INTEGRAL INEQUALITIES FOR EQUIMEASURABLE REARRANGEMENTS

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1. Introduction. For a real-valued function f on the domain $[0, b]$, the equimeasurable decreasing rearrangement f^* of f is defined as a function μ^{-1} inverse to μ, where $\mu(y)$ is the measure of the set $\{x | f(x) > y\}$. Inequalities connected with rearrangements of sequences as well as functions play a considerable part in various branches of analysis, and, for example, the concluding chapter of Hardy, Littlewood, and Pólya [**3**] is devoted to rearrangement inequalities. Equimeasurable rearrangements of functions are also used by Zygmund [**6**, Vol. II, Chapters I and XII].

One well-known feature of the rearrangement operation is its variation reducing property. This has been extensively utilized for example, by Pólya and Szegö [**4**] in their work on isoperimetric inequalities. A study of the variation reducing property for infinite sequences, and functions defined on non-compact domains, has been made in [**2**], a typical result being an inequality for the derivative functions $f^{*\prime}$ as follows:

$$\int_0^b |f^{*\prime}(x)|^p \, dx \leqq \int_0^b |f'(x)|^p \, dx, \qquad p > 1. \tag{1.1}$$

This inequality has been extended to almost everywhere differentiable functions by Ryff [**5**]. While such inequalities are sharp for monotonic functions, they are evidently far from best possible for functions that oscillate rapidly. Our aim in this paper is to present improved versions of such inequalities by means of index or multiplicity functions. It is found that stronger inequalities sharp for a much larger class of functions can be obtained in this way. Indeed, if $n(y) = n(f)$ is the multiplicity function for f, enumerating the roots $x_1, \ldots, x_n$ of $f(x) = y$, then (1.1) is strengthened to

$$\int_0^b |f^{*\prime}(x)|^p \, dx \leqq \int_0^b \left| \frac{f'(x)}{n(f(x))} \right|^p dx, \qquad p > 1. \tag{1.2}$$

The technique used in obtaining (1.2) also opens the way to similar inequalities involving different ranges for p, functions of several variables, and other generalizations.

The outline of the work is as follows. A basic one-dimensional relation is established in § 2, followed by the study in § 3 of a special function and its range. The principal inequalities and corollaries are given in § 4. A particular

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case of interest is the relation for integral geometric means, which corresponds to the case $p = 0$ of the basic inequality. We then give the generalization to spherically symmetric rearrangements in m dimensions, where the multiplicity factor n is now replaced by an integrated level surface area. A corresponding but somewhat less strong inequality for sequences is derived in § 6.

The notion of an equivariational transform is then introduced, and a basic arc length inequality established, in § 7. Inequalities for the equivariational transform containing suitable convex functions, and others involving vector norms, are constructed. These types are then combined in a single extended vector norm inequality. Corresponding results in m dimensions are subsequently given, including a generalization of the arc length result to surface area.

2. The multiplicity function and the basic relation. Given are a function f with domain $[0, b]$ and a point y of the range of f.

Definition. The multiplicity $n(y)$ of f at the level y is the number of roots $x_k = x_k(y)$, $k = 1, \ldots, n(y)$, of the equation

$$y = f(x). \tag{2.1}$$

If the number of roots is infinite, we set $n(y) = \infty$. The reader can easily make his own interpretation of the following results in such circumstances.

Suppose now that $f \in C^1[0, b]$, and that h is so small that $n(y + h) = n(y)$. This is true if, for instance, $f'(x_k)$ exists and is not zero for all $k = 1, \ldots, n$. Denoting by $x = x^*(y)$ the abscissa $M(y)$ corresponding to the value y of the rearranged decreasing function f^*, we note that since the rearrangement is equimeasurable, we have

$$|x^*(y) - x^*(y + h)| = \sum_{k=1}^{n} |x_k(y + h) - x_k(y)|. \tag{2.2}$$

Dividing by h and letting $h \to 0$, we obtain the basic relation for the derivative function,

$$\frac{1}{|f^{*\prime}(x)|} = \sum_{k=1}^{n} \frac{1}{|f'(x_k)|}. \tag{2.3}$$

As can be seen from this derivation, this relation expresses the equimeasurable property of the rearrangement.

More precisely, we have the following preliminary result.

LEMMA 1. *If $f \in C^1[0, b]$, then the basic relation* (2.3) *holds for almost all $x \in [0, b]$.*

Proof. Since f^* is monotonic, the derivative $f^{*\prime}(x)$ exists almost everywhere in x. For any such x, let $x_1, \ldots, x_n$ be the roots of $f(x) = f^*(x)$, with n a positive integer or infinity. We must consider four cases, according as n is finite or infinite, and whether or not any of the values of the derivative are zero.

Suppose first that n is finite, and that $f'(x_k) \neq 0$, $k = 1, \ldots, n$. Setting $y = f^*(x^*) = f(x_k)$, we note that for sufficiently small h the number of roots of $f(x) = y + h$ will also be equal to n. Denote these roots by $x_k(y + h)$, so that $x_k = x_k(y)$. Since the rearrangement is equimeasurable, by (2.2) we can divide by h and let h tend to zero. The relation (2.3) then follows at once.

Suppose next that one or more of the $f'(x_k)$ are equal to zero, including say $f'(x_l)$, and let $x_l(y + h)$ approach $x_l = x_l(y)$ as h approaches zero. From (2.2) after division by h, we easily deduce, as $h \to 0$,

$$(2.4) \quad \left|\frac{x^*(y) - x^*(y+h)}{h}\right| \geqq \left|\frac{x_l(y+h) - x_l(y)}{h}\right| = \left|\frac{y + h - y}{x_l(y+h) - x_l(y)}\right|^{-1} \to \frac{1}{|f'(x_l)|} = \infty.$$

Consequently, the left side of (2.4) also tends to infinity as h approaches zero. We conclude that $f^{*\prime}(x^*)$ exists and has the value zero. The basic relation (2.3) therefore holds in this sense, that if the right-hand side is infinite, so is the left-hand side.

Consider now the case when n is infinite. If one or more values of $f'(x_k)$ are zero, then the reasoning just given for n finite still holds.

Finally, suppose that n is infinite, and that all $f'(x_k)$ are different from zero. Since $f'(x) \in C[0, b]$, $f'(x)$ is bounded, say $|f'(x)| < M$. Hence each term in the sum on the right exceeds $1/M$, and the "series" will not converge. More precisely, to each crossing "k", there corresponds an $h_k > 0$ such that $x_k(y + h)$ is defined for $|h| < h_k$, while by the fundamental theorem of calculus

$$h = \int f' \, dx \leqq M|x_k(y + h) - x_k(y)|,$$

so that

$$(2.5) \quad \left|\frac{x_k(y+h) - x_k(y)}{h}\right| \geqq \frac{1}{M}.$$

Now for any given N choose h less than $h_1, \ldots, h_N$. Observe that

$$(2.6) \quad \left|\frac{x^*(y+h) - x^*(y)}{h}\right| \geqq \sum_{k=1}^{N} \left|\frac{x_k(y+h) - x_k(y)}{h}\right| \geqq \frac{N}{M}$$

by (2.5). Since N can be chosen arbitrarily large as h approaches zero, it follows that the quotient on the left tends to infinity. Consequently, $f^{*\prime}(x^*)$ has the value zero, and the formula holds in this sense. This completes the proof of the lemma.

