{v_0}, L^\infty_{v_1})$$

for some weight functions u_0, u_1, v_0, v_1 on Ω, then there exists $\theta \in [0,1]$ such that

$$u_0(t)^\theta u_1(t)^{1-\theta} \approx v_0(t)^\theta v_1(t)^{1-\theta} \text{ on } \Omega.$$

Remark 2 Note that

$$\varphi(L^1_u, L^\infty_v) = L^M_v\Big(\frac{u}{v}dt\Big),$$

where function M is defined by $M\big(\varphi(s,1)\big) = s$ and the last space is a weighted Orlicz space generated by the norm

$$\|x\|_{L^M_v(\frac{u}{v}dt)} = \inf\Big\{ \lambda > 0 : \int_\Omega M\big(v(t)|x(t)|/\lambda\big) \frac{u(t)}{v(t)}\, dt \le 1 \Big\}.$$

Similarly, $\varphi(L^1_w, L^\infty) = L^M(wdt)$. In the case when $v = 1$ and $M \in \Delta_2$ globally, that is, $M(2u) \le CM(u)$ for all $u > 0$, it is known that $L^M(udt) = L^M(wdt)$ if and only if $u \approx w$ on $(0,\infty)$ or on a measurable subset Ω of R^n of a positive measure (see [24]). In the case when v is not equivalent to a constant, then the technique from [24] does not work. On the other hand, if we look for these spaces as special cases of the Musielak–Orlicz spaces generated by the functions $\mathcal{M}(a,t) = M\big(v(t)a\big)\frac{u(t)}{v(t)}$ and $\mathcal{N}(a,t) = M(a)w(t)$ and use the criterion for the equality $L^{\mathcal{M}} = L^{\mathcal{N}}$ with equivalent norms (see [40]), then these general conditions seem to be not helpful in proving the corresponding equivalence of the weights as in Corollary 4.

We give an example showing that for a certain non-power function $\varphi \in \mathcal{U}$ and some weights u, v we can have equality $\varphi(L^1_u, L^\infty_v) = \varphi(L^1, L^\infty)$ with equivalent norms.

Example 1 Consider the concave function on $(0,\infty)$ defined by $\psi(t) = t^{1/2}\ln^{1/2}(1+t)$ and let $\varphi(s,t) = t\psi(s/t) = s^{1/2}t^{1/2}\ln^{1/2}(1+\frac{s}{t})$. Then $\rho_\varphi(a) = a^{1/2}$. We will show that there exists a weight u on $\Omega = I = (0,1)$ such that

$$\varphi\big(L^1_{1/u}(I), L^\infty_u(I)\big) = \varphi\big(L^1(I), L^\infty(I)\big)$$

with equivalent norms. Assume that the weight u satisfies $u(t) \ge 1$ a.e. on I and $\int_0^1 u(t)\, dt \le 2$.

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Observe that for $a, b \geq 0$ we have the inequality

$$a \ln(1 + ab^2) \leq 2(a + b) \ln(1 + a + b). \tag{7}$$

In fact, if $0 \leq a \leq 1$, then

$$\begin{aligned} a \ln(1 + ab^2) &\leq a \ln(1 + b^2) \leq a \ln(1 + b)^2 \\ &= 2a \ln(1 + b) \leq 2(a + b) \ln(1 + a + b), \end{aligned}$$

and if $a \geq 1$, then

$$\begin{aligned} a \ln(1 + ab^2) &\leq a \ln(a + ab^2) \leq a \ln(1 + a + b)^2 \\ &= 2a \ln(1 + a + b) \leq 2(a + b) \ln(1 + a + b). \end{aligned}$$

We show first the imbedding $\varphi(L^1_{1/u}, L^\infty_u) \subset \varphi(L^1, L^\infty)$. If $x \in \varphi(L^1_{1/u}, L^\infty_u)$ and the norm is < 1, then

$$|x| \leq \varphi\Big(|x_0|u, \frac{1}{u}\Big) \text{ with } \|x_0\|_{L^1} \leq 1$$

and, by (7),

$$\begin{aligned} |x| \leq \varphi(|x_0|u, \frac{1}{u}) &= |x_0|^{1/2} \ln^{1/2}(1 + |x_0|u^2) \leq \sqrt{2}(|x_0| + u)^{1/2} \ln^{1/2}(1 + |x_0| + u) \\ &= \sqrt{2}\varphi(|x_0| + u, 1) \leq 3\sqrt{2}\varphi\Big(\frac{|x_0| + u}{3}, 1\Big). \end{aligned}$$

This means that $x \in \varphi(L^1, L^\infty)$ with norm $\leq 3\sqrt{2}$. Therefore we have a continuous imbedding

$$\varphi(L^1_{1/u}, L^\infty_u) \overset{3\sqrt{2}}{\hookrightarrow} \varphi(L^1, L^\infty).$$

Secondly, we prove the reverse imbedding $\varphi(L^1, L^\infty) \subset \varphi(L^1_{1/u}, L^\infty_u)$. Let $x \in \varphi(L^1, L^\infty)$ with norm < 1, that is,

$$|x| \leq \varphi(|x_0|, 1) \text{ and } \|x_0\|_{L^1} \leq 1.$$

Then, since the weight u satisfies $u(t) \geq 1$ a.e on I, it follows that

$$\begin{aligned} |x| \leq \varphi(|x_0|, 1) &= |x_0|^{1/2} \ln^{1/2}(1 + |x_0|) \\ &\leq |x_0|^{1/2} \ln^{1/2}(1 + |x_0|u^2) = \varphi\Big(|x_0|u, \frac{1}{u}\Big), \end{aligned}$$

and so $x \in \varphi(L^1_{1/u}, L^\infty_u)$ with norm ≤ 1. Thus we have a continuous imbedding

$$\varphi(L^1, L^\infty) \overset{1}{\hookrightarrow} \varphi(L^1_{1/u}, L^\infty_u).$$

As concrete weight u on $I = (0, 1)$ for which $\int_0^1 u(t)\,dt \leq 2$ and $u(t) \geq 1$ for all $t \in (0, 1)$ we can take $u(t) = t^{-1/2}$ on $(0, 1)$.

3 Factorization of Positive Sublinear Operators in $X^{(p)}$ Spaces

Let X be either $L^0(\mu)$ or a Banach ideal space on (Ω,μ). An operator $T\colon X\to L^0(\nu)$ is called *positive* if $0\le x\in X$ implies that $0\le Tx\in L^0(\nu)$; T is called *sublinear* if, for all $x,y\in X$ and any $a\in\mathbf{R}$,

$$|T(x+y)(t)|\le|Tx(t)|+|Ty(t)| \text{ and } |T(ax)(t)|=|a|\,|Tx(t)| \quad \nu\text{-a.e.}$$

Classical examples of positive linear operators are integral operators $Kx(t)=\int_\Omega k(s,t)x(s)\,ds$ with positive measurable kernels $k(s,t)\ge 0$ and as positive sublinear operators we can take maximal operators and $Lx(t)=\int_\Omega|l(s,t)|x(s)\,ds$ with measurable kernel $l(s,t)$.

If $1\le p<\infty$ and $1/p+1/p'=1$, then for any positive sublinear operator T the pointwise Hölder–Rogers[1] inequality is true:

$$T(|x|^{1/p}|y|^{1-1/p})(t)\le[T(|x|)(t)]^{1/p}[T(|y|)(t)]^{1-1/p} \quad \nu\text{-a.e.},$$

which can be rewritten as

$$T(|x|\,|y|)(t)\le[T(|x|^p)(t)]^{1/p}[T(|y|^{p'})(t)]^{1/p'} \quad \nu\text{-a.e.} \tag{8}$$

for any $x,y\in X$. This estimate follows directly from the equality

$$a^{1/p}b^{1-1/p}=\inf_{\varepsilon>0}\Big[\frac{1}{p}\varepsilon^{\frac{1}{p}-1}a+\Big(1-\frac{1}{p}\Big)\varepsilon^{\frac{1}{p}}b\Big],$$

which is true for any real positive numbers a, b. Note that more general pointwise estimates for positive sublinear operators can be proved. In fact, this was used (but not explicitly written) for positive linear operators in the proof of the fact that the Calderón–Lozanovskiĭ spaces are exact interpolation spaces for positive linear operators (see [3, 32, 50]; see also [35, Theorem 15.13]). It was also noted in [37] that the same estimate is true for positive sublinear operators. We include the proof here.

Lemma 4 *Let X be either $L^0(\mu)$ or a Banach ideal space on (Ω,μ) and let $T\colon X\to L^0(\nu)$ be a positive sublinear operator. If $\varphi\in\mathcal{U}$, then for any $x,y\in X$*

$$T\big(\varphi(|x|,|y|)\big)(t)\le\varphi\big(T(|x|)(t),T(|y|)(t)\big) \quad \nu\text{-a.e.} \tag{9}$$

Proof Since for arbitrary $a>0$, $b>0$,

$$\varphi(|x|,|y|)\le\frac{a|x|+b|y|}{\hat{\varphi}(a,b)},$$

it follows that

$$T\big(\varphi(|x|,|y|)\big)\le\frac{aT(|x|)+bT(|y|)}{\hat{\varphi}(a,b)}$$

[1]The classical Hölder inequality should historically correctly be called the Hölder–Rogers inequality (*cf.* [36]).

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for arbitrary $a, b > 0$, and so

$$T\big(\varphi(|x|, |y|)\big) \le \hat{\hat{\varphi}}\big(T(|x|), T(|y|)\big) = \varphi\big(T(|x|), T(|y|)\big). \qquad \blacksquare$$

Lemma 5 *Let X be either $L^0(\mu)$ or a Banach ideal space on (Ω, μ). Assume that $T: X \to L^0(\nu)$ is a positive sublinear operator. Then, for any weights w_0, w_1 on Ω and $1 \le p < \infty$, the operator defined by*

$$T_p x(t) = [w_0(t) T(|x|^p w_1)(t)]^{1/p}$$

is positive and sublinear.

Proof We have, by using the Hölder-Rogers inequality (8) similarly as in the proof of the Minkowski inequality,

$$\begin{aligned}
[T_p(x+y)(t)]^p &= w_0(t) T(|x+y|^p w_1)(t) \\
&\le w_0(t) T(|x|\,|x+y|^{p-1} w_1 + |y|\,|x+y|^{p-1} w_1)(t) \\
&\le w_0(t) T(|x|\,|x+y|^{p-1} w_1) + T(|y|\,|x+y|^{p-1} w_1)(t) \\
&\le w_0(t) T(|x|^p w_1)^{1/p} T(|x+y|^{(p-1)p'} w_1)^{1/p'} \\
&\qquad + w_0(t) T(|y|^p w_1)^{1/p} T(|x+y|^{(p-1)p'} w_1)^{1/p'} \\
&= w_0(t) T(|x|^p w_1)^{1/p} T(|x+y|^p w_1)^{1/p'} \\
&\qquad + w_0(t) T(|y|^p w_1)^{1/p} T(|x+y|^p w_1)^{1/p'} \\
&= [T_p(x)(t) + T_p(y)(t)] T_p(x+y)(t)]^{p/p'},
\end{aligned}$$

which gives the subadditivity of T_p. Moreover, $T_p(ax)(t) = |a| T_p(x)(t)$ and the proof is complete. $\blacksquare$

Let us start with a result to which the idea in the L^p-spaces was given by Gagliardo [15, Lemma 3.I] and by Rubio de Francia [48].

Lemma 6 *Let X be a Banach ideal space on (Ω, μ) and let $T: X \to X$ be a bounded positive sublinear operator. Then there exists $u \in X, u(t) > 0$ μ-a.e. on Ω such that $Tu(t) \le Cu(t)$ for $t \in \Omega$ μ-a.e., where $C = (1+\varepsilon)\|T\|_{X\to X}$ with any $\varepsilon > 0$.*[2]

Proof Take $x_0 \in X$ with $x_0(t) > 0$ for $t \in \Omega$ μ-a.e. and put

$$u(t) = \sum_{k=0}^{\infty} C^{-k} T^k x_0(t), \quad \text{where } T^0 = \mathrm{Id}.$$

[2]As usual, for the norm $\|T\|_{X\to X}$ of a sublinear operator T we mean $\sup_{\|x\|_X \le 1} \|Tx\|_X$.

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Since

$$\sum_{k=0}^{\infty} C^{-k}\|T^k x_0\|_X \le \sum_{k=0}^{\infty} C^{-k}\|T\|_{X\to X}^k \|x_0\|_X = \sum_{k=0}^{\infty}(1+\varepsilon)^{-k}\|x_0\|_X = \Big(1+\frac{1}{\varepsilon}\Big)\|x_0\|_X,$$

it follows that $u \in X$ and $\|u\|_X \le (1+\frac{1}{\varepsilon})\|x_0\|_X$. Moreover, by the positivity of the operator T, we have

$$0 < x_0(t) \le x_0(t) + Tx_0(t)/C + T^2x_0(t)/C^2 + \cdots = u(t) \quad \text{for } t \in \Omega\ \mu\text{-a.e.},$$

and

$$\begin{aligned} Tu(t) &\le \sum_{k=0}^{\infty} C^{-k}T^{k+1}x_0(t) = C\sum_{k=1}^{\infty} C^{-k}T^k x_0(t) \\ &\le C\Big[x_0(t) + \sum_{k=1}^{\infty} C^{-k}T^k x_0(t)\Big] = Cu(t). \end{aligned}$$

■

Now we are ready to state and prove the fundamental factorization theorem in weighted Banach ideal $X^{(p)}$ spaces.