The basic relation (2.3) was used in [**2**, § 7] as a starting point for the construction of inequalities for higher-order derivatives.

3. The range of a special function on the positive orthant. To derive an inequality for the pth power of $f^{*\prime}(x)$, we shall proceed as follows. By (2.3) we have, for $p \geqq 1$,

$$|f^{*\prime}(x)|^{p-1} = \left(\sum_{k=1}^{n} \frac{1}{|f'(x_k)|}\right)^{1-p} \leqq C \sum_{k=1}^{n} |f'(x_k)|^{p-1}, \tag{3.1}$$

where the constant C is chosen as the maximum value of

$$\left(\sum \frac{1}{|f'(x_k)|}\right)^{1-p} \cdot \frac{1}{\sum |f'(x_k)|^{p-1}}. \tag{3.2}$$

Here all real values are permitted for the derivatives $f'(x_k)$ and hence all real non-negative values for the $|f'(x_k)|$.

To compute the best value for c, we replace $1/|f'(x_k)|$ by a_k, where $0 < a_k < \infty$, and determine the maximum of

$$c(a) = \frac{1}{\left(\sum_{k=1}^{n} a_k\right)^{p-1} \sum_{k=1}^{n} a_k^{1-p}}. \tag{3.3}$$

We may also consider other real values of p. For $p \leqq 1$ it will be possible to obtain both upper and lower bounds for integrals containing $|f^{*\prime}(x)|^p$, since the function $c(a)$ then has a positive minimum value.

Let C_p denote the maximum (or, if appropriate, supremum) value of $c(a)$ for a given p, in the non-negative orthant $0 \leqq a_k < \infty$, and let c_p denote the minimum (or infimum) value there.

LEMMA 2. *The extreme values of* (3.3) *in the orthant* $0 \leqq a_k < \infty$ *are as follows:*

	C_p (*maximum*)	c_p (*minimum*)
$p > 1$	$\frac{1}{n^p}$	0 (*infimum*)
$p = 1$	1	1
$0 < p < 1$	1	$\frac{1}{n^p}$
$p = 0$	1	1
$p < 0$	$n^{\lvert p\rvert} = \frac{1}{n^p}$	1

Proof. Since the expression for $c(a)$ is homogeneous of degree zero in the a_k $(k = 1, \ldots, n)$, we may without loss of generality set

$$F \equiv \sum_{k=1}^{n} a_k - 1 = 0. \tag{3.4}$$

The problem is thus reduced to finding the constrained extrema of $(\sum_{k=1}^{n} a_k{}^{1-p})^{-1}$, that is, to finding the constrained extrema of

$$G = \sum_{k=1}^{n} a_k{}^{1-p}. \tag{3.5}$$

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With a Lagrangian multiplier λ, we have

$$\frac{\partial}{\partial a_k}(G + \lambda F) = -(p-1)\frac{1}{a_k{}^p} + \lambda = 0, \qquad k = 1, \ldots, n,$$

which shows that there is one stationary point $a_k = 1/n$ $(k = 1, \ldots, n)$. Since

$$(3.6) \qquad \left(\frac{\partial^2 G}{\partial a_k \partial a_l}\right) = \left(\frac{p(p-1)}{a_k{}^{p+1}}\delta_{kl}\right)\begin{cases} \gg 0, & p > 1, p < 0, \\ \ll 0, & 0 < p < 1, \end{cases}$$

it is easily verified that this extreme point is of the type indicated in the table. Returning to the original expression (3.3) for $c(a)$ we observe that in these cases the extreme value is $c(a) = n^{-p}$. Note that (3.6) shows there is at most one interior extremum on the hyperplane $\sum a_k = 1$. Denote by Π the portion of this hyperplane lying in the non-negative orthant $0 \leqq a_k < \infty$.

For $p \geqq 1$ the minimum value must be taken on the boundary $b\Pi$. This value is easily seen to be zero since the sum $\sum_{k=1}^n a_k{}^{1-p}$ becomes unbounded as $b\Pi$ is approached (when one or more of the a_k tends to zero).

For $p = 1$, $c(a) \equiv 1$.

For $0 < p < 1$, the inequality (3.6) is reversed, so that the interior extremum $1/n^p$ is now a minimum value. To determine the maximum, we examine $c(a)$ on the boundary simplexes of descending dimension of $b\Pi$. If, for instance, q of the as are zero, we find the same problem with n replaced by $n - q$. On such a boundary simplex the minimum is $(n - q)^{-p}$. Thus the maximum value must be taken at a vertex such as $(1, 0, 0, \ldots)$ and by inspection this maximum value is seen to be 1.

For $p = 0$, $c(a) \equiv 1$.

For $p < 0$, the interior extremum value $1/n^p$ is again a maximum, while the minimum is found as in the preceding case to be 1. This completes the proof of Lemma 2.

4. Integral inequalities in one dimension. While integral inequalities concerning the derivative of a rearranged function can be formulated at various levels of generality, the arithmetic inequalities considered in the preceding section lead naturally to the consideration of pth power integrals.

THEOREM 1. *Let $f^*(x)$ be the equimeasurable decreasing rearrangement of $f \in C^1[0, b]$, and let $n(y)$ be the multiplicity of f at the level y. Then the following inequalities hold for the indicated ranges of p:*

(a) *for* $p \geqq 1$,

$$(4.1) \qquad \int_0^b |f^{*\prime}(x)|^p\, dx \leqq \int_0^b \left|\frac{f'(x)}{n(f(x))}\right|^p dx,$$

(b) *for* $0 < p \leqq 1$,

$$(4.2) \qquad \int_0^b \left|\frac{f'(x)}{n(f(x))}\right|^p dx \leqq \int_0^b |f^{*\prime}(x)|^p\, dx \leqq \int_0^b |f'(x)|^p\, dx,$$

(c) *for* $p < 0$,

$$\int_0^b \frac{1}{|f'(x)|^{|p|}}\,dx \leqq \int_0^b \frac{1}{|f^{*\prime}(x)|^{|p|}}\,dx \leqq \int \frac{n(f(x))^{|p|}}{|f'(x)|^{|p|}}\,dx. \tag{4.3}$$

Equality holds on the right in (a) *and* (c), *and on the left in* (b), *if and only if the values of* $|f'(x_k)|$ *are independent of* k, $k = 1, 2, \ldots, n$, *for almost all* x_k.