Theorem 5 *For some weight functions v_0, v_1, w_0, w_1 on Ω and some $p_0, p_1, q_0, q_1 \in [1,\infty)$, let the operators $T_0\colon (X_{v_0})^{(p_0)} \to (X_{v_1})^{(p_1)}$ and $T_1\colon (X_{w_0})^{(q_0)} \to (X_{w_1})^{(q_1)}$ be positive sublinear and bounded with the corresponding norms C_0 and C_1. Assume that we have continuous imbeddings $X^{(p_1)} \subset X^{(p_0)}$ and $X^{(q_1)} \subset X^{(q_0)}$ with the norms not exceeding C_2 and C_3, respectively. Then:*

(i) *There exists a positive weight $u \in X^{(p_0 q_0)}$ such that*

$$v_1 T_0(u^{q_0} v_0^{-1}) \le C^{q_0} u^{q_0} \text{ and } w_1 T_1(u^{p_0} w_0^{-1}) \le C^{p_0} u^{p_0},$$

with $C = 2(C_0^{1/q_0} C_2^{1/q_0} + C_1^{1/p_0} C_3^{1/p_0})$ or, equivalently, we have that

$$T_0\colon L^\infty_{v_0 u^{-q_0}} \to L^\infty_{v_1 u^{-q_0}} \text{ and } T_1\colon L^\infty_{w_0 u^{-p_0}} \to L^\infty_{w_1 u^{-p_0}}$$

are bounded with norms not exceeding C^{q_0} and C^{p_0}, respectively.

(ii) *There exists a positive weight $u \in X^{(p_1 q_1)}$ such that*

$$v_1 T_0(u^{q_1} v_0^{-1}) \le D^{q_1} u^{q_1} \text{ and } w_1 T_1(u^{p_1} w_0^{-1}) \le D^{p_1} u^{p_1}$$

with $D = 2(C_0^{1/q_1} C_2^{1/q_1} + C_1^{1/p_1} C_3^{1/p_1})$ or, equivalently, we have that

$$T_0\colon L^\infty_{v_0 u^{-q_1}} \to L^\infty_{v_1 u^{-q_1}} \text{ and } T_1\colon L^\infty_{w_0 u^{-p_1}} \to L^\infty_{w_1 u^{-p_1}}$$

are bounded with norms not exceeding D^{q_1} and D^{p_1}, respectively.

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Proof (i) Using the given operator T_0 we can construct a new positive sublinear operator S_0 by

$$S_0x = [v_1 T_0(|x|^{q_0} v_0^{-1})]^{1/q_0}.$$

Of course, S_0 is positive, and by Lemma 5 it is sublinear. The operator S_0 is also bounded from $X^{(p_0q_0)}$ into $X^{(p_1q_0)}$ with the norm $\leq C_0^{1/q_0}$. Indeed,

$$\begin{aligned}\|S_0x\|_{X^{(p_1q_0)}} &= \big\| [v_1 T_0(|x|^{q_0} v_0^{-1})]^{p_1} \big\|_X^{\frac{1}{p_1q_0}} \\ &\leq C_0^{1/q_0} \big\| [v_0 |x|^{q_0} v_0^{-1}]^{p_0} \big\|_X^{\frac{1}{p_0q_0}} \\ &= C_0^{1/q_0} \big\| |x|^{p_0q_0} \big\|_X^{\frac{1}{p_0q_0}} = C_0^{1/q_0} \|x\|_{X^{(p_0q_0)}}.\end{aligned}$$

Similarly, the operator S_1 given by

$$S_1x = [w_1 T_1(|x|^{p_0} w_0^{-1})]^{1/p_0}$$

is positive, sublinear and bounded from $X^{(p_0q_0)}$ into $X^{(p_0q_1)}$ with norm $\leq C_1^{1/p_0}$.

Since we have imbeddings $X^{(p_1q_0)} \subset X^{(p_0q_0)}$ and $X^{(p_0q_1)} \subset X^{(p_0q_0)}$, it follows that the operator $S = S_0 + S_1$ is bounded from $X^{(p_0q_0)}$ into $X^{(p_0q_0)}$ with norm $\leq C$, and applying Lemma 6 to S we obtain the required estimates.

(ii) The proof here is similar. We should only consider the operators

$$L_0x = [v_1 T_0(|x|^{q_1} v_0^{-1})]^{1/q_1} \text{ and } L_1x = [w_1 T_1(|x|^{p_1} w_0^{-1})]^{1/p_1},$$

use the embeddings $X^{(p_1q_1)} \subset X^{(p_0q_1)}, X^{(p_1q_1)} \subset X^{(p_1q_0)}$ and apply Lemma 6 to the operator $L = L_0 + L_1$ which is bounded from $X^{(p_1q_1)}$ into itself. The proof is complete. ∎

In some cases we do not need the above imbeddings. We can then formulate a generalization to $X^{(p)}$ spaces of the result of Rubio de Francia type. This result gives the factorization theorem through weighted L^∞ spaces.

Corollary 5 *Assume that for some weight functions v, w on Ω and some $p_0, p_1 \in [1, \infty)$ the operators*

$$T_0 \colon (X_v)^{(p_0)} \to (X_v)^{(p_0)} \text{ and } T_1 \colon (X_w)^{(p_1)} \to (X_w)^{(p_1)}$$

are positive, sublinear and bounded with the corresponding norms C_0 and C_1. Then there exists a positive weight $u \in X^{(p_0p_1)}$ such that

$$vT_0(u^{p_1} v^{-1}) \leq C^{p_1} u^{p_1} \text{ and } wT_1(u^{p_0} w^{-1}) \leq C^{p_0} u^{p_0},$$

with $C = 2(C_0^{1/p_1} + C_1^{1/p_0})$. The last estimates mean that the operators

$$T_0 \colon L^\infty_{vu^{-p_1}} \to L^\infty_{vu^{-p_1}} \text{ and } T_1 \colon L^\infty_{wu^{-p_0}} \to L^\infty_{wu^{-p_0}}$$

are bounded with norms not exceeding C^{p_1} and C^{p_0}, respectively.

As special cases of Theorem 5 and Corollary 5 we obtain the factorization results of Hernández (see [17, Theorem 2.1] and [18, Theorem 1]).

Corollary 6 *If $T_0\colon L^{p_0} \to L^{p_0}$ and $T_1\colon L^{p_1} \to L^{p_1}$ are bounded, positive sublinear operators, then there exists a positive weight $u \in L^{p_0 p_1}$ such that $T_0 u^{p_1} \le C u^{p_1}$ and $T_1 u^{p_0} \le C u^{p_0}$ or, equivalently, we have that the operators $T_0\colon L^\infty_{u^{-p_1}} \to L^\infty_{u^{-p_1}}$ and $T_1\colon L^\infty_{u^{-p_0}} \to L^\infty_{u^{-p_0}}$ are bounded.*

In Theorem 5 and Corollaries 5 and 6 there are two operators T_0 and T_1 but in applications sometimes as an operator T_1 is taken the associated operator T_0' (sometimes also called the dual operator in the sense of Köthe) to T_0. If $T_0 \in \mathcal{K}$, then the associated operator does not always exist. Here by $\mathcal{K}$ we denote the class of positive sublinear operators T defined on $L^0(\mu)$ with values in $L^0(\mu)$ and for $T \in \mathcal{K}$ we consider the notion of the associated operator $T' \in \mathcal{K}$.

For $T \in \mathcal{K}$, an operator $T' \in \mathcal{K}$ is called *associated* to T (in the scale of L^p-spaces) if, for all $1 \le p \le \infty$ and all weights u we have that $T\colon L^p_u \to L^p_u$ is bounded if and only if $T'\colon L^{p'}_{1/u} \to L^{p'}_{1/u}$ is bounded, and the estimates

$$\frac{1}{C}\|T\|_{L^p_u \to L^p_u} \le \|T'\|_{L^{p'}_{1/u} \to L^{p'}_{1/u}} \le C\|T\|_{L^p_u \to L^p_u}$$

hold with a constant $C > 0$ independent of p and u.

Note that T' is not necessary unique. If T is a linear operator, then as T' we can take the conjugate operator T^*. Also for a linear operator T the operator $x \mapsto |Tx|$ is sublinear and there is no notion of conjugate operator to it but we can instead take $T'x = |T^* x|$.

We are now ready to formulate the factorization theorem in L^p-spaces with the factorization through the weighted L^1 and L^∞ spaces for operators $T \in \mathcal{K}$ for which an associated operator $T' \in \mathcal{K}$ exists.

Corollary 7 *Let $1 < p < \infty$. Assume that $T \in \mathcal{K}$ and for T there exists $T' \in \mathcal{K}$. Then $T\colon L^p \to L^p$ is bounded if and only if there exists a weight $u \in L^p$ on Ω such that*

$$T\colon L^1_{u^{p-1}} \to L^1_{u^{p-1}} \textit{ and } T\colon L^\infty_{1/u} \to L^\infty_{1/u}$$

is bounded.

Proof If $T\colon L^p \to L^p$ and $T'\colon L^{p'} \to L^{p'}$ are bounded then, by Corollary 6, there exists $w \in L^{pp'}$ such that

$$Tw^{p'} \le Cw^{p'} \text{ and } T'w^p \le Cw^p.$$

Taking $u = w^{p'}$ we have $u \in L^p$ and

$$Tu \le Cu \text{ and } T'u^{p-1} \le Cu^{p-1}$$

or

$$T\colon L^\infty_{1/u} \to L^\infty_{1/u} \text{ and } T\colon L^1_{u^{p-1}} \to T\colon L^1_{u^{p-1}}$$

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is bounded.

Conversely, if T is bounded in $L^1_{u^{p-1}}$ and in $L^\infty_{1/u}$, then T is also bounded in the Calderón spaces $(L^1_{u^{p-1}})^{1/p}(L^\infty_{1/u})^{1-1/p} = L^p$. ■

We will show a little later that if for an operator $T \in \mathcal{K}$ we do not put additional restrictions, (for example, the existence of an associated operator) then the factorization theorem through weighted L^1 and L^∞ spaces cannot be true.

Before giving this counter-example we would like to show that for some class of operators we can prove a factorization theorem where the extreme spaces are Lorentz and Marcinkiewicz spaces determined by weight instead of weighted Lebesgue spaces L^1 and L^∞. Lorentz and Marcinkiewicz spaces are natural extreme spaces in the class of symmetric spaces, *cf.* [25]. All our spaces here are on $(0,\infty)$.

We consider a subclass $\mathcal{K}_*$ of operators $T \in \mathcal{K}$ for which there exists a constant $C > 0$ such that

$$\int_0^t (Tx)^*(s)\,ds \le C \int_0^t Tx^*(s)\,ds$$

for all $t > 0$ and $x \in L^0(0,\infty)$.

As an example of $T \in \mathcal{K}_*$ we can take the Hardy operator $Hx(t) = \frac{1}{t}\int_0^t x(s)\,ds$, Hardy sublinear operator $H^*x(t) = \frac{1}{t}\int_0^t x^*(s)\,ds$, maximal operator M, and integral operator $Tx(t) = \int_0^\infty k(t,s)x(s)\,ds$ with a positive kernel $k(t,s) \ge 0$ which is decreasing in each variable separately.

We recall the definition of Lorentz Λ_{u^*} spaces and Marcinkiewicz M_{u^*} spaces. For the weight function u on $(0,\infty)$, the *Lorentz space* Λ_{u^*} is the space generated by the norm

$$\|x\|_{\Lambda_{u^*}} = \int_0^\infty x^*(t)u^*(t)\,dt,$$

and the *Marcinkiewicz space* M_{u^*} that by the norm

$$\|x\|_{M_{u^*}} = \sup_{t>0} \frac{1}{\int_0^t u^*(s)\,ds} \int_0^t x^*(s)\,ds.$$

Theorem 6 *Let $1 < p < \infty$. Assume that $T \in \mathcal{K}_*$ and for T there exists an associated operator $T' \in \mathcal{K}$. If $T\colon L^p \to L^p$ is bounded, then there exists a positive weight $u \in L^p$ such that*

(i) *The estimates*

$$\frac{1}{t}\int_0^t u^*(s)\,ds \le C_1 u(t), \quad \frac{1}{t}\int_0^t u^*(s)^{p-1}\,ds \le C_2 u(t)^{p-1}$$

and

$$\int_t^\infty \frac{u^*(s)}{s}\,ds \le C_3 u(t), \quad \int_t^\infty \frac{u^*(s)^{p-1}}{s}\,ds \le C_4 u(t)^{p-1}$$

hold for all $t > 0$.

(ii) *The operators $T\colon \Lambda_{u^{*p-1}} \to \Lambda_{u^{*p-1}}$ and $T\colon M_{u^*} \to M_{u^*}$ are bounded.*

Conversely, if conditions (i) *and* (ii) *are satisfied then $T\colon L^p \to L^p$ is bounded.*

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Proof Assume that $T\colon L^p \to L^p$ is bounded. Consider two operators

$$S_0 = T + H^* + H_*\colon L^p \to L^p \text{ and } S_1 = T' + H^* + H_*\colon L^{p'} \to L^{p'},$$

where $H^*x(t) = x^{**}(t) = \frac{1}{t}\int_0^t x^*(s)\,ds$ and $H_*x(t) = \int_t^\infty \frac{x^*(s)}{s}\,ds$.

Note that H^* is bounded in L^p spaces for all $1 < p \leq \infty$ and H_* is bounded in L^p spaces for all $1 \leq p < \infty$. Then $S_0, S_1 \in \mathcal{K}$ and by Corollary 6 we can find a weight $w \in L^{pp'}$ such that the operators

$$S_0\colon L^\infty_{w^{-p'}} \to L^\infty_{w^{-p'}} \text{ and } S_1\colon L^\infty_{w^{-p}} \to L^\infty_{w^{-p}}$$

are bounded, which can be rewritten by taking $u = w^{p'}$ such that $u \in L^p$ and

$$S_0\colon L^\infty_{1/u} \to L^\infty_{1/u} \text{ and } S_1\colon L^\infty_{u^{1-p}} \to L^\infty_{u^{1-p}}$$

are bounded. Since $H^*\colon L^\infty_{1/u} \to L^\infty_{1/u}$ is bounded with norm $\leq A$, it follows that

$$H^*u(t) = u^{**}(t) \leq Au(t)$$

for all $t > 0$, and the first estimate in (i) is proved.

The operator $T\colon L^\infty_{1/u} \to L^\infty_{1/u}$ is also bounded with norm $\leq B$. Therefore $|Tu(t)| \leq Bu(t)$ for all $t > 0$, and so

$$(Tu)^*(t) \leq Bu^*(t) \quad \text{for all } t > 0.$$

If we assume that $x^{**}(t) \leq u^{**}(t)$ for all $t > 0$, then by the assumption $T \in \mathcal{K}_*$ we obtain

$$\begin{aligned}(Tx)^{**}(t) &= \frac{1}{t}\int_0^t (Tx)^*(s)\,ds \leq \frac{C}{t}\int_0^t Tx^*(s)\,ds\\ &\leq \frac{C}{t}\int_0^t Tu^{**}(s)\,ds \leq \frac{AC}{t}\int_0^t Tu(s)\,ds \leq \frac{AC}{t}\int_0^t (Tu)^*(s)\,ds\\ &\leq \frac{ABC}{t}\int_0^t u^*(s)\,ds = ABCu^{**}(t),\end{aligned}$$

and so $T\colon M_{u^*} \to M_{u^*}$ is bounded with the norm $\leq ABC$.

We also have that $H^*\colon L^\infty_{u^{1-p}} \to L^\infty_{u^{1-p}}$ is bounded with the norm $\leq D$ which gives the second estimate in (i)

$$\frac{1}{t}\int_0^t u^*(s)^{p-1}\,ds \leq Du(t)^{p-1} \quad \text{for all } t > 0.$$

The operator $T \in \mathcal{K}_*$ satisfies the estimate $\int_0^t (Tx)^*(s)\,ds \leq C\int_0^t Tx^*(s)ds$ for all $t > 0$. Therefore, by the well-known property of rearrangement (see [25], property

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18^0) and the boundedness of T in $L^1_{u^{p-1}}$ with the norm $\leq E$, we obtain

$$\begin{aligned}\|Tx\|_{\Lambda_{u^*p-1}} &= \int_0^\infty u^*(t)^{p-1}(Tx)^*(t)\,dt\\ &\leq C\int_0^\infty u^*(t)^{p-1}Tx^*(t)\,dt \leq CD\int_0^\infty u(t)^{p-1}Tx^*(t)\,dt\\ &\leq CDE\int_0^\infty u(t)^{p-1}x^*(t)\,dt \leq CDE\int_0^\infty u^*(t)^{p-1}x^*(t)\,dt\\ &= CDE\|x\|_{\Lambda_{u^*p-1}},\end{aligned}$$

and $T\colon \Lambda_{u^*p-1} \to \Lambda_{u^*p-1}$ is bounded with the norm $\leq CDE$.

The boundedness of $H_*\colon L^\infty_{1/u} \to L^\infty_{1/u}$ and $H_*\colon L^\infty_{u^{1-p}} \to L^\infty_{u^{1-p}}$ gives the third and the fourth estimate in (i).

Assume that conditions (i) and (ii) are satisfied. Then it is enough to show that any L^p space can be described from Λ_{u^*p-1} and M_{u^*} by the real method of interpolation (the K-method of interpolation). We have

$$M_{u^*} = (L^1, L^\infty)_{\Phi_1;K} \ \text{ with } \ \Phi_1 = L^\infty_{\frac{1}{v}},\ v(t) = \int_0^t u^*(s)\,ds,$$

and the first with the third estimate in (i) ensures that Φ_1 is a quasi-power parameter, that is, the Calderón operator

$$Sf(t) = \int_0^\infty \min\Big(1, \frac{t}{s}\Big)\,|f(s)|\frac{ds}{s}$$

is bounded in $L^\infty_{1/v}$ (see [10, p. 387] for the definition and examples). In fact,

$$\begin{aligned}Sf(t) &= \int_0^t |f(s)|\frac{ds}{s} + t\int_t^\infty |f(s)|\frac{ds}{s^2}\\ &\leq \|f\|_{L^\infty_{\frac{1}{v}}}\Big(\int_0^t v(s)\frac{ds}{s} + t\int_t^\infty v(s)\frac{ds}{s^2}\Big)\\ &= \|f\|_{L^\infty_{\frac{1}{v}}}\Big(\int_0^t u^{**}(s)\,ds + t\int_t^\infty u^{**}(s)\frac{ds}{s}\Big)\\ &\leq C_1\|f\|_{L^\infty_{\frac{1}{v}}}\left(\int_0^t u^*(s)\,ds + t\int_t^\infty \frac{u^*(s)}{s}\,ds\right)\\ &\leq C_1\|f\|_{L^\infty_{\frac{1}{v}}}\big(v(t) + C_3tu^*(t)\big) \leq C_1(1+C_3)\|f\|_{L^\infty_{\frac{1}{v}}}v(t).\end{aligned}$$

The second and the fourth estimate in (i) ensure that $\Phi_0 = L^1_{\frac{u^*(t)^{p-1}}{t}}$ is a quasi-power

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parameter. In fact,

$$\begin{aligned}\int_0^\infty \frac{u^*(t)^{p-1}}{t} Sf(t)\,dt &= \int_0^\infty \frac{u^*(t)^{p-1}}{t}\Big(\int_0^t |f(s)|\frac{ds}{s} + t\int_t^\infty |f(s)|\frac{ds}{s^2}\Big)\,dt\\ &= \int_0^\infty \Big(\int_s^\infty \frac{u^*(t)^{p-1}}{t}\,dt\Big)|f(s)|\frac{ds}{s}\\ &\quad + \int_0^\infty \Big(\int_0^s u^*(t)^{p-1}\,dt\Big)|f(s)|\frac{ds}{s^2}\\ &\le C_4\int_0^\infty u^*(s)^{p-1}|f(s)|\frac{ds}{s} + C_2\int_0^\infty u^*(s)^{p-1}|f(s)|\frac{ds}{s}\\ &= (C_2+C_4)\|f\|_{L^1_{\frac{u^*(t)^{p-1}}{t}}}.\end{aligned}$$

We also have

$$\Lambda_{u^{*\,p-1}} = (L^1, L^\infty)_{\Phi_0;K}.$$

The last identification of the spaces follows from the estimates

$$\begin{aligned}\|x\|_{\Lambda_{u^{*\,p-1}}} &= \int_0^\infty x^*(t)u^*(t)^{p-1}\,dt\\ &\le \int_0^\infty \Big(\int_0^t x^*(s)\,ds\Big)u^*(t)^{p-1}\frac{dt}{t}\\ &= \|x\|_{\Phi_0;K} = \int_0^\infty \Big(\int_s^\infty \frac{u^*(t)^{p-1}}{t}\,dt\Big)x^*(s)\,ds\\ &\le C_4\int_0^\infty u^*(s)^{p-1}x^*(s)\,ds = C_4\|x\|_{\Lambda_{u^{*\,p-1}}}.\end{aligned}$$

Using now a generalization of the Holmstedt formula with quasi-power parameters Φ_0, Φ_1, proved by Dmitriev–Ovchinnikov (1979) and Brudnyĭ–Krugljak (1981) (see [10, Corollary 7.1.1, p. 466] and [41, p. 30] in the discrete case),

$$K\big(t, a; (A_0,A_1)_{\Phi_0;K}, (A_0,A_1)_{\Phi_1;K}\big) \approx K\big(t, K(\cdot, a; A_0, A_1); \Phi_0, \Phi_1\big)$$

we obtain

$$\begin{aligned}K(t, x; \Lambda_{u^{*\,p-1}}, M_{u^*}) &\approx K\big(t, x : (L^1, L^\infty)_{\Phi_0;K}, (L^1, L^\infty)_{\Phi_1;K}\big)\\ &\approx K\big(t, K(s, x; L^1, L^\infty); L^1_{\frac{u^*(s)^{p-1}}{s}}, L^\infty_{\frac{1}{su^*(s)}}\big)\\ &\approx K\Big(t, \int_0^s x^*(u)\,du; L^1_{\frac{u^*(s)^{p-1}}{s}}, L^\infty_{\frac{1}{su^*(s)}}\Big)\\ &= K\big(t, x^{**}(s); L^1_{u^{*\,p-1}}, L^\infty_{\frac{1}{u^*}}\big).\end{aligned}$$

Thus, for $\theta = 1 - \frac{1}{p}$,

$$\|x\|_{(\Lambda_{u^* p-1}, M_{u^*})_{\theta,p}} \approx \|x^{**}\|_{(L^1_{u^* p-1}, L^\infty_{\frac{1}{u^*}})_{\theta,p}}$$
$$\approx \|x^{**}\|_{L^p_w} = \|x^{**}\|_{L^p} \approx \|x^*\|_{L^p} = \|x\|_{L^p},$$

where the first equality comes from the fact that

$$w(t) = \big(u^*(t)^{p-1}\big)^{1-\theta} \Big(\frac{1}{u^*(t)}\Big)^{\theta} = u^*(t)^{1-1/p} \Big(\frac{1}{u^*(t)}\Big)^{1-1/p} = 1.$$

Thus $(\Lambda_{u^* p-1}, M_{u^*})_{\theta,p} = L^p$ with equivalent norms and the proof is complete. ■

Corollary 8 *If $1 < p < \infty$ and estimates in* (i) *are satisfied, then $(\Lambda_{u^* p-1}, M_{u^*})_{\theta,p} = L^p$ and so L^p is an interpolation space between $\Lambda_{u^* p-1}$ and M_{u^*}. In particular, since $u(t) = t^{-1/p}$ with $1 < p < \infty$ satisfies estimates in* (i) *we obtain $(L^{p,1}, L^{p,\infty})_{1-1/p,p} = L^p$ and so L^p is an interpolation space between Lorentz space $L^{p,1}$ and the Marcinkiewicz weak L^p-space $L^{p,\infty}$.*

Let us consider the factorization theorem of Schur type, that is, a factorization through weighted L^1 and L^∞ spaces. We will show the failure of a Schur type factorization theorem even for powers in symmetric spaces for the positive sublinear Hardy operator.

A symmetric space X on $(0,\infty)$ is a Banach ideal space on $(0,\infty)$ with the additional property that $x^*(t) \le y^*(t)$ for every $t > 0$ and $y \in X$ imply $x \in X$ and $\|x\|_X \le \|y\|_X$, where x^* denotes the decreasing rearrangement of $|x|$ (see [25] for definition and properties). The *fundamental function* $\varphi_X(t)$ of X is defined by $\varphi_X(t) = \|\chi_{(0,t)}\|_X, t > 0$.

Given $\lambda > 0$, the dilation operator σ_λ given by $\sigma_\lambda x(t) = x(t/\lambda)$, $t > 0$, is well defined in every symmetric space X and $\|\sigma_\lambda\|_{X\to X} \le \max(1,\lambda)$. The classical *Boyd indices of* X are defined by (*cf.* [2, 25, 27])

$$\alpha_X = \lim_{\lambda\to 0} \frac{\ln \|\sigma_\lambda\|_{X\to X}}{\ln \lambda}, \ \beta_X = \lim_{\lambda\to\infty} \frac{\ln \|\sigma_\lambda\|_{X\to X}}{\ln \lambda}.$$

In general, $0 \le \alpha_X \le \beta_X \le 1$. For other properties of symmetric spaces and also the Lorentz and Marcinkiewicz spaces we refer to [2, 25, 27].