Proof. We have, by (2.3),

$$|f^{*\prime}(x)|^{p-1} = \left(\sum_{k=1}^{n} \frac{1}{|f'(x_k)|}\right)^{1-p}. \tag{4.4}$$

By the definition (3.3) of $c(a)$, we thus find

$$c_p \sum_{k=1}^{n} |f'(x_k)|^{p-1} \leqq |f^{*\prime}(x)|^{p-1} \leqq C_p \sum_{k=1}^{n} |f'(x_k)|^{p-1}, \tag{4.5}$$

where c_p and C_p denote minimum and maximum values of $c(a)$ in the non-negative orthant. Multiply by $|df^*| = |df|$ and integrate over the range of f. As $y = f = f^*$ traverses the range of f in decreasing order, x^* traverses the domain $[0, b]$ of f^* in increasing order and the set of roots x_k covers the domain $[0, b]$ of f exactly once. We thus find

$$\int c_p \sum_{k=1}^{n} |f'(x_k)|^{p-1}\,|df| \leqq \int_0^b |f^{*\prime}(x)|^{p-1}\,|df^*| \tag{4.6}$$

$$\leqq \int C_p \sum_{k=1}^{n} |f'(x_k)|^{p-1}\,|df|.$$

Since $|df^*| = |f^{*\prime}(x)|\,dx$ and $|df| = |f'(x)|\,dx$, we then obtain

$$\int_0^b c_p |f'(x)|^p\,dx \leqq \int_0^b |f^{*\prime}(x)|^p\,dx \leqq \int_0^b C_p |f'(x)|^p\,dx, \tag{4.7}$$

where

$$c_p = c_p(n(f(x))) \quad \text{and} \quad C_p = C_p(n(f(x))).$$

Replacing c_p and C_p by their values as functions of n as given by Lemma 2 for the various ranges in p, we find the inequalities stated in the theorem. Note that for $p \geqq 1$ the infimum $c_p = 0$ does not yield any lower bound, whereas in the other two cases such a positive lower bound is obtained. (It will be shown below that these lower bounds occur only in the one-dimensional case and are not present for higher dimensions.)

Since the proof of Lemma 2 shows that the interior extremum point is the place where all a_k are equal, it follows that the maximum value C_p in cases (a) and (c), and the minimum value c_k in case (b), are achieved when the values of all $|f'(x_k)|$ are equal, $k = 1, \ldots, n = n(f(x))$. The concluding statement of the theorem now follows, and the proof is complete.

Since $n(f) \geqq 1$, (4.1) implies (1.1) for $p > 1$. Note that the right-hand inequality in (4.2) implies (1.1) for $0 < p < 1$. Thus, as noted by Ryff [**4**], (1.1) holds for $p > 0$.

A more precise characterization of the class of functions for which equality on the right in (4.1) and (4.3) and on the left in (4.2) will hold is the following.

COROLLARY 1.1. *If $|f'(x_k)|$ has the same value for all x_k such that $f(x_k) = y$, and if this value determines a once differentiable function $F(y)$ of y, then $f(x)$ satisfies a second-order differential equation in normal form*

$$f''(x) = F(f(x))F'(f(x)). \tag{4.8}$$

Conversely, if $f(x)$ satisfies a second-order differential equation of the type

$$f'' = G(f), \tag{4.9}$$

then equality will hold for f in Theorem 1 on the right for $p > 1$, $p < 0$ and on the left if $0 < p < 1$.

Proof. Since all values of $|f'(x_k)|$ are equal, we have

$$|f'(x)| = F(f(x)),$$

and, squaring, we find

$$f'^2(x) = F^2(f(x)).$$

Differentiation with respect to x yields

$$2f'f'' = 2F(f)F'(f)f'.$$

Cancellation of the factor $2f'$ (which will vanish at isolated points only unless F vanishes in an open set of values of f) proves (4.8).

Conversely, suppose that a second-order equation (4.9) holds for some function G. It is then directly possible to show by integration that $|f'(x)| = F_1(f)$, where $F_1(f) = (2\int G(f)\,df + K)^{1/2}$ and K is a constant of integration. Then equality will surely hold for f as indicated. This completes the proof.

If $p = 1$, it is obvious from the proof of the theorem that equality will always hold. Indeed, we then obtain from (4.1) and the left-hand inequality of (4.2) an obvious formal result

$$\int_0^b |df^*| = \int_0^b |f^{*\prime}(x)|\,dx = \int_0^b \frac{|f'(x)|}{n(f(x))}\,dx = \int_0^b \frac{|df|}{n}$$

in which the common value of the integrals is the measure of the range of f. This proves the following result.

COROLLARY 1.2.

$$\int_0^b |df^*| = \int_0^b \frac{1}{n(f)}\,|df|. \tag{4.10}$$

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We may compare (4.10) with the slightly different result

$$\int_0^b n|df^*| = \int_0^b n|f^{*\prime}(x)|\,dx = \int_0^b |f'(x)|\,dx = \int_0^b |df| = \mathrm{TV}(f) \tag{4.11}$$

wherein the common value is the total variation of f. This latter formula was studied by Banach [**1**, Theorem 2] and will be derived again below.

For the limiting case $p = 0$ we employ the *integral geometric mean*

$$G(f) = \exp\left(\frac{1}{b}\int_0^b \log f\,dx\right) \tag{4.12}$$

[**3**, p. 136].

COROLLARY 1.3. *Suppose that the integrals in either* (4.2) *or* (4.3) *exist for sufficiently small* p. *Then*

$$\frac{G(|f'|)}{G(n)} \leqq G(|f^{*\prime}|) \leqq G(|f'|). \tag{4.13}$$

Proof. Starting for example from (4.2), we take the $(1/p)$th power of each expression. Setting

$$M_p(f) = \left(\frac{1}{b}\int_0^b f^p\,dx\right)^{1/p}$$

as in [**3**, p. 134], we obtain

$$M_p\left(\frac{|f'|}{n}\right) \leqq M_p(|f^{*\prime}|) \leqq M_p(|f'|). \tag{4.14}$$

But by [**3**, Theorem 187], we have

$$G(f) = \lim_{p\to 0} M_p(f).$$

Noting the "geometric" property $G(f_1f_2) = G(f_1)G(f_2)$ and the corresponding formula for quotients, and letting $p \to 0$ in (4.14), we obtain the results stated in Corollary 1.3.

In view of the entries for $p = 0$ in the table of Lemma 2, it is tempting to conjecture that equality on the left in (4.13) might hold in general. This however is not known one way or the other.

Recall now that by definition the rearranged function f^* is the inverse of μ, where $\mu(y)$ is the measure of the set $\{x|\,f(x) > y\}$. Consequently, the first derivatives of f^* and of μ are reciprocals.

Let us refer to the function

$$\varphi(y) \equiv |\mu'(y)| \equiv \frac{1}{|f^{*\prime}(x)|} \equiv \sum_{k=1}^{n} \frac{1}{|f'(x_k)|} \tag{4.15}$$

as the *value density of* f *at the level* y. Let the range set of f be denoted by R.

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COROLLARY 1.4. *For $\nu > 1$, the value density function φ satisfies*

$$\int_0^b \frac{1}{|f'(x)|^{\nu-1}}\,dx \leqq \int_R \varphi(y)^\nu\,dy \leqq \int_0^b \frac{n(f(x))^{\nu-1}}{|f'(x)|^{\nu-1}}\,dx. \tag{4.16}$$

Proof. The inequalities follow at once from (4.3) with $\nu = |p| + 1$, when we note that

$$dx = \frac{1}{|f^{*\prime}(x)|}\,dy = \varphi(y)\,dy. \tag{4.17}$$

The two following corollaries indicate different possible generalizations of the main theorem.

In the next corollary we shall state the results that correspond to the case $p > 1$ of the theorem, leaving the remaining cases $0 < p < 1$ and $p < 0$ to the reader.