Theorem 7 *Let $0 < \theta < 1$, X be a symmetric space on $(0,\infty)$ and for weights u, v on $(0,\infty)$ we have $u(t)^{1/\theta} v(t)^{1/(1-\theta)} = 1$ for all $t > 0$. Consider the sublinear Hardy operator $H^*x(t) = x^{**}(t) = \frac{1}{t}\int_0^t x^*(s)\,ds$. Then both $H^*\colon X_u \to X_u$ and $H^*\colon L^\infty_v \to L^\infty_v$ are bounded if and only if $u(t) \approx v(t) \approx \text{constant}$ and $\beta_X < 1$.*

Proof If the Hardy sublinear operator $H^*\colon L^\infty_v \to L^\infty_v$ is bounded, then

$$\Big(\frac{1}{v}\Big)^{**}(t) = \frac{1}{t}\int_0^t \Big(\frac{1}{v}\Big)^*(s)\,ds \le C_1 \frac{1}{v(t)} \quad \text{for all } t > 0. \tag{10}$$

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From the Semenov imbedding theorem (see [25, Theorem 5.7]),

$$\sup_{t>0} \frac{\varphi_X(t)}{t} \int_0^t x^*(s)\,ds \le \|x\|_X \quad \text{for all } x \in X,$$

and the boundedness of $H^*\colon X_u \to X_u$, we obtain that

$$\sup_{t>0} \frac{\varphi_X(t)}{t} \int_0^t (uH^*x)^*(s)\,ds \le C_0 \|xu\|_X \quad \text{for all } x \in X.$$

Using the assumption on weights and (10) we have

$$\begin{aligned}
\int_0^t (uH^*x)^*(s)\,ds &= \int_0^t (v^{-\frac{\theta}{1-\theta}} H^*x)^*(s)\,ds \\
&\ge \int_0^t \Big[C_1 \Big(H^*\Big(\frac{1}{v}\Big)\Big)^{\frac{\theta}{1-\theta}} H^*x \Big]^*(s)\,ds \\
&= C_1 \int_0^t H^*x(s) \Big(H^*\Big(\frac{1}{v}\Big)(s)\Big)^{\frac{\theta}{1-\theta}} ds
\end{aligned}$$

from which we obtain

$$\sup_{t>0} \frac{\varphi_X(t)}{t} \int_0^t H^*x(s) \Big[H^*\Big(\frac{1}{v}\Big)(s)\Big]^{\frac{\theta}{1-\theta}} ds \le \frac{C_0}{C_1} \|xu\|_X \quad \text{for all } x \in X. \tag{11}$$

Assume first that $\lim_{t\to\infty} (\frac{1}{v})^*(t) = 0$.

For any $\varepsilon > 0$ we can find a set $A_\varepsilon \subset \{s > 0 : \frac{1}{v(s)} < \varepsilon\}$ such that its measure $m(A_\varepsilon) = 1$. For the functions $x_\varepsilon = \chi_{A_\varepsilon}$ we have

$$\begin{aligned}
\sup_{t>0} \frac{\varphi_X(t)}{t} \int_0^t H^*x_\varepsilon(s) \Big[H^*\Big(\frac{1}{v}\Big)(s)\Big]^{\frac{\theta}{1-\theta}} ds \\
&\ge \varphi_X(1) \int_0^1 \frac{\min\big(s, m(A_\varepsilon)\big)}{s} \Big[H^*\Big(\frac{1}{v}\Big)(s)\Big]^{\frac{\theta}{1-\theta}} ds \\
&= \varphi_X(1) \int_0^1 \Big[\frac{1}{s} \int_0^s \Big(\frac{1}{v}\Big)^*(\xi)\,d\xi \Big]^{\frac{\theta}{1-\theta}} ds \\
&\ge \varphi_X(1) \int_0^1 \Big(\frac{1}{v}\Big)^*(s)^{\frac{\theta}{1-\theta}} ds
\end{aligned}$$

and putting it into (11) we get

$$\varphi_X(1) \int_0^1 \Big(\frac{1}{v}\Big)^*(s)^{\frac{\theta}{1-\theta}} ds \le \frac{C_0}{C_1} \|u\chi_{A_\varepsilon}\|_X = \frac{C_0}{C_1} \Big\| \frac{1}{v^{\frac{\theta}{1-\theta}}} \chi_{A_\varepsilon} \Big\|_X \le \frac{C_0}{C_1} \varepsilon^{\frac{\theta}{1-\theta}} \varphi_X(1)$$

or

$$\int_0^1 \Big(\frac{1}{v}\Big)^*(s)^{\frac{\theta}{1-\theta}} ds \le \frac{C_0}{C_1} \varepsilon^{\frac{\theta}{1-\theta}},$$

which gives that $\frac{1}{v(s)} = 0$ a.e. But this is impossible.

If $\lim_{t\to\infty}(\frac{1}{v})^*(t) = c > 0$, then by (10)

$$\frac{1}{v(t)} \geq \frac{c}{C_1} \quad \text{for all } t > 0$$

which means that v is bounded.

Assume that $\lim_{t\to 0^+}(\frac{1}{v})^*(t) = \infty$. For any $k, n \in N$ let $A_k = \{t > 0 : 2^k < u(t) \leq 2^{k+1}\}$ and $B_n = \{t > 0 : (\frac{1}{v})^*(t) > 2^n\}$.

Let us choose $k \in N$ such that $m(A_k) > 0$. For any $n \in N, n > k$, we can find $D_n \subset A_k$ such that $0 < m(D_n) < m(B_n)$. Putting $x_n = \chi_{D_n}$ into (11), we obtain the estimates

$$\begin{aligned}
\sup_{t>0} \frac{\varphi_X(t)}{t} \int_0^t H^* x_n \Big[H^*\Big(\frac{1}{v}\Big)(s)\Big]^{\frac{\theta}{1-\theta}} ds
&\geq \sup_{t>0} \frac{\varphi_X(t)}{t} \int_0^t H^* x_n \Big[\Big(\frac{1}{v}\Big)^*(s)\Big]^{\frac{\theta}{1-\theta}} ds \\
&\geq \frac{\varphi_X\big(m(D_n)\big)}{m(D_n)} \int_0^{m(D_n)} \frac{\min\big(s, m(D_n)\big)}{s} \Big[\Big(\frac{1}{v}\Big)^*(s)\Big]^{\frac{\theta}{1-\theta}} ds \\
&= \frac{\varphi_X\big(m(D_n)\big)}{m(D_n)} \int_0^{m(D_n)} \Big[\Big(\frac{1}{v}\Big)^*(s)\Big]^{\frac{\theta}{1-\theta}} ds \\
&\geq \varphi_X\big(m(D_n)\big)\,(2^n)^{\frac{\theta}{1-\theta}}
\end{aligned}$$

and

$$\|x_n u\|_X \leq 2^{k+1}\|\chi_{D_n}\|_X = 2^{k+1}\varphi_X\big(m(D_n)\big),$$

which give

$$(2^n)^{\frac{\theta}{1-\theta}}\varphi_X\big(m(D_n)\big) \leq 2^{k+1}\varphi_X\big(m(D_n)\big),$$

and we come to a contradiction.

If $\lim_{t\to 0^+}(\frac{1}{v})^*(t) = c < \infty$, then $\|\frac{1}{v}\|_{L^\infty} = c$, and we obtain estimation of v from below. Hence, the weight v is equivalent to a constant, and so the weight u is also equivalent to a constant, which means that $X_u = X$. By using the theorem on boundedness of the H^* operator in a symmetric space X (*cf.* [25, Theorem 6.6]) we have that this is equivalent with the condition $\beta_X < 1$. This result gives also the reverse implication and the proof is complete. ■

Corollary 9 *Let X be a symmetric space on $(0,\infty)$ with $\beta_X = 1$. Then for every $1 < p < \infty$ there are no weights u, v such that $H^*\colon L^\infty_v \to L^\infty_v$ is bounded and $H^*\colon X_u \to X_u$ is also bounded, and $(X_u)^{1/p}(L^\infty_v)^{1-1/p} = X^{(p)}$.*

Proof Since $\|\sigma_\lambda\|_{X^{(p)}\to X^{(p)}} = \|\sigma_\lambda\|^{1/p}_{X\to X}$ it follows that $\beta_{X^{(p)}} = \frac{1}{p}\beta_X < 1$ and by using the theorem on boundedness of the H^* operator in a symmetric space X (*cf.*

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[25, Theorem 6.6]) we have that $H^*\colon X^{(p)} \to X^{(p)}$ is bounded and $H^*\colon X \to X$ is unbounded.

Assume that for H^* we can find weights u, v such that $H^*\colon L^\infty_v \to L^\infty_v$ is bounded and $H^*\colon X_u \to X_u$ is also bounded, and $(X_u)^{1/p}(L^\infty_v)^{1-1/p} = X^{(p)}$.

Since the equality $X^{(p)} = (X_u)^{1/p}(L^\infty_v)^{1-1/p}$ gives $(X_u)^{1/p}(L^\infty_v)^{1-1/p} = X^{(p)} = X^{1/p}(L^\infty)^{1-1/p}$, by Theorem 3,

$$u^{1/p}v^{1-1/p} = u^{1-\theta}v^{\theta} \approx 1 \text{ or } u^{\frac{1}{\theta}}v^{\frac{1}{1-\theta}} \approx 1.$$

By Theorem 7 we come to the conclusion that $\beta_X < 1$, which is a contradiction with the assumption on X that $\beta_X = 1$. ■

Corollary 9 gives the following

Remark 3 The Schur test for the sublinear Hardy operator H^* in the $X^{(p)}$ spaces with $\beta(X) = 1$ through the weighted X and weighted L^∞ spaces does not hold even for $\theta = 1 - 1/p$.

This example of the Hardy positive sublinear operator shows that without any additional assumptions on the operator the factorization theorem through weighted L^1 and weighted L^∞ spaces cannot be true.

4 On the Failure of the Factorization Theorem for the Volterra Operator in Some Calderón–Lozanovskiĭ Spaces

We will show here that the factorization theorem of the Rubio de Francia type is not true in general in Calderón–Lozanovskiĭ spaces for a simple integral operator such as the Volterra operator (sometimes also called the *integration operator*) $Vx(t) = \int_0^t x(s)\,ds$.

Let us formulate the *main factorization problem*: Let $\varphi \in \mathcal{U}$, four weights u_0, u_1, v_0, v_1 and a bounded positive linear (or sublinear) operator

$$T\colon \varphi(L^1_{u_0}, L^\infty_{u_1}) \to \varphi(L^1_{v_0}, L^\infty_{v_1}) \tag{12}$$

be given. Can we find four weights w_0, w_1, h_0, h_1 such that

$$\varphi(L^1_{u_0}, L^\infty_{u_1}) = \varphi(L^1_{w_0}, L^\infty_{w_1}),\ \varphi(L^1_{v_0}, L^\infty_{v_1}) = \varphi(L^1_{h_0}, L^\infty_{h_1}) \tag{13}$$

and

$$T\colon L^1_{w_0} \to L^1_{h_0},\ T\colon L^\infty_{w_1} \to L^\infty_{h_1} \tag{14}$$

are bounded?

The answer to this problem is negative already for the Volterra operator. We will find a function $\varphi \in \mathcal{U}$ and four weights u_0, u_1, v_0, v_1 for which (12) is true for the Volterra operator but it is not possible to find weights satisfying (13) and (14). By the

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interpolation property of the Calderón–Lozanovskiĭ construction we have that the assumptions (13) and (14) imply boundedness in (12).

We need first some lemmas.

Lemma 7 *Let $\varphi \in \mathcal{U}$ and weights u_0, u_1, v_0, v_1 on $(0,\infty)$ be given. For $t > 0$ put*

$$w_0(t) = \operatorname{ess}\sup_{0<s\le t} \frac{1}{u_0(s)}, \ w_1(t) = \int_0^t \frac{1}{u_1(s)}\, ds.$$

If $\varphi(w_0, w_1) \in \varphi(L^1_{v_0}, L^\infty_{v_1})$, then for the Volterra operator $Vx(t) = \int_0^t x(s)\, ds$ we have

$$\|V\|_{\varphi(L^1_{u_0}, L^\infty_{u_1}) \to \varphi(L^1_{v_0}, L^\infty_{v_1})} < \infty.$$

Proof If $x \in \varphi(L^1_{u_0}, L^\infty_{u_1})$ and the norm is < 1, then

$$|x| \le \varphi(|x_0|, |x_1|), \text{ with } \|x_0\|_{L^1_{u_0}} \le 1, \ \|x_1\|_{L^\infty_{u_1}} \le 1.$$

Thus, by Lemma 4 and the definition of weights, we have

$$\begin{aligned}
|Vx| \le V(|x|) \le V\big(\varphi(|x_0|, |x_1|)\big) &\le \varphi\big(V(|x_0|), V(|x_1|)\big) \\
&= \varphi\left(\int_0^t \frac{1}{u_0(s)} |x_0(s)| u_0(s)\, ds, \int_0^t |x_1(s)|\, ds\right) \\
&\le \varphi\left(w_0(t) \int_0^t |x_0(s)| u_0(s)\, ds, \|x_1 u_1\|_{L^\infty} \int_0^t \frac{1}{u_1(s)}\, ds\right) \\
&\le \varphi\big(w_0(t) \|x_0 u_0\|_{L^1}, w_1(t) \|x_1 u_1\|_{L^\infty}\big) \le \varphi\big(w_0(t), w_1(t)\big).
\end{aligned}$$

Hence $Vx \in \varphi(L^1_{v_0}, L^\infty_{v_1})$ and

$$\|V\|_{\varphi(L^1_{u_0}, L^\infty_{u_1}) \to \varphi(L^1_{v_0}, L^\infty_{v_1})} \le \|\varphi(w_0, w_1)\|_{\varphi(L^1_{v_0}, L^\infty_{v_1})} < \infty. \qquad \blacksquare$$

Let $\psi(t) = \frac{t}{\ln(1+t)}$ for $t > 0$. This is a concave function on $(0,\infty)$ with $\lim_{t\to 0^+} \psi(t) = 1$. Then the function ψ given by

$$\psi(s,t) = s\psi(t/s) = \frac{s}{\ln(1+\frac{s}{t})} \tag{15}$$

belongs to $\mathcal{U}$.

Lemma 8 *For the function ψ from* (15) *we can find weights u_0, u_1, v_0, v_1 on $(0,\infty)$ such that for the Volterra operator $Vx(t) = \int_0^t x(s)\, ds$ we have*

$$\|V\|_{L^1_{u_0} \to L^1_{v_0}} = \infty, \ \|V\|_{L^\infty_{u_1} \to L^\infty_{v_1}} < \infty$$

and

$$\|V\|_{\psi(L^1_{u_0}, L^\infty_{u_1}) \to \psi(L^1_{v_0}, L^\infty_{v_1})} < \infty.$$

Proof Choose weights u_1, v_0 such that

(i) $\int_0^t \frac{1}{u_1(s)}\, ds < \infty$ for all $t > 0$,
(ii) $\int_t^\infty v_0(s)\, ds < \infty$ for all $t > 0$,
(iii) $\Theta(t) := \int_0^t \frac{1}{u_1(s)}\, ds \int_t^\infty v_0(s)\, ds$ is an increasing function and $\lim_{t\to\infty} \Theta(t) = \infty$

and put

$$u_0(t) = v_1(t) = \frac{1}{\Theta(t)} \int_t^\infty v_0(s)\, ds = \frac{1}{\int_0^t \frac{1}{u_1(s)}\, ds}.$$

For general weights u, v the norms of an operator V between weighted L^1 and weighted L^∞ spaces are known:

$$\|V\|_{L^1_u \to L^1_v} = \operatorname*{ess\,sup}_{t>0} \frac{1}{u(t)} \int_t^\infty v(s)\, ds$$

and

$$\|V\|_{L^\infty_u \to L^\infty_v} = \operatorname*{ess\,sup}_{t>0} v(t) \int_0^t \frac{1}{u(s)}\, ds.$$

In our case of special weights we obtain

$$\|V\|_{L^1_{u_0} \to L^1_{v_0}} = \operatorname*{ess\,sup}_{t>0} \frac{1}{u_0(t)} \int_t^\infty v_0(s)\, ds = \operatorname*{ess\,sup}_{t>0} \Theta(t) = \infty,$$

$$\|V\|_{L^\infty_{u_1} \to L^\infty_{v_1}} = \operatorname*{ess\,sup}_{t>0} v_1(t) \int_0^t \frac{1}{u_1(s)}\, ds = 1.$$

Therefore the first two conditions on V are satisfied. To show the third one we use Lemma 7.

Since u_0 is a decreasing function it follows that

$$w_0(t) = \operatorname*{ess\,sup}_{0<s\le t} \frac{1}{u_0(s)} = \frac{1}{u_0(t)} = \int_0^t \frac{1}{u_1(s)}\, ds = w_1(t),$$

and so

$$\psi\big(w_0(t), w_1(t)\big) = \psi(1,1) w_1(t) = \frac{1}{\ln 2} w_1(t).$$

If $x_0(t) > 0$ a.e. on $(0,\infty)$, then

$$\psi\Big(\frac{x_0(t)}{v_0(t)}, \frac{1}{v_1(t)}\Big) = \frac{x_0(t)}{v_0(t) \ln\big(1 + \frac{x_0(t)}{v_0(t)} v_1(t)\big)} \ge \frac{1}{v_1(t)} = w_1(t).$$

Thus,

$$\psi(w_0, w_1) = \frac{1}{\ln 2} w_1 \le \psi\Big(\frac{x_0}{v_0}, \frac{1}{v_1}\Big) \in \psi(L^1_{v_0}, L^\infty_{v_1})$$

and, according to Lemma 7, the operator V is bounded from $\psi(L^1_{u_0}, L^\infty_{u_1})$ into $\psi(L^1_{v_0}, L^\infty_{v_1})$.

As function $x_0(t) > 0$ for which $\|x_0\|_{L^1} \le 1$ we can take, for example,

$$x_0(t) = \frac{1}{3} \sum_{k=-\infty}^{\infty} \min(1, 2^{-2k}) \chi_{(2^k, 2^{k+1})}. \qquad \blacksquare$$

Theorem 8 *Let ψ and weights u_0, u_1, v_0, v_1 on $(0,\infty)$ be the same as in Lemma 8. There are no weights w_0, w_1, h_0, h_1 on $(0,\infty)$ that satisfy*

$$\psi(L^1_{w_0}, L^\infty_{w_1}) = \psi(L^1_{u_0}, L^\infty_{u_1}),\ \psi(L^1_{h_0}, L^\infty_{h_1}) = \psi(L^1_{v_0}, L^\infty_{v_1})$$

and the Volterra operator V is bounded between

$$V : L^1_{w_0} \to L^1_{h_0} \text{ and } V : L^\infty_{w_1} \to L^\infty_{h_1}.$$

Proof Assume conversely that such weights exist. For the function ψ we have

$$\rho_\psi(a) = \limsup_{t\to\infty} \frac{\psi(at,1)}{\psi(t,1)} = \limsup_{t\to\infty} \frac{at\ln(1+t)}{t\ln(1+at)} = a.$$

Therefore, if $\psi(L^1_{w_0}, L^\infty_{w_1}) = \psi(L^1_{u_0}, L^\infty_{u_1})$, then, by Theorem 4 and Lemma 3 with $\theta = 1$, we have $w_0 \approx u_0$. Similarly, since $\psi(L^1_{h_0}, L^\infty_{h_1}) = \psi(L^1_{v_0}, L^\infty_{v_1})$ then again by, Corollary 4 with $\theta = 1$, we have $h_0 \approx v_0$. According to Lemma 8

$$\|V\|_{L^1_{w_0}\to L^1_{h_0}} = \|V\|_{L^1_{u_0}\to L^1_{v_0}} = \infty,$$

which is a contradiction. $\blacksquare$

Immediately from Lemma 8 and Theorem 8 we have the following example:

Example 2 Let $\psi(s,t) = \frac{s}{\ln(1+\frac{s}{t})}$. We can find weights u_0, u_1, v_0, v_1 on $(0,\infty)$ such that the factorization theorem for the Volterra operator V from the space $\psi(L^1_{u_0}, L^\infty_{u_1})$ into $\psi(L^1_{v_0}, L^\infty_{v_1})$ does not hold.

Remark 4 Theorem 8 and Example 2 are still true if we take any function $\psi \in \mathfrak{U}$ which satisfies two conditions:

$$\lim_{s\to 0^+} \psi(s,1) = c > 0 \qquad \text{and} \qquad \rho_\psi(a) = \limsup_{t\to\infty} \frac{\psi(at,1)}{\psi(t,1)} = a$$

for all $a > 0$.

The failure of the factorization theorem for the operator V was given for the function ψ with the property that $\lim_{s\to 0^+} \psi(s,1) = c > 0$ (see Corollary 9 and Remark 4). We will also present a result for when the function ψ satisfies $\lim_{s\to 0^+} \psi(s,1) = 0$. It is enough to prove a lemma corresponding to Lemma 8.

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For fixed $0 < \theta < 1$, let

$$\psi_\theta(t) = \begin{cases} t^\theta & \text{if } 0 \le t \le \tau(\theta), \\ \text{linear} & \text{if } \tau(\theta) \le t \le 1, \\ \psi(t) & \text{if } t > 1, \end{cases}$$

where $\psi(t) = \frac{t}{\ln(1+t)}$ and $\tau = \tau(\theta)$ is a point in $(0,1)$ such that ψ_θ is concave on $(0,\infty)$, *i.e.*, $\psi'(1) < \frac{\psi(1)-\tau}{1-\tau} < \theta\tau^\theta$. Such a point exists since $\psi'(1) = 1/\ln 2 - 1/[2(\ln 2)]^2 < 1/\ln 2 = \psi(1)$ and $\lim_{\tau\to 0^+} \frac{\psi(1)-\tau}{1-\tau} = \psi(1)$, $\lim_{\tau\to 0^+} \tau^{\theta-1} = \infty$. Let

(16) $$\psi_\theta(s,t) = t\psi_\theta(s/t) \text{ with the above function } \psi_\theta(t).$$

Then $\psi_\theta \in \mathcal{U}$, $\lim_{s\to 0^+} \psi_\theta(s,1) = 0$ and $\rho_{\psi_\theta}(a) = a$ for all $a > 0$.

Lemma 9 *Let $0 < \theta < 1$ be fixed and let the function $\psi_\theta \in \mathcal{U}$ be given by* (16). *Then there exist weights u_0, u_1, v_0, v_1 on $(0,\infty)$ such that, for the Volterra operator $Vx(t) = \int_0^t x(s)\,ds$, we have*

$$\|V\|_{L^1_{u_0}\to L^1_{v_0}} = \infty,\ \|V\|_{L^\infty_{u_1}\to L^\infty_{v_1}} = 1$$

and

$$\|V\|_{\psi_\theta(L^1_{u_0},L^\infty_{u_1})\to\psi_\theta(L^1_{v_0},L^\infty_{v_1})} \le 2.$$

Proof For fixed $\alpha \in (0,1)$ define weights u_0, v_0 by

$$u_0(t) = \min\big(t,\tau(\theta)\big)^\alpha \text{ and } v_0(t) = \tau(\theta)^{1-\alpha}t^{-2}.$$

Then

$$\begin{aligned} \|V\|_{L^1_{u_0}\to L^1_{v_0}} &= \operatorname*{ess\,sup}_{t>0} \frac{1}{u_0(t)} \int_t^\infty v_0(s)\,ds \\ &= \operatorname*{ess\,sup}_{t>0} \big[\min\big(t,\tau(\theta)\big)^{-\alpha}\tau(\theta)^{1-\alpha}t^{-1}\big] \\ &= \operatorname*{ess\,sup}_{t>0} \max\Big(\Big(\frac{\tau(\theta)}{t}\Big)^{1-\alpha}, \frac{\tau(\theta)}{t}\Big) = \infty. \end{aligned}$$

Choose as weight u_1 a function which satisfies

$$\int_0^0 \frac{1}{u_1(s)}\,ds \ge \frac{1}{\tau(\theta)u_0(t)} \quad \text{for } t \in \big(0,\tau(\theta)\big]$$

and $u_1(t) = \tau(\theta)^{-\alpha}$ for $t > \tau(\theta)$. Let

$$\frac{1}{v_1(t)} = \begin{cases} f(t)\int_0^t \frac{ds}{u_1(s)} & \text{if } 0 < t \le \tau(\theta), \\ \int_0^t \frac{ds}{u_1(s)} & \text{if } t > \tau(\theta), \end{cases}$$

where $f(t)$ is a function that for $0 < t \le \tau(\theta)$ satisfies

$$f(t) \ge \max\Big(\frac{1}{\tau(\theta) v_0(t) \int_0^t u_1(s)\, ds}, \ \Big(\frac{v_0(t)}{u_0(t)}\Big)^{\theta/(1-\theta)} \Big).$$

As u_1 and f we can take for example, on $(0, \tau(\theta)]$,

$$u_1(t) = t^{1-\alpha}/\big(\alpha\tau(\theta)\big) \text{ and } f(t) = \tau(\theta)^{\alpha-1} t^{\theta(1-\alpha)/(1-\theta)}.$$

Then

$$\begin{aligned} \|V\|_{L^\infty_{u_1} \to L^\infty_{v_1}} &= \operatorname*{ess\,sup}_{t>0} v_1(t) \int_0^t \frac{1}{u_1(s)}\, ds \\ &= \max\Big(\operatorname*{ess\,sup}_{0<t\le\tau(\theta)} \frac{1}{f(t)}, 1\Big) \\ &\le \max\big(\tau(\theta)^{\theta/(1-\theta)}, 1\big) = 1. \end{aligned}$$

Consider the functions

$$x_0(t) = \begin{cases} \frac{1}{v_0(t)} & \text{if } 0 < t \le \tau(\theta), \\ \frac{1}{u_0(t)} & \text{if } t > \tau(\theta), \end{cases}$$

and $x_1(t) = \frac{1}{v_1(t)}$. Then

$$\|x_0\|_{L^1_{v_0}} = \tau(\theta) + 1, \ \|x_1\|_{L^1_{v_1}} = 1,$$

and for $0 < t \le \tau(\theta)$,

$$\begin{aligned} \psi_\theta\Big(\frac{1}{u_0(t)}, \int_0^t \frac{ds}{u_1(s)}\Big) &= u_0(t)^{-\theta} \Big(\int_0^t \frac{ds}{u_1(s)}\Big)^{1-\theta} \\ &\le f(t)^{1-\theta} v_0(t)^{-\theta} \Big(\int_0^t \frac{ds}{u_1(s)}\Big)^{1-\theta} \\ &= \psi_\theta\Big(\frac{1}{v_0(t)}, f(t) \int_0^t \frac{ds}{u_1(s)}\Big) \\ &= \psi_\theta\Big(\frac{1}{v_0(t)}, \frac{1}{v_1(t)}\Big) = \psi_\theta\big(x_0(t), x_1(t)\big). \end{aligned}$$

For $t > \tau(\theta)$,

$$\psi_\theta\Big(\frac{1}{u_0(t)}, \int_0^t \frac{ds}{u_1(s)}\Big) = \psi_\theta\Big(\frac{1}{u_0(t)}, \frac{1}{v_1(t)}\Big) = \psi_\theta\big(x_0(t), x_1(t)\big).$$

Thus, by similar considerations as those in Lemma 7, we find that

$$\|V\|_{\psi_\theta(L^1_{u_0}, L^\infty_{u_1}) \to \psi_\theta(L^1_{v_0}, L^\infty_{v_1})} \le 1 + \tau(\theta) \le 2.$$

and Lemma 9 is proved. ■

Analogously as in the proof of Theorem 8 and using instead of Lemma 8 the just proved Lemma 9, we can formulate a similar theorem under the assumption that $\lim_{s\to 0^+} \psi(s,1) = 0$.