COROLLARY 1.5. *If $\chi(f) \geqq 0$, then for $p \geqq 1$ we have*

$$\int_0^b \chi(f^*)|f^{*\prime}(x)|^p\,dx \leqq \int_0^b \chi(f)\left|\frac{f'(x)}{n(f(x))}\right|^p dx. \tag{4.18}$$

Proof. We have only to modify the calculation leading to (4.1) by inserting the factor $\chi(f^*)$ on the left and the equal factor $\chi(f)$ on the right before integrating over the range of f.

Since in Corollary 1.5 χ may be any non-negative function on the range of f, that is, any non-negative function of f itself, we may use the fact that the multiplicity function $n(y)$ has this property, and choose

$$\chi(y) = n(y)^p. \tag{4.19}$$

Hence

$$\int_0^b n(f^*)^p|f^{*\prime}(x)|^p\,dx \leqq \int_0^b |f'(x)|^p\,dx, \qquad p \geqq 1, \tag{4.20}$$

and corresponding inequalities hold for other values of p.

Indeed, whenever

$$\chi(y) = \chi_1(y)\,n(y)^p,$$

with $\chi_1(y)$ independent of n, the right-hand side in (4.18) will not contain the multiplicity function n.

Finally, we note that the method used in Theorem 1 yields the following generalization, which we state without further proof.

COROLLARY 1.6. *If $\Phi(s) = s\Psi(s) > 0$, for $s > 0$, and if $C(n)$ and $c(n)$ denote, respectively, the maximum and minimum values in the non-negative orthant $0 < a_k$, $k = 1, \ldots, n$, of*

$$\frac{\Psi\left(\left(\sum_{k=1}^n a_k\right)^{-1}\right)}{\sum_{n=1}^n \Psi(a_k^{-1})},$$

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then

$$(4.21)\qquad \int_0^b c(n)\Phi(|f'(x)|)\,dx \leqq \int_0^b \Phi(|f^{*\prime}(x)|)\,dx \leqq \int_0^b C(n)\Phi(|f'(x)|)\,dx.$$

Note that Theorem 1 is in effect the particular case $\Psi(s) = s^{p-1}$ of this result.

5. Rearrangements of functions of several variables. Given a function f on an m-dimensional domain D in a Cartesian m-space with coordinates $\mathbf{x} = (x_1, \ldots, x_m)$, we can apply the basic construction for the equimeasurable decreasing rearranged function with little alteration from the one-dimensional case. Denote the range of f by R. Let

$$(5.1)\qquad \mu(z) = \text{meas}\{(x_1, \ldots, x_m) |\ f(x_1, \ldots, x_m) > z\}$$

and let the function f^* of one variable x be defined by

$$(5.2)\qquad f^*(x) = \mu^{-1}(x).$$

Then f^* has the range R and the domain $0 \leqq x \leqq \text{meas}(D)$.

Observe that this construction yields a one-dimensional rearranged function. However, spherically symmetrical functions of any number of variables can be used. For instance, if k variables are chosen, and the rearranged function f_k^* depends on $r = (x_1^2 + \ldots + x_k^2)^{1/2}$, then

$$\mu = x = \omega_k k^{-1} r^k$$

and

$$(5.3)\qquad d\mu = dx = \omega_k r^{k-1}\,dr.$$

Here ω_k denotes the surface area of the unit sphere in k-space, thus $\omega_1 = 2t$ $\omega_2 = 2\pi$, $\omega_3 = 4\pi, \ldots$. Note that the spherically symmetric rearrangemen- in the case $k = 1$ is the symmetric or even equimeasurable monotonic re. arrangement of f, not the decreasing equimeasurable rearrangement [**3**, p. 278],

A derivative $f^{*\prime}(x)$ would be replaced by $f_k^{*\prime}(r)/\omega_k r^{k-1}$, so that the replacement for an integral such as $\int |f^{*\prime}(x)|^p\,dx$ is

$$(5.4)\qquad \int |f_k^{*\prime}(r)|^p (\omega_k r^{k-1})^{1-p}\,dr.$$

In the following theorems we shall state the results for the one-dimensional rearranged function $f^*(x) = f_1^*(x)$, leaving the replacements given above to the reader. The case $k = m$ is a natural one to consider for many of the applications. Note that the domain for r above will usually but not necessarily terminate at $r = 0$.

We next require analogues for several variables of Lemmas 1 and 2. For the basic relation we again assume that $f \in C^1$ on some given closed domain D in E^m.

LEMMA 3 (Basic relation in E^m). *Let $f \in C^1(D)$ and let f^* be the one-dimensional equimeasurable decreasing rearrangement of f. Then*

$$\frac{1}{|f^{*\prime}(x)|} = \sum \int_{f=f^*(x)} \frac{dS}{|\nabla f|}, \tag{5.5}$$

where the integration and summation on the right run over all components of the level surfaces $f = f^*(x)$ in D, and where ∇f denotes the gradient vector of f.

Proof. Since f is constant on the level surface, the gradient ∇f is parallel to the normal, and we have

$$|\nabla f| = \left| \frac{df}{dn} \right|, \tag{5.6}$$

whence

$$dn = \frac{df}{|\nabla f|}. \tag{5.7}$$

From (5.1) we have

$$\mu(z) = \int_{f \geqq z} dV \tag{5.8}$$

from which follows

$$d\mu = \sum \int_{f=z} dn\, dS, \tag{5.9}$$

where the integration again extends over the level surfaces $f = z$. From (5.7) we now find

$$d\mu = \sum \int \frac{dS}{|\nabla f|} df. \tag{5.10}$$

However we also have, for the rearranged function in the one-dimensional case, and reckoning df^* as negative, since f^* is a decreasing function of x,

$$d\mu = dx = -\frac{df^*}{|f^{*\prime}(x)|}. \tag{5.11}$$

Since $|df^*| = |df|$, we find on comparing (5.10) and (5.11) that (5.5) holds. This completes the proof of Lemma 3.

To employ Lemma 3 in an integral inequality, we shall be led to consider the integral analogue of (3.2), namely

$$c\left[\frac{1}{|\nabla f|}\right] = \left(\sum \int \frac{dS}{|\nabla f|}\right)^{1-p} \left(\sum \int |\nabla f|^{p-1}\, dS\right)^{-1}. \tag{5.12}$$

Setting

$$g(s) = \frac{1}{|\nabla f|} \geqq 0 \tag{5.13}$$

and noting again that (5.12) is homogeneous of degree zero in $g(s)$, we may normalize the variational problem by imposing the constraint

$$\sum \int g(s)\, dS = 1. \tag{5.14}$$

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Consider therefore the extreme values of

$$c[g] = \left(\sum \int g(s)\, dS\right)^{1-p} \left(\sum \int \frac{dS}{g(s)^{p-1}}\right)^{-1}. \tag{5.15}$$

The results may be summarized as follows, where the level surface area

$$S = \sum \int dS \tag{5.16}$$

now corresponds to the multiplicity n.