Theorem 8′ *Let ψ and weights u_0, u_1, v_0, v_1 on $(0,\infty)$ be the same as in Lemma 9. There are no weights w_0, w_1, h_0, h_1 on $(0,\infty)$ that satisfy*

$$\psi(L^1_{w_0}, L^\infty_{w_1}) = \psi(L^1_{u_0}, L^\infty_{u_1}),\ \psi(L^1_{h_0}, L^\infty_{h_1}) = \psi(L^1_{v_0}, L^\infty_{v_1})$$

and so that the Volterra operator V is bounded between

$$V : L^1_{w_0} \to L^1_{h_0} \text{ and } V : L^\infty_{w_1} \to L^\infty_{h_1}.$$

5 Factorization of the Averaging Operator

We shall now consider a factorization theorem for the averaging operator on $[0,\infty)$. Let

$$A_I x(t) = \frac{1}{|I|}\int_I x(s)\, ds \chi_I(t), \quad \text{where } I = [a,b] \text{ with } a,b > 0.$$

We can easily see that

$$\|A_I\|_{L^1_u \to L^1_u} = \frac{1}{|I|}\int_I u(s)\, ds \operatorname*{ess\,sup}_{t\in I} \frac{1}{u(t)}$$

and

$$\|A_I\|_{L^\infty_v \to L^\infty_v} = \frac{1}{|I|}\int_I \frac{1}{v(s)}\, ds \operatorname*{ess\,sup}_{t\in I} v(t).$$

We show that the analogue of the factorization theorem of Muckenhoupt's A_p-condition (*cf.* [39]) for the operator A and the space $\psi(L^1_u, L^\infty_v)$ does not hold.

Theorem 9 *Let $\psi \in \mathfrak{U}$ be such a function that $\lim_{s\to 0^+} \psi(s,1) = c > 0$ and $\rho_\psi(a) = \limsup_{t\to\infty} \frac{\psi(at,1)}{\psi(t,1)} = a$ for all $a > 0$. If $u(t) = \sum_{k=0}^{\infty} 2^k \chi_{[k,k+1)}(t)$ for $t \ge 0$, then for the averaging operator A_I we have*

$$\sup_I \|A_I\|_{\psi(L^1_u, L^\infty) \to \psi(L^1_u, L^\infty)} = C < \infty,$$

and there are no weights w_0, w_1, h_0, h_1 on $(0,\infty)$ that satisfy the conditions

$$\psi(L^1_{w_0}, L^\infty_{w_1}) = \psi(L^1_u, L^\infty),\ \psi(L^1_{h_0}, L^\infty_{h_1}) = \psi(L^1_u, L^\infty)$$

with

$$\sup_I \|A_I\|_{L^1_{w_0} \to L^1_{h_0}} < \infty,\ \sup_I \|A_I\|_{L^\infty_{w_1} \to L^\infty_{h_1}} < \infty.$$

Proof We have

$$\sup_I \|A_I\|_{L^1_u \to L^1_u} = \frac{1}{|I|}\int_I u(s)\,ds\, \operatorname*{ess\,sup}_{t>0} \frac{1}{u(t)}$$
$$\geq \sup_{n\in\mathbf{N}} \frac{1}{n}\int_0^n u(s)\,ds\, \operatorname*{ess\,sup}_{t\in[0,n]} \frac{1}{u(t)} = \sup_{n\in\mathbf{N}} \frac{2^n-1}{n} = \infty.$$

Of course, $\sup_I \|A_I\|_{L^\infty\to L^\infty} = 1$. We show that

$$\sup_I \|A_I\|_{\psi(L^1_u, L^\infty)\to\psi(L^1_u,L^\infty)} = C < \infty.$$

If $|I| \leq 1$ and $I \cap [i, i+1) \neq \varnothing$, then $2^{i-1} \leq u(t) \leq 2^{i+1}$ for all $t \in I$ and

$$\sup_I \|A_I\|_{\psi(L^1_u, L^\infty)\to\psi(L^1_u,L^\infty)} \leq 4.$$

If $|I| > 1$, then for all $\|x_0\|_{L^1_u} \leq 1$ and $\|x_1\|_{L^\infty} \leq 1$, we have

$$A_I\big(\psi\big(|x_0(t)|, |x_1(t)|\big)\big) \leq \psi\big(A_I\big(|x_0(t)|\big), A_I\big(|x_1(t)|\big)\big) \leq \psi\big(A_I(|x_0(t)|), 1\big).$$

Since $u(t) \geq 1$ it follows that $a = \frac{1}{|I|}\int_I |x_0(s)|\,ds \leq \frac{1}{|I|}\int_I |x_0(s)|u(s)\,ds \leq \frac{1}{|I|} < 1$ and so $\psi(a,1) \leq \psi(1,1)$. Thus, choosing $x_0(t) > 0$ a.e. we obtain

$$A_I\big(\psi\big(|x_0(t)|, |x_1(t)|\big)\big) \leq \psi(1,1) \leq \frac{\psi(1,1)}{c}\psi\big(|x_0(t)|, 1\big)$$

which means that

$$\sup_I \|A_I\|_{\psi(L^1_u, L^\infty)\to\psi(L^1_u,L^\infty)} \leq \frac{\psi(1,1)}{c}$$

for $|I| > 1$. Thus $C = \max(4, \frac{\psi(1,1)}{c})$.

The rest of the proof is similar to the proof of Theorem 8. ■

6 The Failure for the Hardy Operator of the Factorization and the Schur Lemma in Some Reflexive Orlicz Spaces

The Schur lemma for an integral operator $Kx(t) = \int k(t,s)x(s)\,ds$ with a positive kernel $k(t,s) \geq 0$ saying that the operator K is bounded in $L^p(1 < p < \infty)$ if and only if there exists a positive function u such that

$$Ku^q(t) \leq Cu^q(t) \text{ and } K' u^p(t) \leq Cu^p(t),$$

where $1/p + 1/q = 1$ and K' is the formal associate operator. We can rewrite this in factorization form: there exists a positive function u such that

$$K\colon L^1_{u^p} \to L^1_{u^p} \text{ and } K\colon L^\infty_{u^{-p'}} \to L^\infty_{u^{-p'}}$$

is bounded. The last statement is a factorization theorem for the operator K.

Theorem 8 shows that a similar factorization theorem for the Volterra operator between weighted Orlicz spaces L^M is not possible, but these Orlicz spaces are not reflexive. We will show below that the classical Hardy operator $Hx(t) = \frac{1}{t}\int_0^t x(s)\,ds$, which is a bounded operator in any reflexive Orlicz space L^M, has in some of them, no factorization through weighted L^1 and weighted L^∞ spaces.

Theorem 10 *There exist reflexive Orlicz spaces L^M on $(0,\infty)$ for which there are no weights u_0, u_1, v_0, v_1 on $(0,\infty)$ that satisfy*

$$\varphi(L^1_{u_0}, L^\infty_{u_1}) = \varphi(L^1, L^\infty) = L^M,\ \varphi(L^1_{v_0}, L^\infty_{v_1}) = \varphi(L^1, L^\infty) = L^M$$

and such that the Hardy operator H is bounded between

$$H\colon L^1_{u_0} \to L^1_{v_0} \text{ and } H\colon L^\infty_{u_1} \to L^\infty_{v_1}.$$

In particular, the Schur lemma does not hold for the Hardy operator H in some reflexive Orlicz spaces.

Proof Since the Orlicz space L^M is reflexive, it follows that the function M and its complementary M^* satisfy the Δ_2-condition, that is, $M(2t) \le CM(t)$ and $M^*(2t) \le CM^*(t)$ for all $t > 0$. For a new Orlicz function M_1 defined by

$$M_1(t) = \int_0^t \frac{M(s)}{s}\,ds, \quad t > 0,$$

we have that M_1 is strictly increasing on $(0,\infty)$. Furthermore M_1 is equivalent to M since $M(t/2) \le M_1(t) \le M(t)$ for all $t > 0$, and so $L^{M_1} = L^M$.

The important step now is a construction of the function M or φ with $\varphi(t) = \varphi(t,1) = M^{-1}(t)$ for which $\rho_\varphi(a)$ from Theorem 4 is not equivalent to a power function for all $a > 0$. Such constructions we will do later on in Examples 2 and 3 but we continue our proof with the function M having such a property.

Assume conversely that we can find weights u_0, u_1, v_0, v_1 on $(0,\infty)$ that satisfy

$$\varphi(L^1_{u_0}, L^\infty_{u_1}) = \varphi(L^1, L^\infty) = L^{M_1},\ \varphi(L^1_{v_0}, L^\infty_{v_1}) = \varphi(L^1, L^\infty) = L^{M_1},$$

and the Hardy operator $H\colon L^1_{u_0} \to L^1_{v_0}$ and $H\colon L^\infty_{u_1} \to L^\infty_{v_1}$ is bounded.

Then neither u_1 nor v_1 is equivalent to a constant function. If u_1 is equivalent to a constant function, then, observing that $\varphi(s,1) = M_1^{-1}(s)$ is strictly increasing, we can use Theorem 4, which gives that u_0 is equivalent to a constant function, and consequently the Hardy operator will be bounded in L^1, which is not the case. Similarly with v_1.

Assume now that both weights u_1 and v_1 are not equivalent to a constant function. Then, again by Theorem 4, we obtain that

$$u_0^\theta u_1^{1-\theta} \approx 1 \text{ and } v_0^\theta v_1^{1-\theta} \approx 1,$$

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where θ is the number such that $\rho_\varphi(a) \approx a^\theta$ for all $a > 0$. Thus any $\varphi \in \mathcal{U}$ such that $\rho_\varphi(a)$ has different indices $\alpha_\varphi \neq \beta_\varphi$ gives a counterexample.

We now only need to give an example of such a function. We give below three such examples, but the proofs we put into Appendix A at the end of the paper. ■

Example 3 Let $0 < \theta_0 < \theta_1 < 1$ and $1 = a_1 < a_2 < a_3 < \cdots$ be a sequence such that the quotient $\frac{a_{n+1}}{a_n}$ is increasing to infinity. Put $\varphi(t) = t^{\theta_1}$ for $0 \leq t \leq 1$ and

$$\varphi(t) = \begin{cases} (\frac{t}{a_{2n-1}})^{\theta_0}\varphi(a_{2n-1}) & \text{if } a_{2n-1} \leq t \leq a_{2n}, \\ (\frac{t}{a_{2n}})^{\theta_1}\varphi(a_{2n}) & \text{if } a_{2n} \leq t \leq a_{2n+1}. \end{cases}$$

Then φ is a quasi-concave function on $(0, \infty)$, *i.e.*, φ is increasing and $\frac{\varphi(t)}{t}$ is decreasing on $(0, \infty)$. It is well known that there exists a concave function $\tilde{\varphi}$ such that $\varphi(t) \leq \tilde{\varphi}(t) \leq 2\varphi(t)$ (see [25]). Moreover,

$$\rho_\varphi(a) = \max(a^{\theta_0}, a^{\theta_1})$$

for any $a > 0$ and the indices are $\alpha_\varphi = \theta_0$, $\beta_\varphi = \theta_1$.

Example 4 For small $\alpha > 0$ and $\sqrt{2}\alpha < \theta \leq 1 - \sqrt{2}\alpha$ let

$$\varphi(t) = \begin{cases} t^\theta & \text{if } 0 \leq t \leq e, \\ t^{\theta+\alpha\sin(\ln\ln t)} & \text{if } t \geq e. \end{cases}$$

Then φ is a quasi-concave function on $(0, \infty)$, $\rho_\varphi(a) = \max(a^{\theta-\sqrt{2}\alpha}, a^{\theta+\sqrt{2}\alpha})$ and the indices are $\alpha_\varphi = \theta - \sqrt{2}\alpha$, $\beta_\varphi = \theta + \sqrt{2}\alpha$.

Example 5 (*cf.* [35, pp. 93–94] for t near zero) For $k > 0$ and $p > \sqrt{2}k + 2$ let

$$M(t) = \begin{cases} t^p & \text{if } 0 \leq t \leq e, \\ t^{p+k\sin(\ln\ln t)} & \text{if } t \geq e. \end{cases}$$

Then M is a convex increasing function on $(0, \infty)$, $\rho_M(a) = \max(a^{p-\sqrt{2}k}, a^{p+\sqrt{2}k})$ and the indices are $\alpha_\varphi = \theta - \sqrt{2}k$, $\beta_\varphi = \theta + \sqrt{2}k$.

A Proofs of the Statements in Examples 3, 4 and 5

Proof in Example 3 From the definition of φ we have

$$\varphi(t) = \begin{cases} t^{\theta_0}\Big(\prod_{k=1}^n \dfrac{a_{2k-1}}{a_{2k-2}}\Big)^{\theta_1-\theta_0} & \text{if } a_{2n-1} \leq t \leq a_{2n}, \\ t^{\theta_1}\Big(\prod_{k=1}^n \dfrac{a_{2k-1}}{a_{2k}}\Big)^{\theta_1-\theta_0} & \text{if } a_{2n} \leq t \leq a_{2n+1}, \end{cases}$$

for $n = 1, 2, \ldots$, where $a_0 = a_1 = 1$.

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We show that φ is a quasi-concave function on $(0,\infty)$. If either $a_{2n-1} \le s < t \le a_{2n}$ or $a_{2n} \le s < t \le a_{2n+1}$, then clearly $\varphi(s) < \varphi(t)$ and $\frac{\varphi(t)}{t} < \frac{\varphi(s)}{s}$. In the remaining case, $a_{2n-1} \le s \le a_{2n} < t < a_{2n+1}$, we have

$$\begin{aligned}\varphi(s) &= \Big(\frac{s}{a_{2n-1}}\Big)^{\theta_0}\varphi(a_{2n-1}) = s^{\theta_0}\Big(\prod_{k=1}^{n}\frac{a_{2k-1}}{a_{2k-2}}\Big)^{\theta_1-\theta_0}\\ &\le a_{2n}^{\theta_0}\Big(\prod_{k=1}^{n}\frac{a_{2k-1}}{a_{2k-2}}\Big)^{\theta_1-\theta_0} < t^{\theta_1}\Big(\prod_{k=1}^{n}\frac{a_{2k-1}}{a_{2k}}\Big)^{\theta_1-\theta_0}\\ &= \Big(\frac{t}{a_{2n}}\Big)^{\theta_1}\varphi(a_{2n}) = \varphi(t)\end{aligned}$$

and

$$\begin{aligned}\frac{\varphi(t)}{t} &= t^{\theta_1-1}\Big(\prod_{k=1}^{n}\frac{a_{2k-1}}{a_{2k}}\Big)^{\theta_1-\theta_0} < a_{2n}^{\theta_1-1}\Big(\prod_{k=1}^{n}\frac{a_{2k-1}}{a_{2k}}\Big)^{\theta_1-\theta_0}\\ &\le s^{\theta_0-1}\Big(\prod_{k=1}^{n}\frac{a_{2k-1}}{a_{2k}}\Big)^{\theta_1-\theta_0} = \frac{\varphi(s)}{s}\end{aligned}$$

and this shows that the function φ is increasing and $\frac{\varphi(t)}{t}$ is decreasing on $(0,\infty)$. Note that

$$\varphi'_{-}(a_{2n}) = \theta_0 a_{2n}^{\theta_0-1}\Big(\prod_{k=1}^{n}\frac{a_{2k-1}}{a_{2k}}\Big)^{\theta_1-\theta_0} < \theta_1 a_{2n}^{\theta_1-1}\Big(\prod_{k=1}^{n}\frac{a_{2k-1}}{a_{2k}}\Big)^{\theta_1-\theta_0} = \varphi'_{+}(a_{2n}),$$

which means that the function φ is not concave on $(0,\infty)$.

Let $a > 1$. We want to show that

$$\liminf_{t\to\infty}\frac{\varphi(at)}{\varphi(t)} = a^{\theta_0} \text{ and } \limsup_{t\to\infty}\frac{\varphi(at)}{\varphi(t)} = a^{\theta_1}. \tag{17}$$

Consider several cases, where $n = 2, 3, \dots$ is arbitrary but fixed:

1. If $t, at \in [a_{2n}, a_{2n+1}]$, then $\frac{\varphi(at)}{\varphi(t)} = a^{\theta_1} \ge a^{\theta_0}$ and so $\limsup_{t\to\infty}\frac{\varphi(at)}{\varphi(t)} \ge a^{\theta_1}$.
2. If $t, at \in [a_{2n-1}, a_{2n}]$, then $\frac{\varphi(at)}{\varphi(t)} = a^{\theta_0} \le a^{\theta_1}$ and so $\liminf_{t\to\infty}\frac{\varphi(at)}{\varphi(t)} \le a^{\theta_0}$.
3. If $t \in (a_{2n-2i}, a_{2n-2i+1}]$, $i = 1, 2, \dots, n$ and $at \in [a_{2n}, a_{2n+1}]$, then

$$\frac{\varphi(at)}{\varphi(t)} = a^{\theta_1}\Big(\frac{\prod_{k=1}^{n}\frac{a_{2k-1}}{a_{2k}}}{\prod_{k=1}^{n-i}\frac{a_{2k-1}}{a_{2k}}}\Big)^{\theta_1-\theta_0} = a^{\theta_1}\Big(\prod_{k=n-i+1}^{n}\frac{a_{2k-1}}{a_{2k}}\Big)^{\theta_1-\theta_0} \le a^{\theta_1}$$

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and

$$\frac{\varphi(at)}{\varphi(t)} = a^{\theta_1}\Big(\prod_{k=n-i+1}^{n}\frac{a_{2k-1}}{a_{2k}}\Big)^{\theta_1-\theta_0} = a^{\theta_0}\Big(a\frac{a_{2n-2i+1}}{a_{2n-2i+2}}\prod_{k=n-i+2}^{n}\frac{a_{2k-1}}{a_{2k}}\Big)^{\theta_1-\theta_0}$$

$$\geq a^{\theta_0}\Big(\frac{at}{a_{2n-2i+2}}\prod_{k=n-i+2}^{n}\frac{a_{2k-1}}{a_{2k}}\Big)^{\theta_1-\theta_0} \geq a^{\theta_0}\Big(\frac{a_{2n}}{a_{2n-2i+2}}\prod_{k=n-i+2}^{n}\frac{a_{2k-1}}{a_{2k}}\Big)^{\theta_1-\theta_0}$$

$$\geq a^{\theta_0}\Big(\frac{a_{2n-2i+3}}{a_{2n-2i+2}}\cdot\frac{a_{2n-2i+5}}{a_{2n-2i+4}}\cdots\frac{a_{2n-1}}{a_{2n-2}}\Big)^{\theta_1-\theta_0} \geq a^{\theta_0}.$$

4. If $t \in [a_{2n-2i-1}, a_{2n-2i}]$, $i = 0, 1, 2, \ldots, n-1$ and $at \in [a_{2n}, a_{2n+1}]$, then

$$\frac{\varphi(at)}{\varphi(t)} = a^{\theta_1}t^{\theta_1-\theta_0}\Big(\frac{\prod_{k=1}^{n}\frac{a_{2k-1}}{a_{2k}}}{\prod_{k=1}^{n-i}\frac{a_{2k-1}}{a_{2k}}\cdot a_{2n-2i}}\Big)^{\theta_1-\theta_0}$$

$$= a^{\theta_1}t^{\theta_1-\theta_0}\Big(\prod_{k=n-i+1}^{n}\frac{a_{2k-1}}{a_{2k}}\frac{1}{a_{2n-2i}}\Big)^{\theta_1-\theta_0}$$

$$= a^{\theta_1}\Big(\frac{t}{a_{2n-2i}}\prod_{k=n-i+1}^{n}\frac{a_{2k-1}}{a_{2k}}\Big)^{\theta_1-\theta_0} \leq a^{\theta_1}$$

and

$$\frac{\varphi(at)}{\varphi(t)} = a^{\theta_1}\Big(\frac{t}{a_{2n-2i}}\prod_{k=n-i+1}^{n}\frac{a_{2k-1}}{a_{2k}}\Big)^{\theta_1-\theta_0}$$

$$= a^{\theta_0}\Big(\frac{at}{a_{2n-2i}}\cdot\frac{a_{2n-2i+1}}{a_{2n-2i+2}}\cdot\frac{a_{2n-2i+3}}{a_{2n-2i+4}}\cdots\frac{a_{2n-1}}{a_{2n}}\Big)^{\theta_1-\theta_0}$$

$$\geq a^{\theta_0}\Big(\frac{a_{2n}}{a_{2n-2i}}\cdot\frac{a_{2n-2i+1}}{a_{2n-2i+2}}\cdot\frac{a_{2n-2i+3}}{a_{2n-2i+4}}\cdots\frac{a_{2n-1}}{a_{2n}}\Big)^{\theta_1-\theta_0}$$

$$= a^{\theta_0}\Big(\frac{a_{2n-2i+1}}{a_{2n-2i}}\cdot\frac{a_{2n-2i+3}}{a_{2n-2i+2}}\cdots\frac{a_{2n-1}}{a_{2n-2}}\Big)^{\theta_1-\theta_0} \geq a^{\theta_0}$$

5. If $t \in [a_{2n-2i}, a_{2n-2i+1}]$, $i = 1, 2, \ldots, n$, and $at \in [a_{2n-1}, a_{2n}]$, then

$$\frac{\varphi(at)}{\varphi(t)} = a^{\theta_0}t^{\theta_0-\theta_1}\frac{\Big(\frac{a_1\cdot a_3\cdots a_{2n-1}}{a_0\cdot a_2\cdots a_{2n-2}}\Big)^{\theta_1-\theta_0}}{\Big(\frac{a_1\cdot a_3\cdots a_{2n-2i-1}}{a_0\cdot a_2\cdot\cdots\cdot a_{2n-2i}}\Big)^{\theta_1-\theta_0}}$$

$$= a^{\theta_0}t^{\theta_0-\theta_1}\Big(\frac{a_{2n-2i+1}}{a_{2n-2i+2}}\cdot\frac{a_{2n-2i+3}}{a_{2n-2i+4}}\cdots\frac{a_{2n-3}}{a_{2n-2}}\cdot a_{2n-1}\Big)^{\theta_1-\theta_0}$$

$$= a^{\theta_0}\Big(\frac{a_{2n-2i+1}}{t}\cdot\frac{a_{2n-2i+2}}{a_{2n-2i+3}}\cdots\frac{a_{2n-1}}{a_{2n-2}}\Big)^{\theta_1-\theta_0} \geq a^{\theta_0}$$

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and

$$\frac{\varphi(at)}{\varphi(t)} = a^{\theta_0} t^{\theta_0-\theta_1} \Big(\frac{a_{2n-2i+1}}{a_{2n-2i+2}} \cdot \frac{a_{2n-2i+3}}{a_{2n-2i+4}} \cdots \frac{a_{2n-3}}{a_{2n-2}} \cdot a_{2n-1} \Big)^{\theta_1-\theta_0}$$
$$= a^{\theta_0} \Big(\frac{a_{2n-1}}{t} \cdot \frac{a_{2n-2i+1}}{a_{2n-2i+2}} \cdot \frac{a_{2n-2i+3}}{a_{2n-2i+4}} \cdots \frac{a_{2n-3}}{a_{2n-2}} \Big)^{\theta_1-\theta_0} \le a^{\theta_1}.$$

6. If $t \in [a_{2n-2i-1}, a_{2n-2i}]$, $i = 1, 2, \ldots, n-1$, and $at \in [a_{2n-1}, a_{2n}]$, then

$$\frac{\varphi(at)}{\varphi(t)} = a^{\theta_0} \Big(\frac{\prod_{k=1}^{n} \frac{a_{2k-1}}{a_{2k-2}}}{\prod_{k=1}^{n-i} \frac{a_{2k-1}}{a_{2k-2}}} \Big)^{\theta_1-\theta_0}$$
$$= a^{\theta_0} \Big(\prod_{k=n-i+1}^{n} \frac{a_{2k-1}}{a_{2k-2}} \Big)^{\theta_1-\theta_0} \ge a^{\theta_0}$$

and

$$\frac{\varphi(at)}{\varphi(t)} = a^{\theta_0} \Big(\frac{a_{2n-2i+1} \cdot a_{2n-2i+3} \cdots a_{2n-1}}{a_{2n-2i} \cdot a_{2n-2i+2} \cdots a_{2n-2}} \Big)^{\theta_1-\theta_0}$$
$$= a^{\theta_0} \Big(\frac{a_{2n-1}}{a_{2n-2i}} \cdot \frac{a_{2n-2i+1}}{a_{2n-2i+2}} \cdot \frac{a_{2n-2i+1}}{a_{2n-2i+2}} \cdots \frac{a_{2n-3}}{a_{2n-2}} \Big)^{\theta_1-\theta_0}$$
$$\le a^{\theta_0} \Big(\frac{a_{2n-1}}{a_{2n-2i}} \Big)^{\theta_1-\theta_0} \le a^{\theta_0} \Big(\frac{at}{t} \Big)^{\theta_1-\theta_0} = a^{\theta_1}.$$

From all these cases we see that (17) is true and the proof of Lemma 2 is complete.