LEMMA 4. *The extreme values of* (5.15) *in the function space orthant* $g(s) \geqq 0$ *are as follows:*

	maximum	*minimum*
$p > 1$	$\frac{1}{S^p}$	0 (*infimum*)
$p = 1$	1	1
$0 < p < 1$	∞ (*supremum*)	$\frac{1}{S^p}$
$p = 0$	1	1
$p < 0$	$S^{\lvert p \rvert} = \frac{1}{S^p}$	0 (*infimum*)

Proof. With the constraint (5.14), we have in effect to find the extrema of

$$\sum \int \frac{dS}{g(s)^{p-1}}, \tag{5.17}$$

that is, to find the free extrema of

$$\sum \int \left[\frac{1}{g(s)^{p-1}} + \lambda g(s)\right] dS, \tag{5.18}$$

where λ is a Lagrange multiplier. Variation of g at once yields the equation

$$\frac{-(p-1)}{g(s)^p} + \lambda = 0;$$

whence

$$g(s) = \text{const} = 1/S. \tag{5.19}$$

The sole interior extremum is then given by (5.19), entirely in analogy with the results of Lemma 2. The second variation of (5.17) with respect to $g(s)$ is

$$p(p-1) \sum \int \frac{(\delta g(s))^2}{g(s)^{p-1}}\, dS \tag{5.20}$$

which is positive for $p > 1$ and $p < 0$ and negative for $0 < p < 1$. It follows that a maximum of (5.15) is attained for $p > 1$ and $p < 0$ while a minimum is attained for $0 < p < 1$.

For these ranges of p the opposite extremes are however not analogous to those of Lemma 2, essentially because we are now considering an integral variational problem rather than a finite-dimensional maximum or minimum problem. Consider any subset S_ϵ of S of measure ϵ, where $\epsilon > 0$, and let

$$g_\epsilon(s) = \begin{cases} \epsilon^{-1} + b & \text{on } S_\epsilon, \\ \epsilon & \text{on } S - S_\epsilon, \end{cases}$$

where the constant b is bounded independently of ϵ for ϵ small and is chosen so that (5.14) holds. (Actually $b = -S + \epsilon$.)

Then we have

$$\int_S g_\epsilon(s)^{1-p}\, dS = \int_{S_\epsilon} \left(\frac{1}{\epsilon} + b\right)^{1-p} dS + \int_{S-S_\epsilon} \epsilon^{1-p}\, dS$$
$$= \left(\frac{1}{\epsilon} + b\right)^{1-p} \epsilon + \epsilon^{1-p}(S - \epsilon).$$

For $p > 1$ the second term tends to infinity as ϵ tends to zero. For $0 < p < 1$ the integral is

$$O(\epsilon^p) + O(\epsilon^{1-p}),$$

and hence tends to zero with ϵ. Finally, for $p < 0$ the first term above tends to infinity as ϵ tends to zero. From these cases the infimum of 0 for $p > 1$, $p < 0$ follows together with the supremum ∞ for $0 < p < 1$ as indicated in the lemma.

The cases $p = 0$ and $p = 1$ are trivial, and this completes the proof of Lemma 4.

The main result for functions of m variables may now be stated as follows, where $b = V = \int_D dV$.

THEOREM 2. *Let $f(x) \in C^1(D)$, where D is a closed domain in E^m, $\mathbf{x} = (x_1, \ldots, x_m)$, and let f^* be the one-dimensional equimeasurable decreasing rearrangement of f. Then we have*

$$\text{(5.21)} \qquad \int_0^b |f^{*\prime}(x)|^p\, dx \leqq \int_D \frac{|\nabla f|^p}{S(f)^p}\, dV, \qquad p \geqq 1, p \leqq 0,$$

and the opposite inequality holds for $0 \leqq p \leqq 1$.

Remark. Observe that for $m > 1$ we have only a single inequality for each range of values of p. The reason is as stated in the proof of Lemma 4 and can be interpreted as follows: The surface measure of the set whereon f takes a given value need not have any positive lower bound, whereas in one dimension one is the minimal multiplicity.

Proof. We may confine attention to the case $p > 1$, leaving the two similar remaining cases to the reader. By Lemma 3 we have

$$(5.22)\qquad |f^{*\prime}(x)|^{p-1} = \left(\sum \int \frac{dS}{|\nabla f|}\right)^{1-p} \leqq C \sum \int |\nabla f|^{p-1}\, dS,$$

where C denotes a maximum or least upper bound for the functional $c[|\nabla f|^{-1}]$ of (5.12). By Lemma 4, for $p > 1$ we have $C = S^{-p}$. Therefore, from (5.22) and (5.7) we find successively

$$(5.23)\qquad \int |f^{*\prime}(x)|^p\, dx = \int |f^{*\prime}(x)|^{p-1}\, |df^*| \leqq \int S^{-p} \sum \int |\nabla f|^{p-1}\, dS |df^*|$$

$$= \int S(f)^{-p} \sum \int |\nabla f|^{p-1}\, dS |df| = \int \int \frac{|\nabla f|^{p-1}}{S(f)^p}\, dS |\nabla f|\, dn = \int \frac{|\nabla f|^p}{S(f)^p}\, dV,$$

which proves the result for $p > 1$, as required.

This completes the proof of Theorem 2. The cases of equality are, as before, those in which the value of $|\nabla f|$ depends only on the value of f.

COROLLARY 2.1. *Equality holds in Theorem* 4 *if and only if the level surfaces of f form a family of wave fronts, that is, if and only if there exists a function g depending only on f and satisfying the eikonal equation*

$$(5.24)\qquad (\nabla g)^2 = 1.$$

Proof. We suppose that f is not constant in any open set in D, or if so we consider only intervals of the range of f not containing such constant values. Thus equality holds if and only if the values of $|\nabla f|$ depend only on the values of f: that is

$$(5.25)\qquad (\nabla f)^2 = F(f),$$

where $F(f)$ denotes a positive function of f. Define

$$(5.26)\qquad g(f) = \int^f (F(s))^{-1/2}\, ds;$$

then $g(f)$ will satisfy the eikonal equation.

In the converse case, (5.24) implies (5.25) and hence equality. This completes the proof.

By analogy with Corollary 1.2, we have the following result.

COROLLARY 2.2. *For* $f \in C^1(D)$ *we have*

$$(5.27)\qquad \int |df^*| = \int_D \frac{|\nabla f|}{S(f)}\, dV,$$

the common value of these integrals being the measure of the range of f.

Proof. Either from the case $p = 1$ of Theorem 2, or directly as follows:

$$\text{(5.28)}\quad \text{meas (range} f) = \int |df^*| = \int |df| = \sum \int \int |df| \frac{dS}{S(f)}$$
$$= \sum \int \int \frac{|\nabla f|}{S(f)} dndS = \int_D \frac{|\nabla f|}{S(f)} dV.$$

Remark. A related formula involves the total variation $\text{TV}(f)$ as follows:

$$\text{(5.29)}\quad \text{TV}(f) \equiv \int |\nabla f| \, dV = \int \sum \int \frac{|df|}{|dn|} dndS$$
$$= \int \sum \int |df| \, dS = \int S(f^*) \, |df^*|.$$

For the m-dimensional geometric mean

$$\text{(5.30)}\qquad G_m(f) = \exp\left(\frac{1}{V} \int \log f \, dV\right),$$

we find the following single inequality.