Proof in Example 4 The function φ is quasi-concave on $(0, \infty)$. It is enough to see that for $t \ge e$,

$$\varphi'(t) = \frac{\varphi(t)}{t} [\theta + \alpha(\sin \ln \ln t + \cos \ln \ln t)] = \frac{\varphi(t)}{t} \Big[\theta + \sqrt{2}\alpha \sin\Big(\ln \ln t + \frac{\pi}{4} \Big) \Big]$$

and

$$\big(\varphi(t)/t \big)' = \frac{\varphi(t)}{t} \Big[\theta - 1 + \sqrt{2}\alpha \sin\Big(\ln \ln t + \frac{\pi}{4} \Big) \Big].$$

We show now that $\rho_\varphi(a) = \max(a^{\theta-\sqrt{2}\alpha}, a^{\theta-\sqrt{2}\alpha})$. If $a > 1$ and $t > e$, then

$$\frac{\varphi(at)}{\varphi(t)} = a^{\theta+\alpha \sin(\ln \ln(at))} t^{\alpha[\sin(\ln \ln(at)) - \sin(\ln \ln t)]}$$
$$= a^{\theta+\alpha \sin(\ln \ln(at))} t^{2\alpha \sin[(\ln \ln(at) - \ln \ln t)/2] \cos[(\ln \ln(at) + \ln \ln t)/2]}$$
$$= a^{\theta+\alpha \sin[\ln \ln(at)]} t^{2\alpha \sin[\ln(1+\ln a/\ln t)/2] \cos[\ln \ln t + \ln(1+\ln a/\ln t)/2]}.$$

Since for $|u| \le \frac{1}{2}$ we have $\ln(1+u) = ub(u)$ where $|b(u)| \le 2$ and by the Lagrange mean-value theorem,

$$\cos(x+h) = \cos x + c(x,h)h \text{ with } |c(x,h)| \le 1,$$

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it follows that for large t and all $a > 1$,

$$\begin{aligned}\cos\Big[\ln\ln t + \frac{1}{2}\ln\Big(1+\frac{\ln a}{\ln t}\Big)\Big] &= \cos\Big[\ln\ln t + \frac{1}{2}\frac{\ln a}{\ln t}b(a,t)\Big]\\ &= \cos(\ln\ln t) + \frac{1}{2}\frac{\ln a}{\ln t}b(a,t)c(a,t)\\ &= \cos(\ln\ln t) + \frac{1}{2}\frac{\ln a}{\ln t}d(a,t),\end{aligned}$$

where $|d(a,t)| \leq 2$. Thus

$$\begin{aligned}\frac{\varphi(at)}{\varphi(t)} &= a^{\theta+\alpha\sin[\ln\ln(at)]}t^{2\alpha\sin((1+\ln a/\ln t)/2[\cos[\ln\ln t)+d(a,t)\ln a/2\ln t]}\\ &= a^{\theta+\alpha\sin[\ln\ln(at)]+2\alpha\frac{\ln t}{\ln a}\sin\sin((1+\ln a/\ln t)/2[\cos[\ln\ln t)+d(a,t)\ln a/2\ln t]}.\end{aligned}$$

Since

$$\lim_{t\to\infty}\frac{\ln t}{\ln a}2\sin\frac{\ln(1+\frac{\ln a}{\ln t})}{2} = 1$$

it follows that

$$\frac{\varphi(at)}{\varphi(t)} = a^{\theta+\alpha\sin[\ln\ln(at)]+\alpha\cos[\ln\ln(at)]+e(a,t)},$$

where $\lim_{t\to\infty} e(a,t) = 0$ and so

$$\begin{aligned}\limsup_{t\to\infty}\frac{\varphi(at)}{\varphi(t)} &= a^{\theta+\alpha\limsup_{t\to\infty}\{\sin[\ln\ln(at)]+\cos[\ln\ln(at)]\}}\\ &= a^{\theta+\alpha\limsup_{u\to\infty}(\sin u+\cos u)} = a^{\theta+\sqrt{2}\alpha}\end{aligned}$$

and

$$\liminf_{t\to\infty}\frac{\varphi(at)}{\varphi(t)} = a^{\theta-\sqrt{2}\alpha}.$$

The proof is complete.

Proof in Example 5 For $t \geq e$ we have

$$M'(t) = \frac{M(t)}{t}\Big[p + \sqrt{2}p\sin\Big(\ln\ln t + \frac{\pi}{4}\Big)\Big],$$

and

$$\begin{aligned}M''(t) = \frac{M(t)}{t^2}\Big\{&[p-1+\sqrt{2}k\sin\Big(\ln\ln t + \frac{\pi}{4}\Big)\Big]\Big[p-1+\sqrt{2}k\sin\Big(\ln\ln t + \frac{\pi}{4}\Big)\Big]\\ &-\frac{\sqrt{2}k}{\ln t}\sin\Big(\ln\ln t - \frac{\pi}{4}\Big)\Big\}.\end{aligned}$$

Also $M'(e^-) = pe^{p-1} \leq (p+k)e^{p-1} = M'(e^-)$. Therefore we can show that M is increasing, convex on $(0,\infty)$ and the proof of the fact that $\rho_M(a) = \max(a^{\theta-\sqrt{2}k}, a^{\theta-\sqrt{2}k})$ for all $a > 0$ is similar to Example 3.

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References

[1] I. Asekritova and N. Krugljak, *On equivalence of K- and J-methods for $(n+1)$-tuples of Banach spaces*. Studia Math. **122**(1997), 99–116.
[2] C. Bennett and R. Sharpley, *Interpolation of Operators*. Academic Press, Boston 1988.
[3] E. I. Berezhnoĭ, *Interpolation of positive operators in the spaces $\varphi(X_0, X_1)$*. In: Qualitative and Approximate Methods for the Investigation of Operator Equations, Yaroslav. Gos. Univ., Yaroslavl 1981, pp. 3–12 (Russian).
[4] ———, *Theorems on the representation of spaces and Schur's lemma*. Dokl. Akad. Nauk **344**(1995), 727–729; English Transl. in Doklady Math. **52**(1995), 252–254.
[5] ———, *An inverse problem in the theory of the interpolation of operators*. Mat. Zametki **59**(1996), 323–333; English Transl. in Math. Notes **59**(1996), 227–233.
[6] E. I. Berezhnoĭ and M. Mastyło, *On Calderón–Lozanovskiĭ construction*. Bull. Polish Acad. Sci. Math. **37**(1989), 23–33.
[7] ———, *The Lions problem for Gustavson–Peetre functor*. Publ. Math. **34**(1990), 175–180.
[8] J. Bergh and J. Löfström, *Interpolation Spaces*. Springer, Berlin 1976.
[9] S. Bloom, *Solving weighted norm inequalities using the Rubio de Francia algorithm*. Proc. Amer. Math. Soc. **101**(1987), 306–312.
[10] Yu. A. Brudnyĭ and N. Ya. Krugljak, *Interpolation Functors and Interpolation Spaces I*. North-Holland, Amsterdam 1991.
[11] A. P. Calderón, *Intermediate spaces and interpolation, the complex method*. Studia Math. **24**(1964), 113–190.
[12] M. Christ, *Weighted norm inequalities and Schur's lemma*. Studia Math. **78**(1984), 309–319.
[13] R. Coifman, P. W. Jones, and J. L. Rubio de Francia, *Constructive decomposition of* BMO *functions and factorization of A_p weights*. Proc. Amer. Math. Soc. **87**(1983), 675–676.
[14] M. Cwikel, P. G. Nilsson and G. Schechtman, *Interpolation of Weighted Banach Lattices/ A Characterization of Relatively Decomposable Banach Lattices.* Mem. Amer. Math. Soc. 787, 2003.
[15] E. Gagliardo, *On integral transformations with positive kernel*. Proc. Amer. Math. Soc. **16**(1965), 429–434.
[16] J. Garcia-Cuerva and J. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*. North-Holland, Amsterdam, 1985.
[17] E. Hernández, *Factorization and extrapolation of pairs of weights*. Studia Math. **45**(1989), 179–193.
[18] ———, *Weighted inequalities through factorization*. Publ. Math. **35**(1991), 141–153.
[19] S. Janson, P. Nilsson, and J. Peetre, *Notes on Wolff's note on interpolation spaces*. Proc. London Math. Soc. **48**(1984), 283–299.
[20] G. B. Jawerth, *Weighted inequalities for maximal operators: linearization, localization and factorization*. Amer. J. Math. **108**(1986), 361–414.
[21] P. Jones, *Factorization of A_p-weights*. Ann. of Math. **111**(1980), 511–530.
[22] A. Kamińska, L. Maligranda, and L. E. Persson, *Indices, convexity and concavity of Calderón–Lozanovskiĭ spaces*. Math. Scand. **92**(2003), 141–160.
[23] V. B. Korotkov, *Integral Operators*. Nauka Sibirsk. Otdel., Novosibirsk, 1983, (Russian).
[24] M. Krbec and L. Pick, *On imbeddings between weighted Orlicz spaces*. Z. Anal. Anwendungen **10**(1991), 107–117.
[25] S. G. Krein, Yu. I. Petunin, and E. M. Semenov, *Interpolation of Linear Operators*. Nauka, Moscow, 1978; Translations of Mathematical Monographs 54, American Mathematical Society, Providence, RI, 1982.
[26] N. Krugljak and L. Maligranda, *Calderón–Lozanovskiĭ construction on weighted Banach function lattices*. J. Math. Anal. Appl. **288**(2003), 744–757.
[27] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces, II. Function Spaces*. Springer-Verlag, Berlin, New York, 1979.
[28] J. L. Lions and E. Magenes, *Problèmes aux Limites Non Homogènes et Applications I*. Springer, Berlin, 1972.
[29] G. Ya. Lozanovskiĭ, *On some Banach lattices*. Sibirsk. Mat. Z. **10**(1969), 584–599; English Transl. in Siberian. Math. J. **10**(1969), 419–431.
[30] ———, *On some Banach lattices IV*. Sibirsk. Mat. Z. **14**(1973), 140–155; English Transl. in Siberian. Math. J. **14**(1973), 97–108.
[31] ———, *Transformations of ideal Banach spaces by means of concave functions*. In: Qualitative and Approximate Methods for the Investigation of Operator Equations, Yaroslav. Gos. Univ., Yaroslavl' 1978, pp. 122–147 (Russian).
[32] L. Maligranda, *Calderón–Lozanovskiĭ spaces and interpolation of operators*. Semesterbericht Funktionalanalysis, Tübingen **8**(1985), 83–92.

[33] ———, *On commutativity of interpolation with intersection*. Rend. Circ. Mat. Palermo **10**(1985), 113–118.
[34] ———, *A property of interpolation spaces*. Arch. Math. **48**(1987), 82–84.
[35] ———, *Orlicz Spaces and Interpolation*. Sem. Math. **5**, Univ. of Campinas, Campinas SP, Brazil, 1989.
[36] ———, *Why Hölder's inequality should be called Rogers' inequality*. Math. Inequal. Appl. **1**(1998), 69–83.
[37] ———, *Positive bilinear operators in Calderón–Lozanovskiĭ spaces*. Arch. Math. **81**(2003), 26–37.
[38] L. Maligranda and L. E. Persson, *Generalized duality of some Banach function spaces*. Indag. Math. **51**(1989), 323–338.
[39] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*. Trans. Amer. Math. Soc. **165**(1972), 207-226.
[40] J. Musielak, *Orlicz Spaces and Modular Spaces*. Lecture Notes in Math. **1034**, Springer-Verlag, Berlin, 1983.
[41] P. Nilsson, *Reiteration theorems for real interpolation and approximation spaces*. Ann. Mat. Pura Appl. **132**(1982), 291–330.
[42] ———, *Interpolation of Banach lattices*. Studia Math. **82**(1985), 135–154.
[43] V. I. Ovchinnikov, *Interpolation theorems that arise from Grothendieck's inequality*. Funktsional. Anal. i Prilozhen. (4) **10**(1976), 45–54; English transl. in Functional Anal. Appl. **10**(1976), 287–294 (1977).
[44] ———, *The Methods of Orbits in Interpolation Theory*. Math. Reports **1**, Part 2, Harwood Academic Publishers 1984, 349–516.
[45] G. Pisier, *Some applications of the complex interpolation method to Banach lattices*. J. Analyse Math. **35**(1979), 264–281.
[46] S. Reisner, *Some remarks on Lozanovskyi's intermediate normed lattices*. Bull. Polish Acad. Sci. Math. **41**(1993), 189–196 (1994).
[47] R. Rochberg, *Function theoretic results for complex interpolation families of Banach spaces*. Trans. Amer. Math. Soc. **284**(1984), 745–758.
[48] J. L. Rubio de Francia, *A new technigue in the theory of A_p-weights*. In: Topics in Modern Harmonic Analysis, Roma, 1983, pp. 571–579.
[49] ———, *Factorization theory and A_p-weights*. Amer. J. Math. **106**(1984), 533–547.
[50] V. A. Shestakov, *Transformations of Banach ideal spaces and interpolation of linear operators*. Bull. Acad. Polon. Sci. Sér/ Math. **29**(1981), 569–577 (1982) (Russian).
[51] J. D. Stafney, *Analytic interpolation of certain multiplier spaces*. Pacific J. Math. **32**(1970), 241–248.
[52] P. Szeptycki, *Notes on Integral Transformations*. Dissertationes Math. (Rozprawy Mat.) 231, 1984.
[53] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*. VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.
[54] R. Wallstén, *Remarks on interpolation of subspaces*. In: Function Spaces and Applications, Lecture Notes in Math. 1302, Springer, Berlin, 1988, pp. 410–419.

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