COROLLARY 2.3. *The integral geometric means of* ∇f *and* $f^{*\prime}$ *satisfy*

$$\text{(5.31)}\qquad G_m(|\nabla f|) \leqq G_m(S(f)) G_1(|f^{*\prime}|).$$

Proof. Assuming that the integrals in (5.21) exist for, sufficiently small, say, positive p, multiply (5.21) by $1/V$, where $V = \int_D dV$, take the $(1/p)$th power of each side, and let $p \to 0$. Then use the geometric mean property $G_m(|\nabla f|/S) = G_m(|\nabla f|)/G_m(S)$. The one-dimensional geometric mean of $|f^{*\prime}|$ is that defined in (4.12) with $b = V$. The result is thus proved. A similar proof using negative values of p is also available.

Again we may introduce the concept of a value density function ρ, where

$$\text{(5.32)}\qquad \rho(y) = \frac{1}{|f^{*\prime}(x)|} = \sum \int_{f=y} \frac{dS}{|\nabla f|}.$$

COROLLARY 2.4. *For* $\nu > 1$,

$$\text{(5.33)}\qquad \int_R \rho(y)^\nu \, dy \leqq \int \frac{S(f)^{\nu-1}}{|\nabla f|^{\nu-1}} dV.$$

Proof. This follows at once from Theorem 2 for $p < 0$ with $\nu = -p + 1$.

Generalizations similar to Corollaries 1.5 and 1.6 may also be formulated but we shall omit them here.

6. Sequences. The study of rearranged functions and sequences made in [**2**] indicates a certain analogy between the two. However, this analogy cannot be pressed too far; it fails for second differences, and we will show here

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that it only partially holds for inequalities involving the index or counting function n. More precisely, we have the following result (for definitions and notation see [**2**]).

THEOREM 3. *If* $\{a_k^*\}_{k=1}^N$ *is the decreasing rearrangement of* $\{a_k\}_{k=1}^N$, $\Delta a_k = a_{k+1} - a_k$, *and if* n_k^* *denotes the number of intervals of* $\{a_k\}$ *that contain the open interval* (a_k^*, a_{k+1}^*), *then*

$$\sum_{k=1}^N n_k^* |\Delta a_k^*|^p \leqq \sum_{k=1}^N |\Delta a_k|^p, \qquad p \geqq 1, \tag{6.1}$$

and the opposite inequality holds for $p \leqq 1$. *Equality holds if* $p = 1$, *or if all pairs* (a_k, a_{k+1}) *adjacent in* $\{a_k\}$ *are also adjacent in* $\{a_k^*\}$.

Proof. Write $\alpha_k = |\Delta a_k^*|$, so that $a_2^* = a_1^* - \alpha_1$, $a_3^* = a_1^* - \alpha_1 - \alpha_2$, and so on. If $a_k = a_l^*$, $a_{k+1} = a_m^*$ where, for instance, $m > l$, then

$$|\Delta a_k| = |a_m^* - a_l^*| = a_l^* - a_m^* = \alpha_l + \alpha_{l+1} + \ldots + \alpha_{m-1}. \tag{6.2}$$

Observe that for $\alpha_k > 0$, $p \geqq 1$, we have

$$\alpha_l^p + \alpha_{l+1}^p + \ldots + \alpha_{m-1}^p \leqq (\alpha_l + \alpha_{l+1} + \ldots + \alpha_{m-1})^p, \tag{6.3}$$

while if $p < 1$ the opposite inequality holds [**3**, Theorem 16]. In the expression for

$$\sum_{k=1}^N |\Delta a_k|^p = \sum_{k=1}^N (\alpha_{l_k} + \alpha_{l_{k+1}} + \ldots + \alpha_{m_k - 1})^p \tag{6.4}$$

$$\geqq \sum_{k=1}^N (\alpha_{l_k}^{\ p} + \alpha_{l_{k+1}}^p + \ldots + \alpha_{m_k-1}^p), \qquad p \geqq 1,$$

the total number of times each α_k appears will be equal to the number of intervals (a_k, a_{k+1}) that contain (a_h^*, a_{h+1}^*), that is, equal to n_h^*. Consequently, the last sum on the right can be written as

$$\sum_{h=1}^N n_h^* \alpha_h^{\ p} \tag{6.5}$$

and on replacing α_h by $|\Delta a_h^*|$ we obtain the inequality in the theorem for $p \geqq 1$. The proof for $p < 1$ is essentially the same.

Note that equality holds in (6.3) only if $p = 1$ or if there is only one α_k. The presence of two or more α_k in a parenthesis on the right in (6.4) indicates that there is an adjacent pair (a_k, a_{k+1}) in $\{a_k\}$ that are not adjacent in the rearranged sequence $\{a_k^*\}$. This completes the proof of the theorem.

Although this result appears formally less strong than the theorem for integrals, it is evidently the best possible of its kind being sharp in the cases of equality just mentioned.

7. The equivariational transform. We now introduce another function related to the equimeasurable decreasing rearrangement f^* of a given function f

of one real variable x. The variation of this new function F is proportional to the multiplicity index n: thus

$$dF = ndf^* = -n|df|. \tag{7.1}$$

The domain of F is identical with the domain of f^*. To normalize we set $F(b) = 0$, thus $F(x) = \int_x^b n\,|df^*|$. In particular, by (4.11),

$$F(0) = \int_0^b n|df^*| = \int_0^b n|df| = \mathrm{TV}(f). \tag{7.2}$$

Since the total variation $\mathrm{TV}(F)$ is thus equal to $\mathrm{TV}(f)$, we shall call F the *equivariational transform* of f. Note that F, having in general a wider range, is not a rearrangement of f.

The most immediate property of the equivariational transform is the following arc length property. Suppose that f is rectifiable on $[0, b]$.

THEOREM 4. *The arc length of the curve $y = F(x)$ is less than or equal to the arc length of the curve $y = f(x)$. Equality holds in the same cases as in Theorem* 1.

Proof. Denote the n roots of $f(x) = y$ by $x_1, \ldots, x_n$, in decreasing order. Then

$$x^* = x_1 - x_2 + x_3 - x_4 + \ldots \tag{7.3}$$

or

$$x^* = b - x_1 + x_2 - x_3 + \ldots$$

and for definiteness we may assume (7.3) without loss of generality. In this proof we employ a modified version of a technique used in [**4**, p. 183].

For $k = 1, \ldots, n$ we construct the two-component vector

$$v_k = \left(1,\ (-1)^{k-1}\frac{dx_k}{dy}\right). \tag{7.4}$$

Then

$$v^* = \sum_{k=1}^{n} v_k = \left(n, \frac{d}{dy}\sum\ (-1)^{k-1}x_k\right) = \left(n, \frac{dx^*}{dy}\right). \tag{7.5}$$

By the triangle property of the Euclidean vector norm we have

$$|v^*| \leqq \sum_{k=1}^{n} |v_k|, \tag{7.6}$$

that is

$$\left(n^2 + \left(\frac{dx^*}{dy}\right)^2\right)^{1/2} \leqq \sum_{k=1}^{n}\left(1 + \left(\frac{dx_k}{dy}\right)^2\right)^{1/2}. \tag{7.7}$$

Consequently, the arc length expression for the curve $y = F(x)$ becomes

$$\begin{aligned}(7.8)\qquad \int_0^b \left(1+\left(\frac{dF}{dx^*}\right)^2\right)^{1/2} dx^* &= \int_0^b \left(n^2\left(\frac{dy}{dx^*}\right)^2+1\right)^{1/2} dx^* \\ &= \int \left(n^2+\left(\frac{dx^*}{dy}\right)^2\right)^{1/2} dy \leqq \int \sum_{k=1}^n \left(1+\left(\frac{dx_k}{dy}\right)^2\right)^{1/2} dy \\ &= \int \sum_{k=1}^n \left(\left(\frac{dy}{dx_k}\right)^2+1\right)^{1/2} dx_k = \int \left(\left(\frac{dy}{dx}\right)^2+1\right)^{1/2} dx.\end{aligned}$$

This completes the proof of Theorem 4.

A similar proof will hold for any vector norm, not necessarily Euclidean, since the properties assumed for the norm are homogeneity of the first degree $||cw|| = |c|\ ||w||$, as well as the triangle inequality. Denote by $\operatorname{sgn} x$ the quantity

$$(7.9)\qquad \operatorname{sgn} x = \begin{cases} 1, & x > 0, \\ -1, & x < 0. \end{cases}$$

COROLLARY 4.1. *Given the vectors*

$$(7.10)\qquad w = \left(\frac{df}{dx}, \operatorname{sgn}\frac{df}{dx}\right), \qquad w^* = \left(\frac{dF}{dx^*}, 1\right),$$

we have

$$(7.11)\qquad \int_0^b ||w^*||\, dx^* \leqq \int_0^b ||w||\, dx.$$

The proof is omitted.

A different type of result, involving convex functions of F', can be found by adapting a particular case of the reasoning of Theorem 1. Observe that, by (2.3),

$$(7.12)\qquad |F'(x)| = n\left|\frac{df^*}{dx}\right| = n\left(\sum_{k=1}^n \frac{1}{|f'(x_k)|}\right)^{-1}.$$

The expression on the right will be majorized by $K \sum_{k=1}^n |f'(x_k)|$ provided that

$$K = n \cdot \max\left\{\left(\sum_{k=1}^n |f'(x_k)|\right)^{-1}\left(\sum_{k=1}^n \frac{1}{|f'(x_k)|}\right)^{-1}\right\}.$$

From the case $p = 2$ of Lemma 2, we find that $K = 1/n$. Together with (7.12) this implies

$$(7.13)\qquad |F'(x)| \leqq \frac{1}{n}\sum_{k=1}^n |f'(x_k)|.$$

Now let $\Psi(s)$ be an increasing, convex function. Then, using these two properties in succession, we find

$$(7.14)\qquad \Psi(|F'(x)|) \leqq \Psi\left(\frac{1}{n}\sum_{k=1}^n |f'(x_k)|\right) \leqq \frac{1}{n}\sum_{k=1}^n \Psi(|f'(x_k)|).$$

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Multiplying by $|dF| = n|df|$, we obtain

$$\Psi(|F'(x)|)|dF| \leqq \sum_{k=1}^{n} \Psi(|f'(x_k)|)|df|. \tag{7.15}$$

Integrating this inequality over the appropriate ranges for each side, we find an inequality that can be written

$$\int \Psi(|F'(x)|)|F'(x)|\, dx \leqq \int \Psi(|f'(x)|)|f'(x)|\, dx. \tag{7.16}$$

Thus we have established the following result.

THEOREM 5. *Let* $\Psi'(s) \geqq 0$, $\Psi''(s) \geqq 0$, *and let* $\Phi(s) = s\Psi(s)$, *where* $s > 0$. *Then*

$$\int \Phi(|F'(x)|)\, dx \leqq \int \Phi(|f'(x)|)\, dx. \tag{7.17}$$

Note that Corollary 1.2 for $p \geqq 2$ can be obtained from Theorem 5 by the particular choice $\Psi(s) = s^{p-1}$. The arc length function of Theorem 4, however, does not satisfy the conditions of Theorem 5.

A generalization to convex functions of Theorem 4, or rather of Corollary 4.1, can be established by the following argument. Assuming again (7.4), (7.5), and (7.10), and letting $\Psi'(s) > 0$, $\Psi''(s) > 0$, we have

$$\begin{aligned} \int ||w^*||\Psi\left(\frac{||v^*||}{n}\right) dx &= \int ||v^*||\Psi\left(\frac{||v^*||}{n}\right) dy \\ &\leqq \int \left(\sum_{k=1}^{n} ||v_k||\right)\left(\sum_{l=1}^{n} \frac{1}{n} \Psi(||v_l||)\right) dy. \end{aligned} \tag{7.18}$$

Setting $||v_k|| = \alpha_k$, $\Psi(||v_l||) = \beta_l$, we see that the integrand is of the form

$$\frac{1}{n} \sum_{k,\, l=1}^{n} \alpha_k \beta_l. \tag{7.19}$$

Since Ψ is an increasing function, the ordering of the α_k and β_l in decreasing order is the same, and without loss of generality we may suppose that $\alpha_1 \geqq \alpha_2 \geqq \ldots \geqq \alpha_n$, and $\beta_1 \geqq \beta_2 \geqq \ldots \geqq \beta_n$. Write

$$\begin{aligned} \sum \alpha_k \beta_l = \quad & \alpha_1\beta_1 + \alpha_2\beta_2 + \ldots + \alpha_n\beta_n \\ + \quad & \alpha_1\beta_2 + \alpha_2\beta_3 + \ldots + \alpha_n\beta_1 \\ + \quad & \alpha_1\beta_3 + \alpha_2\beta_4 + \ldots + \alpha_n\beta_2 \\ & \cdot \\ & \cdot \\ & \cdot \\ + \quad & \alpha_1\beta_n + \alpha_2\beta_1 + \ldots + \alpha_n\beta_{n-1} \end{aligned}$$

and observe that, by [**3**, p. 261, Theorem 368], each group of n terms in one of the above rows is less than or equal to the group in the first row. Hence

$$\frac{1}{n} \sum_{k,\, l=1}^{n} \alpha_k \beta_l \leqq \sum_{k=1}^{n} \alpha_k \beta_k. \tag{7.20}$$

The integrals in (7.18) therefore do not exceed

$$(7.21)\quad \int \sum_{k=1}^{n} ||v_k||\Psi(||v_k||)\, dy = \int \sum_{k=1}^{n} ||w_k||\Psi(||v_k||)\, dx_k = \int ||w||\Psi(||v||)\, dx.$$

We have thus proved the following result.

THEOREM 6. *If v, v^*, w, w^* are as defined by* (7.4), (7.5), *and* (7.10), *and if* $\Psi'(s) > 0$, $\Psi''(s) > 0$, *then*

$$(7.22)\qquad \int ||w^*||\Psi\left(\frac{||v^*||}{n}\right) dx \leqq \int ||w||\Psi(||v||)\, dx.$$

In a similar way we can establish under the same conditions the following related inequality:

$$(7.23)\qquad \int ||w^*||\Psi(||v^*||)\, dx \leqq \int ||w||\Psi(n||v||)\, dx.$$

8. Equivariational transforms of functions of several variables. Let $f = f(x_1, \ldots, x_m)$ be defined on a domain D as in § 5, and suppose that $f \in C^1(D)$. Our definition of the equivariational transform $F = F(r)$ of f will be based on preservation of the total variation

$$(8.1)\qquad \mathrm{TV}(f) = \int_D |\nabla f|\, dV, \qquad dV = dx_1 \ldots dx_m,$$

introduced in (5.29).

Assuming spherical symmetry in E^m for F, we set

$$(8.2)\qquad \omega_m r^{m-1}\, dF(r) = -S(f)|df|,$$

where $S(y)$ denotes the total surface area of the section $f(x_1, \ldots, x_m) = y$, as in (5.16). We normalize the additive constant in F by setting $F(R) = 0$, where

$$\omega_m R^{m-1} = \int_D dV = V(D).$$

By (5.6) we have

$$(8.3)\qquad \int_D |\nabla f|\, dV = \int\int \left|\frac{df}{dn}\right| dS\, dn = \int\int |df|\, dS.$$

Since f is constant on each level surface, this integral may in view of (8.2) be expressed as

$$(8.4)\qquad \int |df| \int dS = \int S(f)\, |df| = -\int_0^R F'(r)\omega_m r^{m-1}\, dr.$$

As F is spherically symmetric and decreasing in r, we have $-F'(r) = |\nabla F|$, so that the expression becomes

$$(8.5)\qquad \int_0^R |\nabla F|\, dV_r, \qquad dV_r = \omega_m r^{m-1}\, dr.$$

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Thus the total variations of f and F are equal, as suggested by the term equivariational transform.

THEOREM 7. *If Σ is the surface area of $y = f(x_1, \ldots. x_m)$ and Σ^* the surface area of $y^* = F(r)$ in $E^m \times E^1$, then*

(8.6) $$\Sigma^* \leqq \Sigma.$$

Proof. We have, since $F = F(r)$,

(8.7) $$\Sigma^* = \int_0^R (1 + (F'(r))^2)^{1/2} \omega_m r^{m-1} dr = \omega_m \int_0^R r^{m-1}(dr^2 + (dy^*)^2)^{1/2}.$$

Observe that

(8.8) $$dV = dS\, dn$$

and from the equivariational property of F note that

(8.9) $$\omega_m r^{m-1} dy^* = S(y)|dy|.$$

At a typical point P on a level surface consider the vector

(8.10) $$v_p = (dy, dn)$$

and its vector integral over the level surface, namely

(8.11) $$\mathbf{V}(y) = \int_0^{S(y)} v_{p'}\, dS_{p'} = \int_0^{S(y)} (dy, dn)\, dS_{p'}$$
$$= \left(\int_0^{S(y)} dy\, dS_{p'}, \int_0^{S(y)} dn\, dS_{p'} \right) = (S(y)dy, \Delta V).$$

Note that in the first component dy is constant along the level surface, and that ΔV denotes the integral over the level surface of the volume elements $dn\, dS_{p'} = dV$ as in (8.8). Observe that from the definition (5.1) of μ we have

(8.12) $$\Delta V = \omega_m r^{m-1} dr.$$

Hence

(8.13) $$\mathbf{V}(y) = (S(y)dy, \omega_m r^{m-1}\, dr)$$

and from (8.9) we now find

(8.14) $$\mathbf{V}(y) = \omega_{m-1} r^{m-1}(dy^*, dr).$$

Since with the Euclidean vector norm we have

(8.15) $$|\mathbf{V}(y)| = \left| \int v_{p'}\, dS_{p'} \right| \leqq \int |v_{p'}|\, dS_{p'}$$

it follows that

(8.16) $$\Sigma^* = \int_0^R \mathbf{V}(y) \leqq \int_0^R \int |v_{p'}|\, dS_{p'} = \int \int |(dy, dn)|\, dS_{p'}.$$

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But the magnitude of (dy, dn) is ds, the hypotenuse of the infinitesimal right triangle that forms a cross-section of the level surface diagram in $E^m \times E^1$, and which lies on the surface $y = f(x_1, \ldots, x_m)$. Assuming smoothness, we now observe that the surface element $d\Sigma$ may be expressed as

$$d\Sigma = ds\, dS_{p'}. \tag{8.17}$$

Thus, finally, from (8.16) and (8.17) we find

$$\Sigma^* \leqq \iint ds\, dS_{p'} = \iint d\Sigma = \Sigma. \tag{8.18}$$

This completes the proof of the theorem. Note that the one-dimensional case would yield a two-sided or symmetrical rearrangement version of Theorem 4.

A corresponding inequality for the surface area of $y = f^*(r)$ is given by Pólya and Szegö in [**4**, p. 194]. The two results are not directly dependent one upon the other, since the level surface areas $S(y)$ may be large or small. Thus Pólya and Szegö consider an annular domain in order that a lower bound should be placed on $S(y)$.

We shall conclude by establishing a many-variable version of Theorem 5.

THEOREM 8. *Let $\Psi'(s) \geqq 0$, $\Psi''(s) \geqq 0$ and let $\Phi(s) = s\Psi(s)$, where $s > 0$. Then*

$$\int \Phi(|F'(r)|)\, dV_r \leqq \int \Phi(|\nabla f|)\, dV. \tag{8.19}$$

Proof. Since $d\mu = dV = \omega_m r^{m-1} dr$, we have, by (8.2),

$$|F'(r)| = \omega_m r^{m-1} \left|\frac{dF(r)}{d\mu}\right| = S(y) \frac{|df_m^*|}{|d\mu|}. \tag{8.20}$$

Here f_m denotes the m-dimensional rearrangement, as described at the beginning of § 5. Introducing f_1^* we see that

$$|F'(r)| = S(y) \left|\frac{df_1^*}{dx^*}\right| = S(y)\left(\int_0^{S(y)} \frac{dS}{|\nabla f|}\right)^{-1} \tag{8.21}$$

by Lemma 3.

Applying the procedure following (7.12) that was used in the proof of Theorem 5 we find, in analogy to (7.13), that

$$|F'(r)| \leqq \frac{1}{S(y)} \int_0^S |\nabla f|\, dS. \tag{8.22}$$

Since Ψ is assumed increasing and convex, we have

$$\Psi(|F'(r)|) \leqq \Psi\left(\frac{1}{S(y)} \int_0^S |\nabla f|\, dS\right) \leqq \frac{1}{S(y)} \int_0^S \Psi(|\nabla f|)\, dS; \tag{8.23}$$

this last step is the continuous generalization of the usual finite formulation of the convexity property [**3**, p. 72].

Applying (8.2) once again we find, by (5.6),

$$(8.24)\qquad \omega_m r^{m-1}\Psi(|F'(r)|)\,|dF(r)| \leqq \int_0^S \Psi(|\nabla f|)\,dS\,\frac{|df|}{|dn|}\,dn$$
$$= \int_0^S \Psi(|\nabla f|)|\nabla f|\,dS\,dn.$$

As integration over r on the left corresponds to integration over all level surfaces on the right, and since $dV = dS\,dn$, we find

$$(8.25)\qquad \int_0^R \omega_m r^{m-1}\Psi(|F'(r)|)\,|F'(r)|\,dr \leqq \int\int \Psi(|\nabla f|)|\nabla f|\,dV.$$

Denoting by $dV_r = \omega_m r^{m-1}\,dr$ the volume element in the rearranged domain, we obtain (8.19). This concludes the proof of Theorem 8.

If the inequalities for Ψ' and Ψ'' in Theorems 5 and 8 are strict, then equality will hold in these theorems under the same circumstances as in Theorems 1 and 2, respectively.

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